NEGATIVITY-PRESERVING TRANSFORMS OF TUPLES OF SYMMETRIC MATRICES

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Abstract. Compared to the entrywise transforms which preserve positive semidefiniteness, those leaving invariant the inertia of symmetric matrices reveal a surprising rigidity. We achieve the classification of negativity preservers by combining recent advances in matrix analysis with some elementary arguments using well chosen test matrices. We unveil a complete analysis of the multi-variable setting with a striking combination of absolute monotonicity and single-variable rigidity appearing in our conclusions.

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1. Introduction and main results

Our work here is a part of a larger project aimed at identifying those entrywise operations that leave invariant various classes of structured matrices. The importance of entrywise matrix transforms that preserve positive definiteness has been apparent for many years, in settings such as distance geometry and Fourier analysis on groups. Key observations due to Schur, Schoenberg, Rudin, and Herz, and later developments by Loewner, Horn, Christensen and Ressel, and Vasudeva, realized manifold achievements in the classification of this form of matrix operation. Among all of these contributions, Schoenberg's foundational result [17] stands apart. We state his theorem in a form which includes some slight enhancements accumulated over time.

Henceforth, a function $f : I \to \mathbb{R}$ acts entrywise on a matrix $A = (a_{ij})$ with entries in I via the prescription $f[A] := \big(f(a_{ij})\big)$.

Theorem 1.1 ([17, 15, 2]). *Let $I := (-\rho, \rho)$, where $0 < \rho \leq \infty$. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

Date: November 22, 2023.

2010 *Mathematics Subject Classification.* 15B48 (primary); 15A18, 26A48, 32A05 (secondary).

Key words and phrases. symmetric matrix, inertia, entrywise transform, Schoenberg's theorem, absolutely monotone function, Pontryagin space.

(1) *The function f acts entrywise to preserve the set of positive semidefinite matrices of all dimensions with entries in I.*
(2) *The function f is* absolutely monotone, *that is, $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all $x \in I$ with $c_n \geq 0$ for all n.*

As seen above, we use the term *absolutely monotone* to describe functions which have a power-series representation with non-negative Maclaurin coefficients, although the non-negativity of derivatives holds only for a certain subset of the domain (which is $[0, \rho)$ above). For more on this, see Appendix A.

The study of entrywise transforms of positive semidefinite matrices was recently reinvigorated by statisticians. This was motivated by the use of thresholding and other entrywise operations to regularize large correlation matrices which are close to being sparse. For details of this line of enquiry, as well as recent related developments, see [8], [4, Section 5], and the monograph [12].

One way to view Theorem 1.1, which motivates our work here, is that it classifies the entrywise transforms which do not change the number of negative eigenvalues possessed by any positive semidefinite matrix, which is necessarily zero, but allow for the nullity or number of positive eigenvalues to change. (Throughout this work, eigenvalues are counted with multiplicity.) From this perspective, it is natural to seek to classify entrywise operations which do not change the inertia of positive semidefinite matrices. As we will show below, this class of inertia preservers is far smaller than the collection of functions in Theorem 1.1.

1.1. **One-variable inertia preservers.** A step further from Schoenberg's Theorem is the description of entrywise transforms that preserve the inertia of matrices with precisely k negative eigenvalues for some choice of integer k. To state our first major result precisely, we introduce the following notation.

Given non-negative integers n and k, with $n \geq 1$ and $k \leq n$, we let $\mathcal{S}_n^{(k)}(I)$ denote the set of $n \times n$ symmetric matrices with entries in $I \subseteq \mathbb{R}$ having exactly k negative eigenvalues; here and throughout, eigenvalues are counted with multiplicity. Let

$$\mathcal{S}^{(k)}(I) := \bigcup_{n=k}^{\infty} \mathcal{S}_n^{(k)}(I)$$

be the set of real symmetric matrices of arbitrary size with entries in I and exactly k negative eigenvalues. For brevity we let $\mathcal{S}^{(k)} := \mathcal{S}^{(k)}(\mathbb{R})$ and $\mathcal{S}_n^{(k)} := \mathcal{S}_n^{(k)}(\mathbb{R})$.

Note that, for any $n \geq 1$, the sets $\mathcal{S}_n^{(0)}$, $\mathcal{S}_n^{(1)}$, $\ldots$, $\mathcal{S}_n^{(n)}$ are pairwise disjoint and partition the set of $n \times n$ real symmetric matrices.

We now assert

Theorem 1.2. *Let $I := (-\rho, \rho)$, where $0 < \rho \leq \infty$, and let k be a non-negative integer. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

(1) *The entrywise transform $f[-]$ preserves the inertia of all matrices in $\mathcal{S}^{(k)}(I)$.*
(2) *The function is a positive homothety: $f(x) \equiv cx$ for some constant $c > 0$.*

Thus, the class of inertia preservers for the collection of real symmetric matrices of all sizes with k negative eigenvalues is highly restricted, whatever the choice of k: every such map in fact preserves not only the nullity and the total multiplicities of positive and negative eigenvalues, it preserves the eigenvalues themselves, up to simultaneous scaling.

Our second result resolves the dimension-free preserver problem for $\mathcal{S}^{(k)}(I)$. If $k = 0$, Schoenberg's Theorem 1.1 shows that the class of entrywise preservers is far larger than the class of inertia preservers, which contains only the positive homotheties. However, if $k > 0$ then this is no longer the case.

Theorem 1.3. *Let $I := (-\rho, \rho)$, where $0 < \rho \leq \infty$, and let k be a positive integer. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

(1) *The entrywise transform $f[-]$ sends $\mathcal{S}^{(k)}(I)$ to $\mathcal{S}^{(k)}$.*
(2) *The function f is a positive homothety, so that $f(x) \equiv cx$ for some $c > 0$, or, when $k = 1$, we can also have that $f(x) \equiv -c$ for some $c > 0$.*

There is a notable rigidity phenomenon here, in stark contrast to the dimension-free preserver problem for positive semidefinite matrices (the $k = 0$ case). When there is at least one negative eigenvalue, the non-constant transforms leaving invariant the number of negative eigenvalues also conserve the number of positive eigenvalues and the number of zero eigenvalues; more strongly, they preserve the eigenvalues themselves, up to simultaneous scaling. That is, Schoenberg's theorem collapses to just homotheties if $k \geq 2$, with the additional appearance of negative constant functions if $k = 1$ (and the collection of preservers is non-convex in this last case).

It is interesting to compare these results with a theorem obtained about three decades ago by FitzGerald, Micchelli, and Pinkus [7], who classified the entrywise preservers of conditionally positive matrices of all sizes. An $n \times n$ real symmetric matrix A is *conditionally positive* if the corresponding quadratic form is positive semidefinite when restricted to the hyperplane $\mathbf{1}_n^\perp \subseteq \mathbb{R}^n$, where $\mathbf{1}_n := (1, \dots, 1)^T$. That is,

$$\text{if } v \in \mathbb{R}^n \text{ is such that } v^T \mathbf{1}_n = 0 \text{ then } v^T A v \geq 0.$$

The authors showed in [7, Theorem 2.9] that an entrywise preserver $f[-]$ of this class of conditionally positive matrices corresponds to a function that differs from being absolutely monotone by a constant:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all } x \in \mathbb{R}, \text{ where } c_n \geq 0 \text{ for all } n \geq 1. \tag{1.1}$$

From the perspective of the present article, conditionally positive matrices are those real symmetric matrices with at most one negative eigenvalue, with the negative eigenspace (if it exists) constrained to equal $\mathbb{R}\mathbf{1}_n$. If this constraint on the eigenspace is removed, Theorem 1.3 shows that the class of preservers shrinks dramatically. The high level of rigidity for preservers of negative inertia is akin to that occurring for preservers of totally positive and totally non-negative kernels: see the recent work [5] for further details on the latter.

Another way to view Schoenberg's Theorem [17] is as the description of the entrywise transforms that preserve the class of correlation matrices of vectors in Hilbert space. A completely parallel theory is developed in Section 3, providing the classification of self transforms of Gram matrices of vectors belonging to a Pontryagin space (that is, a Hilbert space endowed with an indefinite sesquilinear form with a finite number of negative squares [1]).

A natural variation on the theme of Theorem 1.3 is to classify those entrywise maps leaving invariant the collection of real symmetric matrices with at most k negative eigenvalues. In other words, we seek to classify the endomorphisms of the closure

$$\overline{\mathcal{S}_n^{(k)}}(I) := \bigcup_{j=0}^{k} \mathcal{S}_n^{(j)}(I), \qquad \text{where } n \geq 1. \tag{1.2}$$

Note that the domain remains I and not its closure $\overline{I}$. Similarly to before, we let $\overline{\mathcal{S}_n^{(k)}}$ serve as an abbreviation for $\overline{\mathcal{S}_n^{(k)}}(\mathbb{R})$.

Once again, if $k = 0$ then this is just Schoenberg's Theorem 1.1, which yields a large class of transforms. In contrast, if $k > 0$ then we again obtain a far smaller class.

Theorem 1.4. *Let $I := (-\rho, \rho)$, where $0 < \rho \leq \infty$, and let k be a positive integer. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

1. *The entrywise transform $f[-]$ sends $\overline{\mathcal{S}_n^{(k)}}(I)$ to $\overline{\mathcal{S}_n^{(k)}}$ for all $n \geq k$.*
2. *The function f is either linear and of the form $f(x) \equiv f(0) + cx$, where $f(0) \geq 0$ and $c > 0$, or constant, so that $f(x) \equiv d$ for some $d \in \mathbb{R}$.*

Theorems 1.2, 1.3 and 1.4 are negativity-preserving results. All three turn out to be consequences of the following theorem that we prove in Section 2 below.

Theorem A. *Let $I := (-\rho, \rho)$, where $0 < \rho \leq \infty$, and let k and l be positive integers such that $l = 1$ if $k = 1$ and $l \leq 2k - 2$ if $k \geq 2$. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

1. *The entrywise transform $f[-]$ sends $\mathcal{S}_n^{(k)}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k$.*
2. *Exactly one of the following occurs:*
   (a) *the function f is constant, so that $f(x) \equiv d$ for some $d \in \mathbb{R}$;*
   (b) *it holds that $l \geq k$ and f is linear, with $f(x) \equiv f(0) + cx$, where $c > 0$ and also $f(0) \geq 0$ if $l = k$.*

Moreover, if $k \geq 1$ and $l = 0$ then the entrywise transform $f[-]$ sends $\mathcal{S}_n^{(k)}(I)$ to $\overline{\mathcal{S}_n^{(0)}} = \mathcal{S}_n^{(0)}$ for all $n \geq k$ if and only if $f(x) \equiv c$ for some $c \geq 0$.

Finally, if $k = 0$ and $l \geq 1$ then the entrywise transform $f[-]$ sends $\mathcal{S}_n^{(0)}(I) \to \overline{\mathcal{S}_n^{(l)}}$ for all $n \geq 1$ if and only if

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \textit{for all } x \in (-\rho, \rho), \textit{ where } c_n \geq 0 \textit{ for all } n \geq 1.$$

Note that setting $k = l = 0$ in Theorem A(1) (the missing case) gives exactly hypothesis (1) of Schoenberg's Theorem 1.1.

The class of functions identified in Theorem A when $k = 0$ and $l \geq 1$ is independent of l and coincides with the dimension-free entrywise preservers for two related but distinct constraints: (a) conditional positivity, as noted above (1.1), and (b) Loewner monotonicity, so that $f[A] - f[B] \in \mathcal{S}^{(0)}$ whenever $A - B \in \mathcal{S}^{(0)}$. The latter claim is a straightforward consequence of Schoenberg's Theorem 1.1; see [12, Theorem 19.2].

1.2. **Multi-variable transforms and non-balanced domains.** Given the results described above, it is natural to seek extensions in two directions, aligned to previous work. In the sequel, for any integers m and n with $m \leq n$, we let $[m : n]$ denote the set $\{m, m+1, \dots, n\} = [m, n] \cap \mathbb{Z}$.

- *Functions acting on matrices with entries in $I = (0, \rho)$ (positive entries) or in $I = [0, \rho)$ (non-negative entries).* For preservers of positive semidefiniteness, this problem was considered by Loewner and Horn [10] and Vasudeva [19] for the case $\rho = \infty$, and then in recent work [2] for finite ρ. In each case, the class obtained consists of functions represented by convergent power series with non-negative Maclaurin coefficients.
- *Functions acting on m-tuples of matrices.* A function $f : I^m \to \mathbb{R}$ acts entrywise on m-tuples of matrices with entries in I: if $B^{(p)} = (b_{ij}^{(p)})$ is an $n \times n$ matrix for $p = 1, \dots, m$ then the $n \times n$ matrix $f[B^{(1)}, \dots, B^{(m)}]$ has (i, j) entry
$$f[B^{(1)}, \dots, B^{(m)}]_{ij} = f(b_{ij}^{(1)}, \dots, b_{ij}^{(m)}) \qquad \text{for all } i, j \in [1 : n].$$
In this case, the classification of preservers in the positive-semidefinite setting was achieved by FitzGerald, Micchelli and Pinkus [7] when $I = \mathbb{R}$, and then in recent work [2] over smaller domains.

Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ is the set of non-negative integers, and a point $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, we use the standard notation $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Theorem 1.5 ([2]). *Let $I = (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$, and let m be a positive integer. The function $f : I^m \to \mathbb{R}$ acts entrywise to send m-tuples of positive semidefinite matrices with entries in I of arbitrary size to the set of positive semidefinite matrices if and only if f is represented on I^m by a convergent power series with non-negative coefficients:*
$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \textit{for all } \mathbf{x} \in I^m, \textit{ where } c_{\boldsymbol{\alpha}} \geq 0 \textit{ for all } \boldsymbol{\alpha}. \tag{1.3}$$

Below, we extend this result to the complete classification of negativity-preserving transforms acting on tuples of matrices, in the spirit of the one-variable results above, over the three types of domain: $I = (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$. The key to this is a multi-variable strengthening of Theorem A which also applies to these three different types of domain. For ease of exposition and proof, this is split into two parts, Theorems B and C. Detailed proofs of these theorems are to be found in Sections 4 and 5, together with the necessary supporting results and subsequent corollaries. Section 4 is concerned with the extension of Theorem A to several variables and Section 5 then allows the restriction of I from $(-\rho, \rho)$ to $(0, \rho)$ or $[0, \rho)$.

Notation 1.6. In Theorem A, the parameters k and l control the degree of negativity in the domain and the co-domain, respectively. In the multi-variable setting, the domain parameter k becomes an m-tuple of non-negative integers $\mathbf{k} = (k_1, \dots, k_m)$. Given such a $\mathbf{k}$, we may permute the entries so that any zero entries appear first: more formally, there exists $m_0 \in [0 : m]$ with $k_p = 0$ for $p \in [1 : m_0]$ and $k_p \geq 1$ for $p \in [m_0 + 1 : m]$. We say that $\mathbf{k}$ is admissible in this case and let $k_{\max} := \max\{1, k_p : p \in [1 : m]\}$,
$$\mathcal{S}_n^{(\mathbf{k})}(I) := \mathcal{S}_n^{(k_1)}(I) \times \cdots \times \mathcal{S}_n^{(k_m)}(I), \quad \text{and} \quad \overline{\mathcal{S}_n^{(\mathbf{k})}}(I) := \overline{\mathcal{S}_n^{(k_1)}}(I) \times \cdots \times \overline{\mathcal{S}_n^{(k_m)}}(I).$$

Theorem B (The cases $\mathbf{k} = \mathbf{0}$ and $l = 0$, stated and proved below as Theorem 4.1)**.** *Let $I = (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \le \infty$, and let $\mathbf{k} \in \mathbb{Z}_+^m$ be admissible. Given a function $f : I^m \to \mathbb{R}$, the following are equivalent.*

1. *The entrywise transform $f[-]$ sends $\overline{\mathcal{S}_n^{(\mathbf{k})}}(I)$ to $\overline{\mathcal{S}_n^{(0)}}$ for all $n \ge k_{\max}$.*
2. *The entrywise transform $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\mathcal{S}_n^{(0)}$ for all $n \ge k_{\max}$.*
3. *The function f is independent of $x_{m_0+1}, \dots, x_m$ and is represented on I^m by a convergent power series in the reduced tuple $\mathbf{x}' := (x_1, \dots, x_{m_0})$, with all Maclaurin coefficients non-negative.*

If, instead, $\mathbf{k} = \mathbf{0}$ and $l \ge 1$ then $f[-]$ sends $\mathcal{S}_n^{(\mathbf{0})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ if and only if the function f is represented on I^m by a power series $\sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \ge 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$.

The parallels between Theorem B and Theorem A when $k = 0$ or $l = 0$ are clear, and when $\mathbf{k} = \mathbf{0}$ and $l = 0$ we recover Theorem 1.5.

Our second variation on Theorem A combines both kinds of preservers seen above: the rich class of multi-variable power series and the rigid family of linear homotheties.

Theorem C (The case of $\mathbf{k} \ne \mathbf{0}$ and $l > 0$, stated and proved below as Theorem 4.2)**.** *Let $I := (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \le \infty$. Moreover, let $\mathbf{k}$ be admissible and not equal to $\mathbf{0}$, and suppose $l = 1$ if $k_p = 1$ for some $p \in [1 : m]$ and $l \in [1 : 2K - 2]$ otherwise, where $K = \min\{k_p : p \in [1 : m],\ k_p > 0\}$. Given a function $f : I^m \to \mathbb{R}$, the following are equivalent.*

1. *The map $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \ge k_{\max}$.*
2. *There exist an index $p_0 \in [m_0 + 1 : m]$, a function $F : (-\rho, \rho)^{m_0} \to \mathbb{R}$, and a constant $c \ge 0$ such that*
   (a) *we have the representation*
   $$f(\mathbf{x}) = F(x_1, \dots, x_{m_0}) + c x_{p_0} \qquad \text{for all } \mathbf{x} \in I^m, \tag{1.4}$$
   (b) *the function $\mathbf{x}' \mapsto F(\mathbf{x}') - F(\mathbf{0})$ is absolutely monotone on $[0, \rho)^{m_0}$,*
   (c) *if $c > 0$ then p_0 is unique and $l \ge k_{p_0}$, and*
   (d) *if $c > 0$ and $l = k_{p_0}$ then $F(\mathbf{0}) \ge 0$.*

Again, the parallels with Theorem A are clear, and if $m = 1$ and $I = (-\rho, \rho)$ then Theorems B and C recover this theorem.

Given previous results, the fact that the transforms classified by these theorems are real analytic is to be expected, but their exact structure is surprising. In particular, it is striking that one cannot have more than one of the variables $x_{m_0+1}, \dots, x_m$ appearing in the description (1.4).

Theorems B and C are the building blocks we use to obtain the classification of negativity-preserving transforms in several variables. The classes of transforms do not depend on the choice of one-sided or two-sided domains, akin to the one-variable setting.

1.3. **Organization of the paper.** In Section 2, we prove Theorem A and then deduce from it Theorems 1.2, 1.3 and 1.4. Working in the one-variable setting and with the two-sided domain $I = (-\rho, \rho)$ allows us to introduce several key ideas and techniques in less complex circumstances, and these will then be employed in the multi-variable setting and with one-sided domains.

We next explore the territory of Pontryagin space and classify in Section 3 the entrywise transforms of indefinite Gram matrices in this environment.

In Sections 4 and 5 we prove the several-variables results mentioned above, first over the two-sided product domain and then on the one-sided versions. In each section, this is followed by the classification of negativity preservers, extending the results obtained in the single-variable case. We conclude with an appendix that proves the multivariate Bernstein theorem, that absolutely monotone functions on $(0,\rho)^m$ necessarily have power-series representations with non-negative Maclaurin coefficients.

1.4. **Notation.** We let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}_+ = \{0,1,2,\ldots\}$ the set of non-negative integers. Given a, $b \in \mathbb{Z}_+$ with $a < b$, we let $[a:b] := [a,b] \cap \mathbb{Z}_+$. If $\rho = \infty$ then ρ/a and $\rho - a$ also equal ∞, for any finite $a > 0$. We also set $0^0 := 1$.

2. Inertia preservers for matrices with real entries

We now begin to obtain the results stated in the introduction. In this section, we will consider functions with domain $I = (-\rho,\rho)$. The proofs of our results involve three key ideas: (a) using the translation g of f, where $g(x) := f(x) - f(0)$; (b) a "replication trick" that we will demonstrate shortly; (c) the following lemma.

Lemma 2.1. *Let $h : I \to \mathbb{R}$ be absolutely monotone, where $I := (-\rho,\rho)$ and $0 < \rho \le \infty$, and let $C := \begin{bmatrix} A & B \\ B & A \end{bmatrix}$, where A, $B \in \mathcal{S}_n^{(0)}(I)$ are positive semidefinite matrices.*

(1) *If $A - B \in \mathcal{S}_n^{(k)}$ for some non-negative integer k then $C \in \mathcal{S}_{2n}^{(k)}(I)$.*
(2) *If $h[C] \in \overline{\mathcal{S}_{2n}^{(l)}}$ for some non-negative integer l then $h[A] - h[B] \in \overline{\mathcal{S}_n^{(l)}}$.*

Proof. Note first that

$$J^T \begin{bmatrix} A & B \\ B & A \end{bmatrix} J = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}, \quad \text{where } J := \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{Id}_n & -\mathrm{Id}_n \\ \mathrm{Id}_n & \mathrm{Id}_n \end{bmatrix} \tag{2.1}$$

is orthogonal and Id_n is the $n \times n$ identity matrix. As $A + B$ is positive semidefinite, it follows from (2.1) that if $A - B \in \mathcal{S}_n^{(k)}$ then $C \in \mathcal{S}_{2n}^{(k)}(I)$.

Next, if $h[C] \in \overline{\mathcal{S}_{2n}^{(l)}}$ then another application of (2.1) gives that

$$\begin{bmatrix} h[A] + h[B] & 0 \\ 0 & h[A] - h[B] \end{bmatrix} \in \overline{\mathcal{S}_{2n}^{(l)}}.$$

Schoenberg's Theorem 1.1 gives that $h[A] + h[B]$ is positive semidefinite and therefore $h[A] - h[B] \in \overline{\mathcal{S}_n^{(l)}}$, as claimed. □

In addition to the three key ideas, we will use the following basic consequence of Weyl's interlacing theorem.

Lemma 2.2. *Suppose $A \in \mathcal{S}_n^{(k)}$ for some positive integer n and non-negative integer k. If $B \in \mathcal{S}_n^{(0)}$ has rank 1 then $A + B \in \mathcal{S}_n^{(k-1)} \cup \mathcal{S}_n^{(k)}$ and $A - B \in \mathcal{S}_n^{(k)} \cup \mathcal{S}_n^{(k+1)}$ (where $\mathcal{S}_n^{(-1)} := \emptyset$ and $\mathcal{S}_n^{(n+1)} := \emptyset$).*

Proof. Let $\lambda_1(C) \le \cdots \le \lambda_n(C)$ denote the eigenvalues of the $n \times n$ real symmetric matrix C, repeated according to multiplicity, and let $\lambda_0(C) := -\infty$ and $\lambda_{n+1}(C) := \infty$.

We note first that $C \in \mathcal{S}_n^{(k)}$ if and only if $0 \in (\lambda_k(C), \lambda_{k+1}(C)]$, for $k = 0, 1, \ldots, n$.

If A and B are as in the statement of the lemma, then the following inequalities are a corollary [11, Corollary 4.3.9] of Weyl's interlacing theorem:

$$\lambda_1(A) \leq \lambda_1(A+B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(A+B).$$

It follows that $0 \in (\lambda_k(A), \lambda_{k+1}(A)] = I_1 \cup I_2$, where $I_1 := (\lambda_k(A), \lambda_k(A+B)]$ and $I_2 := (\lambda_k(A+B), \lambda_{k+1}(A)]$.

Then either $k \geq 1$ and $0 \in I_1 \subseteq (\lambda_{k-1}(A+B), \lambda_k(A+B)]$, so $A+B \in \mathcal{S}_n^{(k-1)}$, or $0 \in I_2 \subseteq (\lambda_k(A+B), \lambda_{k+1}(A+B)]$, so $A+B \in \mathcal{S}_n^{(k)}$. The first claim follows.

For the second part, note that $A - B \in \mathcal{S}_n^{(l)}$ for some $l \in \{0, \ldots, n\}$, so the previous working gives that $A = (A-B)+B \in \mathcal{S}_n^{(l-1)} \cup \mathcal{S}_n^{(l)}$. As $A \in \mathcal{S}_n^{(k)}$, it follows that $l-1 = k$ or $l = k$. This completes the proof. $\square$

Proof of Theorem A. We first show that (2) implies (1) (and this holds without any restrictions on the domain $I \subseteq \mathbb{R}$).

If $f(x) \equiv d$ for some $d \in \mathbb{R}$ then $f[A] \in \overline{\mathcal{S}_n^{(1)}} \subseteq \overline{\mathcal{S}_n^{(l)}}$, since the matrix $d\mathbf{1}_{n\times n} = d\mathbf{1}_n\mathbf{1}_n^T$ has nullity at least $n-1$ and the eigenvalue nd. This also gives one implication for the final claim in the statement of the theorem, since $f[A] = d\mathbf{1}_{n\times n} \in \mathcal{S}_n^{(0)}$ if $d \geq 0$. Moreover, if $d < 0$ then $f[A] \in \mathcal{S}_n^{(1)}$ and so f does not map $S_n^{(k)}$ into $\mathcal{S}_n^{(0)}$ in this case.

Next, suppose $f(x) = f(0) + cx$ with $c > 0$, and let $A \in \mathcal{S}_n^{(k)}$. If $f(0) \geq 0$ then $f[A] = f(0)\mathbf{1}_{n\times n} + cA \in \mathcal{S}_n^{(k-1)} \cup \mathcal{S}_n^{(k)}$, by Lemma 2.2, and so $f[A] \in \overline{\mathcal{S}_n^{(l)}}$ as long as $l \geq k$. If instead $f(0) < 0$ then, again by Lemma 2.2, $f[A] = f(0)\mathbf{1}_{n\times n} + cA$ has at most $k+1$ negative eigenvalues and $k+1 \leq l$ as long as $l > k$.

This shows that (2) $\implies$ (1) and its $l = 0$ analogue in the penultimate sentence of Theorem A. For the $k = 0$ analogue in the final sentence, if f has the prescribed form then $g : x \mapsto f(x) - f(0)$ is absolutely monotone, so $g[-] : \mathcal{S}_n^{(0)}(I) \to \mathcal{S}_n^{(0)}$ by Schoenberg's Theorem 1.1. Now Lemma 2.2 implies that $f[-] : \mathcal{S}_n^{(0)}(I) \to \overline{\mathcal{S}_n^{(1)}} \subseteq \overline{\mathcal{S}_n^{(l)}}$, as required.

Henceforth, we suppose that $I = (-\rho, \rho)$, where $0 < \rho \leq \infty$, the non-negative integers k and l are not both zero, and $f : I \to \mathbb{R}$ is such that $f[-] : \mathcal{S}_n^{(k)}(I) \to \overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k$, or for all $n \geq 1$ if $k = 0$.

To show that f has the form claimed in each case, we proceed in a series of steps.

Step 1: Let $g : I \to \mathbb{R}$ be defined by setting $g(x) := f(x) - f(0)$. Then g is absolutely monotone.

Indeed, by Lemma 2.2, $g[-] : \mathcal{S}_N^{(k)}(I) \to \overline{\mathcal{S}_N^{(l+1)}}$ for all $N \geq k$. If $A \in \mathcal{S}_n^{(0)}(I)$, then the block-diagonal matrix

$$D := (-t_0 \operatorname{Id}_k) \oplus A^{\oplus(l+2)} \in \mathcal{S}_N^{(k)},$$

where $t_0 \in (0, \rho)$, there are $l+2$ copies of A along the block diagonal, and we have that $N = k + (l+2)n$. It follows that

$$g[D] = (g(-t_0)\operatorname{Id}_k) \oplus g[A]^{\oplus(l+2)} \in \overline{\mathcal{S}_N^{(l+1)}}. \tag{2.2}$$

Thus, the block-diagonal matrix $g[A]^{\oplus(l+2)}$ can have at most $l+1$ negative eigenvalues. This is only possible if $g[A]$ has no such eigenvalues, which implies that $g[A]$ is positive semidefinite whenever A is. Thus, by Schoenberg's Theorem 1.1, the function g is absolutely monotone. (This is the replication trick mentioned at the beginning of this section.)

If $k = 0$ we are now done, so henceforth we assume that $k \geq 1$.

Step 2: If g is as in Step 1 and $B \in \mathcal{S}_n^{(0)}(I)$ has rank k then $g[B]$ has rank at most l.

Define an absolutely monotone function $h : I \to \mathbb{R}$ by setting $h(x) := f(x) + |f(0)|$. If $C \in \mathcal{S}_n^{(k)}(I)$ then $f[C] \in \overline{\mathcal{S}_n^{(l)}}$, by assumption, and so $h[C] = f[C] + |f(0)|\mathbf{1}_{n\times n} \in \overline{\mathcal{S}_n^{(l)}}$, by Lemma 2.2. Thus $h[-] : \mathcal{S}_n^{(k)}(I) \to \overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k$. Applying Lemma 2.1 with the matrices $A = \mathbf{0}_{n\times n}$ and B as above yields

$$-g[B] = h[\mathbf{0}_{n\times n}] - h[B] \in \overline{\mathcal{S}_n^{(l)}}.$$

However, the matrix $g[B]$ is positive semidefinite, by Schoenberg's Theorem 1.1, so it has no negative eigenvalues, and hence has rank at most l.

We now resolve the $l = 0$ case in Step 3, before working with $l > 0$ in Steps 4 and 5.

Step 3: If $f[-] : \mathcal{S}_n^{(k)}(I) \to \mathcal{S}_n^{(0)}$ for all $n \geq k$ then f is constant.

Suppose this holds; let $t_0 \in (0, \rho)$ and note that the function g from Step 1 applied entrywise sends the matrix $-t_0 \mathrm{Id}_k$ to $g(-t_0)\mathrm{Id}_k$. By Step 2, the latter matrix has rank 0, so $g(-t_0) = 0$. Since g is absolutely monotone and vanishes on $(-\rho, 0)$, it must be zero, by the identity theorem. Hence f is constant.

Step 4: If f is non-constant then f is linear with positive slope.

Let $g(x) = \sum_{j=1}^{\infty} c_j x^j$ and suppose for contradiction that $c_r > 0$ for some $r \geq 2$.

We first consider the case $k \geq 2$, which implies that $l \leq 2k - 2$ and so $2k - 1 > l$. Choose $u \in \mathbb{R}^{2k-1}$ with distinct positive coordinates, and let

$$B := t_0\Big(\mathbf{1}_{2k-1}\mathbf{1}_{2k-1}^T + uu^T + u^{\circ 2}(u^{\circ 2})^T + \cdots + u^{\circ(k-1)}(u^{\circ(k-1)})^T\Big), \tag{2.3}$$

where t_0 is a positive real number taken sufficiently small to ensure B has entries in $(0, \rho)$, and $u^{\circ\alpha} := (u_1^\alpha, \ldots, u_{2k-1}^\alpha)^T$ for any $\alpha \in \mathbb{R}$. The matrix B is positive semidefinite and has rank k, as its column space is spanned by the linearly independent set $\{\mathbf{1}_{2k-1}, u, \ldots, u^{\circ(k-1)}\}$. Using the Loewner ordering ($C \geq D$ if and only if $C - D$ is positive semidefinite) and the Schur product theorem [18], it follows that

$$g[B] \geq c_r B^{\circ r} \geq c_r t_0^r \sum_{j=0}^{r(k-1)} u^{\circ j}(u^{\circ j})^T \geq c_r t_0^r \sum_{j=0}^{2(k-1)} u^{\circ j}(u^{\circ j})^T \geq \mathbf{0}_{(2k-1)\times(2k-1)}.$$

(The second inequality holds because the polynomial $(1 + \cdots + x^{k-1})^r - (1 + \cdots + x^{r(k-1)})$ has non-negative coefficients.) By Vandermonde theory, this last matrix has rank $2k - 1 > l$, whereas $g[B]$ has rank at most l, by Step 2. This is a contradiction since $C \geq D \geq \mathbf{0}$ implies that the column space of D is a subset of that of C. As g is non-constant, it is therefore linear with positive slope, hence so is f.

For the remaining case, we suppose $k = l = 1$ and choose $t_0 \in (0, \rho/5)$, so that

$$A := t_0 \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \in \mathcal{S}_2^{(0)}(I) \quad \text{and} \quad B := t_0 \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \in \mathcal{S}_2^{(0)}(I), \tag{2.4}$$

whereas $A - B = -t_0 \mathbf{1}_{2\times 2} \in \mathcal{S}_2^{(1)}(I)$. As for Step 2, the transform $h[-] : \mathcal{S}_4^{(1)}(I) \to \overline{\mathcal{S}_4^{(1)}}$, where $h(x) := f(x) + |f(0)|$, so $h[A] - h[B] \in \overline{\mathcal{S}_2^{(1)}}$, by Lemma 2.1. Hence $h[B] - h[A]$ has at most one positive eigenvalue. We will now show that if $c_r > 0$ for some $r \geq 2$

then $h[B] - h[A]$ is positive definite, a contradiction, so that f must, in fact, be linear with positive slope.

To see this, suppose $c_r \geq 0$ for some $r \geq 2$, let $A^{\circ\alpha} := (a_{ij}^{\alpha})$ for any $\alpha \in \mathbb{R}$ and note that

$$h[B] - h[A] = g[B] - g[A] = \sum_{j=1}^{\infty} c_j (B^{\circ j} - A^{\circ j}). \tag{2.5}$$

The matrix

$$C_j = B^{\circ j} - A^{\circ j} = t_0^j \begin{bmatrix} 2^j - 1 & 3^j - 2^j \\ 3^j - 2^j & 5^j - 4^j \end{bmatrix}$$

is positive semidefinite for $j = 1$ and positive definite for $j \geq 2$ as

$$t_0^{-2j} \det C_j = 10^j - 9^j - 8^j + 6^j + 6^j - 5^j = \sum_{m=1}^{j} \binom{j}{m} 5^{j-m} (5^m - 4^m - 3^m + 2);$$

if $m = 1$ then the summand vanishes and otherwise the summand is positive, since

$$1 = (3/5)^2 + (4/5)^2 \geq (3/5)^m + (4/5)^m \qquad \text{for any } m \geq 2.$$

Consequently, we see that $h[B] - h[A] \geq c_r (B^{\circ r} - A^{\circ r})$, which is positive definite. Hence so is $h[B] - h[A]$, and this contradiction implies that f must be linear with positive slope in this case as well.

Step 5: Concluding the proof.

We may assume $f(x) = f(0) + cx$, where $c > 0$, and we must show that $l \geq k$ and that $f(0) \geq 0$ if $l = k$.

If $l > k$ there is nothing to prove. If $l < k$ then we claim no such function f exists. Let $\{v_1 := \mathbf{1}_{k+1}, v_2, \ldots, v_{k+1}\}$ be an orthogonal set in $\mathbb{R}^{k+1}$, fix $\delta \in (0, \rho)$, and choose a positive ϵ small enough so that the entries of the matrix

$$A := \delta \mathbf{1}_{k+1} \mathbf{1}_{k+1}^T - \epsilon \sum_{j=2}^{k+1} v_j v_j^T \tag{2.6}$$

lie in $(0, \rho)$. Then $A \in \mathcal{S}_{k+1}^{(k)}(I)$ and

$$f[A] = (f(0) + c\delta) \mathbf{1}_{k+1} \mathbf{1}_{k+1}^T - c\epsilon \sum_{j=2}^{k+1} v_j v_j^T \in \mathcal{S}_{k+1}^{(k)} \cup \mathcal{S}_{k+1}^{(k+1)},$$

as we know the eigenvalues explicitly. Hence $f[-]$ cannot send $\mathcal{S}_{k+1}^{(k)}(I)$ to $\overline{\mathcal{S}_{k+1}^{(l)}}$ if $l < k$.

The final case to consider is when $l = k$ and $f(0) < 0$, but the counterexample (2.6) also works here if we insist that $\delta < |f(0)|/c$, whence $f(0) + c\delta < 0$ and therefore $f[A] \in \mathcal{S}_k^{(k+1)}$; this shows that $f[A] \notin \overline{\mathcal{S}_{k+1}^{(l)}}$. □

With Theorem A at hand, we show the remaining results above.

Proof of Theorems 1.2 and 1.3. It is straightforward to verify that (2) implies (1) for both theorems.

Next, suppose $k \geq 1$. If $f[-]$ preserves the inertia of all matrices in $\mathcal{S}_n^{(k)}(I)$ then $f[-]$ sends $\mathcal{S}_n^{(k)}(I)$ into $\mathcal{S}_n^{(k)}$. If this holds for all n then Theorem A with $l = k$ gives that either $f(x) \equiv d$ for some $d \in \mathbb{R}$, or f is linear, so $f(x) \equiv f(0) + cx$, with $f(0) \geq 0$ and $c > 0$.

If $k \geq 2$ then $f(x) \equiv d$ cannot send $\mathcal{S}^{(k)}(I)$ to $\mathcal{S}^{(k)}$. Moreover, if $k = 1$ then indeed $f(x) \equiv d$ preserves $\mathcal{S}^{(1)}$ for $d < 0$ and does not do so if $d \geq 0$. Finally, if $t_0 \in (0, \rho/2)$ then the matrix $t_0 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ shows that no constant function preserves the inertia of all matrices in $\mathcal{S}^{(1)}(I)$. Thus, we now assume f is of the form $f(x) = f(0) + cx$, with $f(0) \geq 0$ and $c > 0$, and show that $f(0) = 0$.

By hypothesis, if $t_0 \in (0, \rho)$ then

$$f[-t_0 \mathrm{Id}_k] = f(0)\mathbf{1}_{k\times k} - ct_0 \mathrm{Id}_k \in \mathcal{S}_k^{(k)}. \tag{2.7}$$

As $f(0)\mathbf{1}_{k\times k}$ has the eigenvalue $kf(0)$, so $f[-t_0\mathrm{Id}_k]$ has the eigenvalue $kf(0) - ct_0$, which can be made positive if $f(0) > 0$ by taking t_0 sufficiently small. Thus $f(0) = 0$ and this concludes the proof for $k \geq 1$.

It remains to show that (1) implies (2) in Theorem 1.2 when $k = 0$. If $f[-]$ preserves the inertia of positive semidefinite matrices then, by Schoenberg's Theorem 1.1, the function f has a power-series representation $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on I, with $c_n \geq 0$ for all $n \geq 0$. Suppose $c_r > 0$ and $c_s > 0$ for distinct non-negative integers r and s. Applying $f[-]$ to the rank-one matrix $A = t_0 \mathbf{u}\mathbf{u}^T$, where $\mathbf{u} = (1, 1/2)^T$ and $t_0 \in (0, \rho)$, and using the Loewner ordering, we see that

$$f[A] \geq c_r A^{\circ r} + c_s A^{\circ s} = c_r t_0^r \mathbf{u}^{\circ r} (\mathbf{u}^{\circ r})^T + c_s t_0^s \mathbf{u}^{\circ s} (\mathbf{u}^{\circ s})^T,$$

where $\mathbf{u}^{\circ r}$ and $\mathbf{u}^{\circ s}$ are not proportional. Hence $f[A]$ is positive definite, and therefore non-singular, while A is not. This contradicts the hypotheses, so the power series representing f has at most one non-zero term.

Finally, we claim that f is a homothety. If not, say $f(x) = cx^n$ for $c > 0$ and $n \geq 2$, then we apply f to the rank-2 matrix $B = t_0(\mathbf{1}_{3\times 3} + \mathbf{u}\mathbf{u}^T)$, where $\mathbf{u} = (x, y, z)^T$ has distinct positive entries and t_0 is positive and sufficiently small to ensure B has entries in I. The binomial theorem gives that

$$B^{\circ n} = t_0^n \sum_{j=0}^{n} \binom{n}{j} (x^j, y^j, z^j)(x^j, y^j, z^j)^T \geq t_0^n \sum_{j=0}^{2} \binom{n}{j} (x^j, y^j, z^j)(x^j, y^j, z^j)^T =: B',$$

and the column space of B' contains $(x^j, y^j, z^j)^T$ for $j = 0$, 1, and 2. These three vectors are linearly independent, as a Vandermonde determinant demonstrates, so B' is positive definite, hence non-singular. Then so is $B^{\circ n}$, while B has rank 2 by construction. This contradicts the hypothesis, and so $n = 1$ and $f(x) = cx$ as claimed. $\square$

Proof of Theorem 1.4. If f is constant then $f[A] \in \overline{\mathcal{S}_n^{(1)}} \subseteq \overline{\mathcal{S}_n^{(k)}}$ for any $n \times n$ matrix A and any positive integer k. Furthermore, if $f(x) = f(0) + cx$, with $c > 0$ and $f(0) \geq 0$, then $f[A] = f(0)\mathbf{1}_{n\times n} + cA \in \overline{\mathcal{S}_n^{(k)}}$ for any $A \in \overline{\mathcal{S}_n^{(k)}}$, by Lemma 2.2.

Conversely, if $f[-]$ sends $\overline{\mathcal{S}_n^{(k)}}(I)$ to $\overline{\mathcal{S}_n^{(k)}}$ for all $n \geq 1$ then, in particular, it sends $\mathcal{S}_n^{(k)}(I)$ to $\overline{\mathcal{S}_n^{(k)}}$. Theorem A now shows that f has the form claimed. $\square$

3. Entrywise preservers of k-indefinite Gram matrices

3.1. **Gram matrices in Pontryagin space.** In analogy with the standard version of Schoenberg's theorem, we may interpret the main result of this section as a classification of entrywise preservers of finite correlation matrices in a Hilbert space endowed with an indefinite metric. To be more specific, we introduce the following terminology.

Definition 3.1. A *Pontryagin space* is a pair (H, J), where H is a separable real Hilbert space and $J : H \to H$ is a bounded linear operator such that $J = J^*$ and $J^2 = \mathrm{Id}_H$, the identity operator on H. Note that J is an isometric isomorphism.

We write $J = P_+ - P_-$, where P_+ and P_- are orthogonal projections onto the eigenspaces $H_+ := \{x \in H : Jx = x\}$ and $H_- := \{x \in H : Jx = -x\}$, respectively, so that $H = H_+ \oplus H_-$.

We say that the Pontryagin space has *negative index* k if $\dim H_- = k$. We assume henceforth that k is positive and finite.

Here we follow Pontryagin's original convention from [13], that the negative index is taken to be finite, as opposed to that used by Azizov and Iokhvidov [1], where the positive index $\dim H_+$ is required to be finite. In the same way that the sign choice for Lorentz metric is merely a convention, there is no difference between the theories which are obtained.

These spaces provide the framework for a still active, important branch of spectral analysis. We refer the reader to the classic monograph on the subject [1].

Definition 3.2. The Pontryagin space (H, J) carries a continuous symmetric bilinear form $[\cdot, \cdot]$, where

$$[u, v] := \langle u, Jv \rangle \qquad \text{for all } u, v \in H.$$

Given any $u \in H$, let $u_+ := P_+ u$ and $u_- := P_- u$, so that $u = u_+ + u_-$ and $Ju = u_+ - u_-$. Then

$$[u_+ + u_-, u_+ + u_-] = \|u_+\|^2 - \|u_-\|^2.$$

The analogy with Minkowski space endowed with the Lorentz metric is obvious and it goes quite far [1]. We include, for completeness, a proof of the following well known lemma.

Lemma 3.3. (1) *Suppose (H, J) is a Pontryagin space of negative index k. If $(v_1, \ldots, v_n)$ is an n-tuple of vectors in the Hilbert space H and*

$$a_{ij} := [v_i, v_j] \qquad \textit{for all } i, j \in [1 : n]$$

then the $n \times n$ correlation matrix *$A = (a_{ij})_{i,j=1}^n$ is real, symmetric, and admits at most k negative eigenvalues, counted with multiplicity.*

(2) *Conversely, let $A = (a_{ij})_{i,j=1}^n$ be a real symmetric matrix with at most k negative eigenvalues, counted with multiplicity. There exists a Pontryagin space (H, J) of negative index k and an n-tuple of vectors $(v_1, \ldots, v_n)$ in H such that*

$$a_{ij} = [v_i, v_j] \qquad \textit{for all } i, j \in [1 : n].$$

Having reminded the reader that we count eigenvalues with multiplicity, we again leave this implicit.

Proof. (1) That A is real and symmetric is immediate. Let $T : \mathbb{R}^n \to H$ be the linear transform mapping the canonical basis vector e_j into v_j for $j = 1, \ldots, n$. Then

$$a_{ij} = \langle e_i, T^* J T e_j \rangle \qquad \text{for all } i, j \in [1 : n],$$

and the self-adjoint operator $T^*JT = T^*P_+T - T^*P_-T$ has at most k negative eigenvalues.

(2) Let T denote the linear transformation of Euclidean space $\mathbb{R}^n$ corresponding to the given matrix A. Thus

$$a_{ij} = \langle e_i, Te_j\rangle \qquad \text{for all } i, j \in [1:n],$$

where $e_1, \ldots, e_n$ are the canonical basis vectors in $\mathbb{R}^n$. The spectral decomposition of T provides orthogonal projections Q_+ and $Q_- = \mathrm{Id} - Q_+$ on $\mathbb{R}^n$ that commute with T and are such that $TQ_+ \geq 0$, $-TQ_- \geq 0$, and $r := \operatorname{rank} Q_- \leq k$.

Setting $H_+ := Q_+\mathbb{R}^n$ and $H_- := (Q_-\mathbb{R}^n) \oplus \mathbb{R}^{k-r}$, let (H, J) be the Pontryagin space of negative index k obtained by equipping $H = H_+ \oplus H_-$ with the map $J : x_+ + x_- \mapsto x_+ - x_-$ for all $x_+ \in H_+$ and $x_- \in H_-$.

Let $v_i := \sqrt{TQ_+}e_i + \sqrt{-TQ_-}e_i$ for $i = 1, \ldots, n$. Since $\sqrt{TQ_+}$ maps $\mathbb{R}^n$ into H_+ and $\sqrt{-TQ_-}$ maps $\mathbb{R}^n$ into H_-, we have that

$$\begin{aligned}
[v_i, v_j] &= \langle \sqrt{TQ_+}e_i + \sqrt{-TQ_-}e_i, J(\sqrt{TQ_+}e_j + \sqrt{-TQ_-}e_j)\rangle \\
&= \langle \sqrt{TQ_+}e_i, \sqrt{TQ_+}e_j\rangle - \langle \sqrt{-TQ_-}e_i, \sqrt{-TQ_-}e_j\rangle \\
&= \langle e_i, TQ_+e_j\rangle + \langle e_i, TQ_-e_j\rangle \\
&= \langle e_i, T(Q_+ + Q_-)e_j\rangle \\
&= a_{ij}. \qquad \square
\end{aligned}$$

A small variation of the first observation above provides the following stabilization result.

Corollary 3.4. *Let $(v_j)_{j=1}^\infty$ be a sequence of vectors in a Pontryagin space of negative index k. There exists a threshold N such that the number of negative eigenvalues of the Gram matrix $\big([v_i, v_j]\big)_{i,j=1}^n$ is constant for all $n \geq N$.*

Proof. For any $n \geq 1$, the Gram matrix $A_{[n]} := \big([v_i, v_j]\big)_{i,j=1}^n$ can have no more than k negative eigenvalues, by Lemma 3.3(1). Furthermore, by the Cauchy interlacing theorem [11, Theorem 4.3.17], the number of negative eigenvalues in $A_{[n]}$ cannot decrease as n increases. The result follows. $\square$

We now establish a partial converse of the above corollary, employing an infinite-matrix version of Lemma 3.3(2).

Lemma 3.5. *Let $(a_{ij})_{i,j=1}^\infty$ be an infinite real symmetric matrix with the property that every finite leading principal submatrix of it has at most k negative eigenvalues. Then there exists a sequence of vectors $(v_j)_{j=1}^\infty$ in a Pontryagin space of negative index k such that*

$$a_{ij} = [v_i, v_j] \qquad \textit{for all } i, j \geq 1.$$

Proof. One can select successively positive weights $(w_j)_{j=1}^\infty$ so that the infinite matrix

$$B = (b_{ij})_{i,j=1}^\infty := (w_iw_ja_{ij})_{i,j=1}^\infty = \operatorname{diag}(w_1, w_2, \ldots)(a_{ij}) \operatorname{diag}(w_1, w_2, \ldots)$$

has square-summable entries:

$$\sum_{i,j=1}^\infty b_{ij}^2 < \infty.$$

(To do this, choose w_n so that the sum of the squares of the entries of $B_{[n]} = (b_{ij})_{i,j=1}^n$ that do not appear in $B_{[n-1]} = (b_{ij})_{i,j=1}^{n-1}$ is less than 2^{-n}.)

Then the matrix B represents a self-adjoint Hilbert–Schmidt operator on ℓ^2 which we denote by B as well. Since all finite leading principal submatrices of this matrix have at most k negative eigenvalues, the operator B has at most k spectral points, counting multiplicities, on $(-\infty, 0)$. (Suppose otherwise, so that there exists a $(k+1)$-dimensional subspace U of ℓ^2 such that $\langle x, Bx\rangle < 0$ whenever $x \in U \setminus \{0\}$. The truncation $B_n := B_{[n]} \oplus \mathbf{0}_{\infty\times\infty}$ converges to B in the Hilbert–Schmidt norm, so in operator norm, as $n \to \infty$, and therefore $\langle x, B_n x\rangle < 0$ for all $x \in U \setminus \{0\}$ and all sufficiently large n, a contradiction.)

Hence there exists an orthogonal projection P_- on ℓ^2 with $r := \operatorname{rank} P_- \leq k$ that commutes with B and is such that $D := -BP_- \geq 0$ and $C := BP_+ \geq 0$, where $P_+ := I - P_-$. We now proceed essentially as for the proof of Lemma 3.3(2).

Define a Pontryagin space (H, J) of negative index k by setting $H := H_+ \oplus H_-$, where $H_+ := P_+\ell^2$ and $H_- := P_-\ell^2 \oplus \mathbb{R}^{k-r}$, and $J(x_+ + x_-) := x_+ - x_-$ whenever $x_+ \in H_+$ and $x_- \in H_-$. If $(e_j)_{j=1}^\infty$ is the canonical basis for $\ell^2 \subseteq H$ and $v_j := w_j^{-1}(C^{1/2} + D^{1/2})e_j$ for all $j \geq 1$ then

$$[v_i, v_j] = w_i^{-1} w_j^{-1} \langle (C^{1/2} + D^{1/2})e_i, (C^{1/2} - D^{1/2})e_j\rangle = w_i^{-1} w_j^{-1} \langle e_i, (C - D)e_j\rangle = a_{ij}$$

for all i, $j \geq 1$, as required. □

3.2. **Preservers of k-indefinite Gram matrices.** As a variation on Schoenberg's description of endomorphisms of correlation matrices of systems of vectors lying in Hilbert space, we now classify the entrywise preservers of the Gram matrices of systems of vectors in a real Pontryagin space of negative index k. We proceed by first introducing some terminology and notation.

Definition 3.6. Given a non-negative integer k and a sequence of vectors $(v_j)_{j=1}^\infty$ in a Pontryagin space, let $a_{ij} := [v_i, v_j]$. The infinite real symmetric matrix $A = (a_{ij})_{i,j=1}^\infty$ is a *k-indefinite Gram matrix* if the leading principal submatrix $A_{[n]} := (a_{i,j})_{i,j=1}^n$ has exactly k negative eigenvalues whenever n is sufficiently large.

We denote the collection of all k-indefinite Gram matrices by $\mathscr{P}_k$ and we let $\overline{\mathscr{P}}_k := \bigcup_{j=0}^k \mathscr{P}_j$, the collection of j-indefinite Gram matrices for $j = 0, \ldots, k$. By Lemma 3.5, this is also the collection of all infinite real symmetric matrices whose leading principal submatrices have at most k negative eigenvalues.

Our next result classifies the entrywise preservers of $\mathscr{P}_k$ and of $\overline{\mathscr{P}}_k$, in the spirit of Theorems 1.3 and 1.4. The entrywise preservers of Gram matrices in Euclidean space are precisely the absolutely monotone functions, by Schoenberg's Theorem 1.1. However, in the indefinite setting the set of preservers is much smaller.

Theorem 3.7. *Fix a positive integer k and a function $f : \mathbb{R} \to \mathbb{R}$.*

1. *The map $f[-] : \mathscr{P}_k \to \mathscr{P}_k$ if and only if f is a positive homothety, so that $f(x) \equiv cx$ for some $c > 0$, or $k = 1$, in which case we may have a negative constant function, so that $f(x) \equiv -c$ for some $c > 0$.*
2. *The map $f[-] : \overline{\mathscr{P}}_k \to \overline{\mathscr{P}}_k$ if and only if $f(x) \equiv d$ for some real constant d or $f(x) = f(0) + cx$, with $f(0) \geq 0$ and $c > 0$.*

The key idea in the proof is to employ a construction introduced and studied in detail in our recent work [3], which we now recall and study further.

Definition 3.8. Let $\pi = \{I_1, \ldots, I_m\}$ be a partition of the set of positive integers $[1 : N] = \{1, 2, \ldots, N\}$ into m non-empty subsets, so that $m \in [1 : N]$. Define the *inflation* $\Sigma_\pi^\uparrow : \mathbb{R}^{m\times m} \to \mathbb{R}^{N\times N}$ as the linear map such that

$$\mathbf{1}_{\{i\}\times\{j\}} \mapsto \mathbf{1}_{I_i\times I_j},$$

where $\mathbf{1}_{A\times B}$ has (p, q) entry 1 if $p \in A$ and $q \in B$, and 0 otherwise.

Thus, $\Sigma_\pi^\uparrow$ sends every $m \times m$ matrix into one with blocks that are constant on the rectangles defined by the partition π.

Lemma 3.9. *Suppose A is an $m\times m$ real symmetric matrix. Then $\Sigma_\pi^\uparrow(A)$ has the same number of positive eigenvalues and the same number of negative eigenvalues, counted with multiplicity, as A does.*

Proof. Define the *weight matrix* $\mathcal{W}_\pi \in \mathbb{R}^{N\times m}$ to have (i, j) entry 1 if $i \in I_j$ and 0 otherwise. As verified in [3], we have that

$$\Sigma_\pi^\uparrow(A) = \mathcal{W}_\pi A \mathcal{W}_\pi^T \quad \text{for any } A \in \mathbb{R}^{m\times m} \quad \text{and} \quad \mathcal{W}_\pi^T \mathcal{W}_\pi = \operatorname{diag}\big(|I_1|, \ldots, |I_m|\big).$$

Since each set in π is non-empty, the matrix $\mathcal{W}_\pi$ has full column rank and so may be extended to an invertible $N \times N$ matrix $X_\pi = [\mathcal{W}_\pi \ \ C]$ for some matrix $C \in \mathbb{R}^{N\times(N-m)}$. Then

$$X_\pi \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} X_\pi^T = \mathcal{W}_\pi A \mathcal{W}_\pi^T = \Sigma_\pi^\uparrow(A).$$

Since X_π is invertible, it follows from Sylvester's law of inertia [11, Theorem 4.5.8] that $\Sigma_\pi^\uparrow(A)$ and $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ have the same inertia, and the proof is complete. □

With Lemma 3.9 at hand, we can prove Theorem 3.7. First we introduce some notation.

Definition 3.10. Given an $n \times n$ real symmetric matrix

$$A = \begin{bmatrix} A_0 & \mathbf{a} \\ \mathbf{a}^T & a_{nn} \end{bmatrix} \tag{3.1}$$

and a positive integer m, let $A_m := \Sigma_{\pi_m}^\uparrow(A)$ be the inflation of A according to the partition

$$\pi_m := \big\{\{1\}, \ldots, \{n-1\}, \{n, \ldots, n-1+m\}\big\},$$

so that

$$A_m = \begin{bmatrix} A_0 & \mathbf{a} & \cdots & \mathbf{a} \\ \mathbf{a}^T & a_{nn} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}^T & a_{nn} & \cdots & a_{nn} \end{bmatrix}.$$

As $A \in \mathcal{S}_n^{(k)}$ for some $k \in \{0, \ldots, n\}$, it follows from Lemma 3.9 that $A_m \in \mathcal{S}_{n-1+m}^{(k)}$ for all $m \geq 1$. Hence there exists a k-indefinite Gram matrix $\widetilde{A}$ that can be considered as the direct limit of the sequence of matrices $(A_m)_{m=1}^\infty$, where $\widetilde{A} = (g_{ij})_{i,j=1}^\infty \in \mathscr{P}_k$ is such that $\widetilde{A}_{[N]} = (g_{ij})_{i,j=1}^N = A_{N-n+1}$ for all $N \geq n$. In particular, the matrix A is recoverable from $\widetilde{A}$, since $\widetilde{A}_{[n]} = A$.

Below, the collection of all k-indefinite Gram matrices of the form $\widetilde{A}$ for some A as in (3.1) is denoted $\mathscr{P}_k^{\rm fin}$.

Note that $f[A_{[n]}] = f[A]_{[n]}$ for all $n \geq 1$ and $f[A_m] = f[A]_m$ for all $m \geq 1$, which implies that $\widetilde{f[A]} = f[\widetilde{A}]$, whenever these quantities are well defined.

Proof of Theorem 3.7. (1) It is immediately seen that every positive homothety preserves $\mathscr{P}_k$ and every negative constant sends $\mathscr{P}_1$ to itself. Furthermore, if $f[-]$ preserves $\mathscr{P}_k$ then it sends $\mathscr{P}_k^{\rm fin}$ to $\mathscr{P}_k$. We now show that

$$f[-] : \mathscr{P}_k^{\rm fin} \to \mathscr{P}_k \quad \text{if and only if} \quad f[-] : \mathcal{S}_n^{(k)} \to \mathcal{S}_n^{(k)} \text{ for all } n \geq k.$$

Given this, the result follows at once from Theorem 1.3.

Suppose $f[-]$ sends $\mathscr{P}_k^{\rm fin}$ to $\mathscr{P}_k$ and let $A \in \mathcal{S}_n^{(k)}$ for some $n \geq k$. The Gram matrix $\widetilde{A} \in \mathscr{P}_k^{\rm fin}$ so $\widetilde{f[A]} = f[\widetilde{A}] \in \mathscr{P}_k$ by assumption. Hence the leading principal submatrix $\widetilde{f[A]}_{[N]} \in \mathcal{S}_N^{(k)}$ for all sufficiently large N, so for some $N \geq n$, but this matrix equals $f[A]_{N-n+1}$, which has the same number of negative eigenvalues as $f[A]$, by Lemma 3.9. Thus $f[-]$ maps $\mathcal{S}_n^{(k)}$ to itself.

Conversely, suppose $f[-]$ maps $\mathcal{S}_n^{(k)}$ to itself for all $n \geq k$ and let $\widetilde{A} \in \mathscr{P}_k^{\rm fin}$ for some $A \in \mathcal{S}_n^{(k)}$. We know that $f[A] \in \mathcal{S}_n^{(k)}$, by assumption, and therefore $f[\widetilde{A}]_{[N]} = \widetilde{f[A]}_{[N]} = f[A]_{N-n+1} \in \mathcal{S}_n^{(k)}$ for all $N \geq n$, again using Lemma 3.9. Hence $f[\widetilde{A}] \in \mathscr{P}_k^{\rm fin} \subseteq \mathscr{P}_k$, as claimed.

(2) If $f(x) \equiv d$ for some $d \in \mathbb{R}$ then $f[B] = \widetilde{d} \in \overline{\mathscr{P}}_1 \subseteq \overline{\mathscr{P}}_k$ for all $k \geq 1$ and any infinite matrix B. Similarly, if $f(x) = f(0) + cx$, with $f(0) \geq 0$ and $c > 0$, then $f[B] = \widetilde{f(0)} + cB$, so $f[A]_{[n]} = f(0)\mathbf{1}_{n\times n} + cA_{[n]} \in \overline{\mathcal{S}_n^{(k)}}$ for all sufficiently large n if $A \in \overline{\mathscr{P}}_k$, by Lemma 2.2. In both cases we see that $f[-] : \overline{\mathscr{P}}_k \to \overline{\mathscr{P}}_k$. In turn, this condition implies that $f[-] : \mathscr{P}_k^{\rm fin} \to \overline{\mathscr{P}}_k$. We now claim that if $f[-] : \mathscr{P}_k^{\rm fin} \to \overline{\mathscr{P}}_k$, then f is a real constant or linear of the above form. Similarly to the previous part, this follows from Theorem A, given the following claim:

$$f[-] : \mathscr{P}_k^{\rm fin} \to \overline{\mathscr{P}}_k \quad \text{if and only if} \quad f[-] : \mathcal{S}_n^{(k)} \to \overline{\mathcal{S}_n^{(k)}} \text{ for all } n \geq k.$$

The proof of this equivalence follows the same lines as that in the previous part, so the details are left to the interested reader. $\square$

4. Multi-variable transforms with negativity constraints

We now turn to the analysis of functions of several variables, acting on tuples of matrices with prescribed negativity. In the present section, we focus on functions of the form $f : I^m \to \mathbb{R}$, where $I = (-\rho, \rho)$ for some $0 < \rho \leq \infty$. The next section will address the cases where $I = (0, \rho)$ and $I = [0, \rho)$.

Recall that if $B^{(p)} = (b_{ij}^{(p)})$ is an $n \times n$ matrix with entries in I for all $p \in [1 : m]$ then the function f acts entrywise to produce the $n \times n$ matrix $f[B^{(1)}, \ldots, B^{(m)}]$ with

(i,j) entry

$$f[B^{(1)},\dots,B^{(m)}]_{ij} = f(b_{ij}^{(1)},\dots,b_{ij}^{(m)}) \qquad \text{for all } i,j \in [1:n].$$

The negativity constraints on the domain are described by an m-tuple of non-negative integers $\mathbf{k} = (k_1,\dots,k_m)$. As noted in the introduction, we may permute the entries of $\mathbf{k}$ so that any zero entries appear first: there exists $m_0 \in [0:m]$ with $k_p = 0$ for $p \in [1:m_0]$ and $k_p \geq 1$ for $p \in [m_0+1:m]$. In this case, the m-tuple $\mathbf{k}$ is said to be admissible. We let $k_{\max} := \max\{k_p : p \in [1:m]\}$,

$$\mathcal{S}_n^{(\mathbf{k})}(I) := \mathcal{S}_n^{(k_1)}(I) \times \cdots \times \mathcal{S}_n^{(k_m)}(I), \quad \text{and} \quad \overline{\mathcal{S}_n^{(\mathbf{k})}}(I) := \overline{\mathcal{S}_n^{(k_1)}}(I) \times \cdots \times \overline{\mathcal{S}_n^{(k_m)}}(I).$$

The simplest generalization of the one-variable preserver problem would involve taking $k_1 = \cdots = k_m = l$, but the more general problem is more interesting (and also more challenging). We now recall the first part of our two-part generalization of Theorem A.

Theorem 4.1 (The cases $\mathbf{k} = \mathbf{0}$ and $l = 0$)**.** *Let $I = (-\rho,\rho)$, $(0,\rho)$ or $[0,\rho)$, where $0 < \rho \leq \infty$, and let $\mathbf{k} \in \mathbb{Z}_+^m$ be admissible. Given a function $f : I^m \to \mathbb{R}$, the following are equivalent.*

1. *The entrywise transform $f[-]$ sends $\overline{\mathcal{S}_n^{(\mathbf{k})}}(I)$ to $\overline{\mathcal{S}_n^{(0)}}$ for all $n \geq k_{\max}$.*
2. *The entrywise transform $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\mathcal{S}_n^{(0)}$ for all $n \geq k_{\max}$.*
3. *The function f is independent of x_{m_0+1}, $\dots$, x_m and is represented on I^m by a convergent power series in the reduced tuple $\mathbf{x}' := (x_1,\dots,x_{m_0})$, with all Maclaurin coefficients non-negative.*

If, instead, $\mathbf{k} = \mathbf{0}$ and $l \geq 1$ then $f[-]$ sends $\mathcal{S}_n^{(\mathbf{0})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ if and only if the function f is represented on I^m by a power series $\sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$.

The case of entrywise positivity preservers corresponds to taking $\mathbf{k} = \mathbf{0}$ and $l = 0$. The analogue of Schoenberg's theorem holds, as shown by FitzGerald, Micchelli, and Pinkus in [7]. Before that, Vasudeva [19] had obtained the one-sided, one-variable version of Schoenberg's theorem, which corresponds to taking $m = 1$ and $I = (0,\infty)$. These results were extended to various smaller domains in our previous work: see Theorem 1.5 above, the proof of which employs the multi-variable version of Bernstein's theorem on absolutely monotone functions. This theorem was obtained by Bernstein [6] for $m = 1$, and then by Schoenberg [16] for $m = 2$. For $m > 2$, see Appendix A.

Schoenberg's theorem and these generalizations yield a large collection of preservers. Theorem 4.1 shows that a similarly rich class of preservers is obtained even if one relaxes some (but not all) of the negativity constraints, from none to some, provided that the codomain is still required to be $\mathcal{S}_n^{(0)}$.

We now complete the classification of the multi-variable transforms for the remaining case. The class of functions obtained is given by a combination of the rich class of absolutely monotone functions in the variables for which the corresponding entries of $\mathbf{k}$ are zero and the rigid class of positive homotheties in at most one of the remaining variables.

Theorem 4.2 (The case of $\mathbf{k} \neq \mathbf{0}$ and $l > 0$)**.** *Let $I := (-\rho,\rho)$, $(0,\rho)$ or $[0,\rho)$, where $0 < \rho \leq \infty$, let $\mathbf{k}$ be admissible and not equal to $\mathbf{0}$, and suppose $l = 1$ if $k_p = 1$ for some $p \in [1:m]$ and $l \in [1:2K-2]$ otherwise, where $K = \min\{k_p : p \in [1:m],\ k_p > 0\}$. Given a function $f : I^m \to \mathbb{R}$, the following are equivalent.*

(1) *The map* $f[-]$ *sends* $\mathcal{S}_n^{(\mathbf{k})}(I)$ *to* $\overline{\mathcal{S}_n^{(l)}}$ *for all* $n \geq k_{\max}$.
(2) *There exist an index* $p_0 \in [m_0+1 : m]$, *a function* $F : (-\rho, \rho)^{m_0} \to \mathbb{R}$, *and a constant* $c \geq 0$ *such that*
  (a) *we have the representation*
$$f(\mathbf{x}) = F(x_1, \ldots, x_{m_0}) + c x_{p_0} \qquad \text{for all } \mathbf{x} \in I^m, \tag{4.1}$$
  (b) *the function* $\mathbf{x}' \mapsto F(\mathbf{x}') - F(\mathbf{0})$ *is absolutely monotone on* $[0, \rho)^{m_0}$,
  (c) *if* $c > 0$ *then* p_0 *is unique and* $l \geq k_{p_0}$, *and*
  (d) *if* $c > 0$ *and* $l = k_{p_0}$ *then* $F(\mathbf{0}) \geq 0$.

In the above, we adopt the convention that if $m_0 = 0$ then $F(x_1, \ldots, x_{m_0})$ is the constant $F(\mathbf{0})$ and (2)(b) is vacuously satisfied.

Remark 4.3. As asserted in [2, Remark 9.16] for the $\mathbf{k} = \mathbf{0}$ case, there is no greater generality in considering, for Theorems 4.1 and 4.2, domains of the form
$$(-\rho_1, \rho_1) \times \cdots \times (-\rho_m, \rho_m), \qquad \text{where } 0 < \rho_1, \ldots, \rho_m \leq \infty.$$
For finite ρ_p, one can introduce the scaling $x_p \mapsto x_p/\rho_p$, whereas for infinite ρ_p, one can truncate to $(-N, N)$, scale and then use the identity theorem to facilitate the extension to $(-\infty, \infty)$.

The simplest multi-variable generalization mentioned above, $k_1 = \cdots = k_m = l$, is a straightforward consequence of Theorem 4.2. The classification obtained is rigid and very close to what is seen in Theorems 1.3 and 1.4.

Corollary 4.4. *Let* $I := (-\rho, \rho)$, $(0, \rho)$ *or* $[0, \rho)$, *where* $0 < \rho \leq \infty$, *let* k *and* m *be positive integers, and let* $f : I^m \to \mathbb{R}$.

(1) *The entrywise transform* $f[-]$ *sends* $\mathcal{S}_n^{(k\mathbf{1}_m^T)}(I)$ *to* $\mathcal{S}_n^{(k)}$ *for all* $n \geq k$ *if and only if* $f(\mathbf{x}) = c x_{p_0}$ *for a constant* $c > 0$ *and some* $p_0 \in [1 : m]$, *or, when* $k = 1$, *we may also have* $f(\mathbf{x}) \equiv -c$ *for some* $c > 0$.
(2) *The entrywise transform* $f[-]$ *sends* $\overline{\mathcal{S}_n^{(k\mathbf{1}_m^T)}}(I)$ *to* $\overline{\mathcal{S}_n^{(k)}}$ *for all* $n \geq k$ *if and only if* $f(\mathbf{x}) = c x_{p_0} + d$ *for some* $p_0 \in [1 : m]$, *with either* $c = 0$ *and* $d \in \mathbb{R}$, *or* $c > 0$ *and* $d \geq 0$.

It is not clear how to define inertia preservers when there are multiple matrices to which an entrywise transform is applied. Hence we provide no version of Theorem 1.2 in Corollary 4.4. However, when we work over $(0, \rho)$ or $[0, \rho)$ in the next section, we will give a proof of the analogue of Theorem 1.2 for these domains in the $m = 1$ case.

Proof. We first prove (1). The reverse implication is readily verified; for the forward assertion, apply Theorem 4.2, noting that $m_0 = 0$, to obtain that either $f(\mathbf{x}) \equiv F(\mathbf{0})$ or $f(\mathbf{x}) = F(\mathbf{0}) + c x_{p_0}$ with $F(\mathbf{0}) \geq 0$ and $c > 0$. Now $f \equiv F(\mathbf{0})$ cannot take $\mathcal{S}_n^{(k\mathbf{1}_m^T)}(I)$ to $\mathcal{S}_n^{(k)}$ for any real $F(\mathbf{0})$ if $k \geq 2$, and for non-negative $F(\mathbf{0})$ if $k = 1$. We next assume that $f(\mathbf{x}) = F(\mathbf{0}) + c x_{p_0}$; to complete the proof, we need to show that $F(\mathbf{0}) = 0$. Taking $B^{(1)} = \cdots = B^{(m)} = -t_0 \mathrm{Id}_k$ for $t_0 \in (0, \rho)$, the working around (2.7) gives that $F(\mathbf{0}) = 0$.

Coming to (2), the reverse implication is trivial for $c = 0$, and follows from Lemma 2.2 if $c > 0$ and $d \geq 0$. Conversely, if $f[-] : \overline{\mathcal{S}_n^{(k\mathbf{1}_m^T)}}(I) \to \overline{\mathcal{S}_n^{(k)}}$ then $f[-] : \mathcal{S}_n^{(k\mathbf{1}_m^T)}(I) \to \overline{\mathcal{S}_n^{(k)}}$, and now Theorem 4.2 completes the proof. $\square$

4.1. **The proofs.** The remainder of this section is devoted to establishing Theorems 4.1 and 4.2 for the case $I = (-\rho, \rho)$. In addition to the replication trick and multi-variable analogues of various lemmas used above, a key ingredient is Lemma 3.9. This result is indispensable for producing tuples of test matrices of the same large dimension while preserving the number of negative eigenvalues.

The next proposition provides suitable multi-variable versions of Steps 1 and 2 from the proof of Theorem A. In order to state it clearly, we introduce some notation.

Notation 4.5. Given $n \times n$ test matrices $A^{(1)}, \dots, A^{(m)}$ and constants $\epsilon_1, \dots, \epsilon_m$, we let the m-tuples $\mathbf{A} := (A^{(1)}, \dots, A^{(m)})$ and

$$\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1} := (A^{(1)} + \epsilon_1 \mathbf{1}_{n\times n}, \dots, A^{(m)} + \epsilon_m \mathbf{1}_{n\times n}), \tag{4.2}$$

and similarly for $\mathbf{B}$ and $\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}$.

If $P = \{p_1 < \cdots < p_k\}$ is a non-empty subset of $[1:m]$ then $\mathbf{A}_P$ and $\mathbf{A}_P + \boldsymbol{\epsilon}_P\mathbf{1}$ denote the corresponding k-tuples with entries whose indices appear in P, so that $\mathbf{A}_P = (A^{(p_1)}, \dots, A^{(p_k)})$ and similarly for $\mathbf{A}_P + \boldsymbol{\epsilon}_P\mathbf{1}$.

If $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ then $\mathbf{A}^{\circ\boldsymbol{\alpha}}$ denotes the matrix $(A^{(1)})^{\circ\alpha_1} \circ \cdots \circ (A^{(m)})^{\circ\alpha_m}$, where $\circ$ denotes the Schur product, so that $\mathbf{A}^{\circ\boldsymbol{\alpha}}$ has (i,j) entry

$$\mathbf{A}_{ij}^{\circ\boldsymbol{\alpha}} = (A_{ij}^{(1)})^{\alpha_1} \cdots (A_{ij}^{(m)})^{\alpha_m} \qquad \text{for all } i,j \in [1:n].$$

A function f with power-series representation $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ acts on the m-tuple $\mathbf{A}$ to give the $n \times n$ matrix $f[\mathbf{A}] = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} A^{\circ\boldsymbol{\alpha}}$.

The following result and its corollary are not required when considering functions of a single variable but they are crucial to our arguments in the multivariable setting.

Proposition 4.6. *Let $I := (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$. Suppose $f : I^m \to \mathbb{R}$ is such that the entrywise transform $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k_{\max}$, where $\mathbf{k} \in \mathbb{Z}_+^m$ and l is a non-negative integer.*

1. *The function f is represented on I^m by a power series $\sum_{\boldsymbol{\alpha}\in\mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, where the coefficient $c_{\boldsymbol{\alpha}} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$.*
2. *Suppose the matrices $A^{(1)}, \dots, A^{(m)}, B^{(1)}, \dots, B^{(p)} \in \mathcal{S}_n^{(0)}$ are such that $A^{(p)} - B^{(p)} \in \mathcal{S}_n^{(k_p)}$ for all $p \in [1:m]$. Fix $\epsilon_1, \dots, \epsilon_m \geq 0$ such that the entries of $A^{(p)} + \epsilon_p \mathbf{1}_{n\times n}$ and $B^{(p)} + \epsilon_p \mathbf{1}_{n\times n}$ all lie in I for every $p \in [1:m]$. Then*
$$f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$$
is a positive semidefinite matrix with rank at most l.

Proof of Proposition 4.6 for $I = (-\rho, \rho)$.

1. Define g on I^m by setting $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$. Given a constant $t_0 \in (0, \rho)$ and matrices $A_1, \dots, A_m \in \mathcal{S}_n^{(0)}(I)$, let $N := k_{\max} + (l+2)n$ and
$$B^{(p)} := -t_0 \mathrm{Id}_{k_p} \oplus \mathbf{0}_{(k_{\max}-k_p)\times(k_{\max}-k_p)} \oplus A_p^{\oplus(l+2)} \in \mathcal{S}_N^{(k_p)}(I) \tag{4.3}$$
for all $p \in [1:m]$. By hypothesis and Lemma 2.2, the entrywise transform $g[-]$ sends $\mathcal{S}_N^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_N^{(l+1)}}$, so $g[\mathbf{B}] \in \overline{\mathcal{S}_N^{(l+1)}}$. As the block-diagonal matrix $g[\mathbf{B}]$ contains $l+2$ copies of the matrix $g[\mathbf{A}]$ on the diagonal, this last matrix must be positive semidefinite. It now follows from Theorem 1.5 that g is absolutely monotone.

(2) It follows from the previous part that the function $h : \mathbf{x} \mapsto f(\mathbf{x}) + |f(\mathbf{0})|$ is absolutely monotone, so $h[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$ and $h[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}]$ are positive semidefinite.

By Lemma 2.2, $h[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k_{\max}$. Using (2.1), we see that

$$C^{(p)} := \begin{bmatrix} A^{(p)} + \epsilon_p \mathbf{1}_{n\times n} & B^{(p)} + \epsilon_p \mathbf{1}_{n\times n} \\ B^{(p)} + \epsilon_p \mathbf{1}_{n\times n} & A^{(p)} + \epsilon_p \mathbf{1}_{n\times n} \end{bmatrix} \in \mathcal{S}_{2n}^{(k_p)}(I)$$

for all $p \in [1:m]$, so $h[\mathbf{C}] \in \overline{\mathcal{S}_{2n}^{(l)}}$. Hence, again using (2.1), we have that

$$f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] = h[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}] - h[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] \in \overline{\mathcal{S}_n^{(l)}}.$$

By the Schur product theorem, any absolutely monotone function acting entrywise preserves Loewner monotonicity. Hence $f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$ is positive semidefinite, and so has rank at most l, by the previous working.

□

We now present an application of the second part of the previous proposition after first introducing some notation. Given a set $P = \{p_1 < \cdots < p_k\} \subseteq [1:m]$ and $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, let $P' = [1:m] \setminus P$ and $x_P = (x_{p_1}, \ldots, x_{p_k})$. Further, let $|P|$ denote the cardinality of P, so that $|P| \in [0:m]$.

Let $I := (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$, and suppose $f : I^m \to \mathbb{R}$ is such that

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{x} \in I^m,$$

where $c_{\boldsymbol{\alpha}} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. Given $P \subseteq [1:m]$, we can write

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}_P \in \mathbb{Z}_+^{|P|}} c_{\boldsymbol{\alpha}_P}(\mathbf{x}_{P'}) x_P^{\boldsymbol{\alpha}_P} \qquad \text{for all } \mathbf{x} \in I^m, \tag{4.4}$$

where $c_{\boldsymbol{\alpha}_P}$ is an absolutely monotone function defined on $I^{P'}$ whenever $\boldsymbol{\alpha}_P \neq \mathbf{0}_P$ and $c_{\mathbf{0}_P}(\mathbf{x}_{P'}) \equiv c_{\mathbf{0}}$ is constant.

Corollary 4.7. *Let $I := (-\rho, \rho)$, $(0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$. Suppose $f : I^m \to \mathbb{R}$ is such that the entrywise transform $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geq k_{\max}$, where $\mathbf{k} \in \mathbb{Z}_+^m$ and l is a non-negative integer. Given $P \subseteq [1:m]$, let f have the split representation (4.4).*

For each $p \in [1:m]$, let $B^{(p)} \in \mathcal{S}_n^{(0)}$ have rank k_p and suppose the non-negative constant ϵ_p is such that $B^{(p)} + \epsilon_p \mathbf{1}_{n\times n}$ has all of its entries in I. If $g : I^P \to \mathbb{R}$ is such that

$$g(\mathbf{x}_P) = \sum_{\boldsymbol{\alpha}_P \in \mathbb{Z}_+^{|P|} \setminus \{\mathbf{0}\}} c_{\boldsymbol{\alpha}_P}(\boldsymbol{\epsilon}_{P'}) \mathbf{x}_P^{\boldsymbol{\alpha}_P} \qquad \text{for all } \mathbf{x}_P \in I^P$$

then the matrix $g[\mathbf{B}_P]$ is positive semidefinite and has rank at most l.

Proof of Corollary 4.7 for $I = (-\rho, \rho)$. That $g[\mathbf{B}_P]$ is positive semidefinite is a straightforward consequence of the Schur product theorem. Next, applying Proposition 4.6(2) with $(A^{(p)}, B^{(p)}, \epsilon_p)$ there equal to $(\epsilon'_p \mathbf{1}_{n\times n}, B^{(p)} + \epsilon'_p \mathbf{1}_{n\times n}, \epsilon'_p)$ for all p, where $\epsilon'_p = \epsilon_p/2$, it follows that

$$C := f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\boldsymbol{\epsilon}\mathbf{1}]$$

is positive semidefinite and has rank at most l. Thus, for the final claim it suffices to show that the inequality $C \geq g[\mathbf{B}_P]$ holds. Note first that

$$c_{\alpha_P}[\mathbf{B}_{P'} + \boldsymbol{\epsilon}_{P'}\mathbf{1}] \geq c_{\alpha_P}[\boldsymbol{\epsilon}_{P'}\mathbf{1}] \quad \text{for all } \alpha_P \in \mathbb{Z}_+^{|P|},$$

by Loewner monotonicity, as in the proof of Proposition 4.6(2). Again using the Schur product theorem, it follows that

$$\begin{aligned}
C &= \sum_{\boldsymbol{\alpha}_P \in \mathbb{Z}_+^{|P|}} \Big(c_{\boldsymbol{\alpha}_P}[\mathbf{B}_{P'} + \boldsymbol{\epsilon}_{P'}\mathbf{1}] \circ (\mathbf{B}_P + \boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} - c_{\boldsymbol{\alpha}_P}[\boldsymbol{\epsilon}_{P'}\mathbf{1}] \circ (\boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} \Big) \\
&\geq \sum_{\boldsymbol{\alpha}_P \in \mathbb{Z}_+^{|P|}} c_{\boldsymbol{\alpha}_P}[\boldsymbol{\epsilon}_{P'}\mathbf{1}] \circ \Big((\mathbf{B}_P + \boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} - (\boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} \Big) \\
&= \sum_{\alpha_P \neq \mathbf{0}} c_{\boldsymbol{\alpha}_P}(\boldsymbol{\epsilon}_{P'}) \Big((\mathbf{B}_P + \boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} - (\boldsymbol{\epsilon}_P \mathbf{1})^{\circ \boldsymbol{\alpha}_P} \Big) \\
&\geq \sum_{\alpha_P \neq \mathbf{0}} c_{\boldsymbol{\alpha}_P}(\boldsymbol{\epsilon}_{P'}) \mathbf{B}_P^{\circ \boldsymbol{\alpha}_P} \\
&= g[\mathbf{B}_P].
\end{aligned}$$

□

With Proposition 4.6 and Corollary 4.7 at hand, we show the two theorems above.

Proof of Theorem 4.1 for $I = (-\rho, \rho)$. We begin by showing a cycle of implications for the three equivalent hypotheses. The Schur product theorem gives that (3) implies (1) and it is immediate that (1) implies (2). We now assume (2) and deduce that (3) holds.

Note first that the case $m_0 = m$ is Theorem 1.5, so we assume henceforth that $m_0 < m$. Proposition 4.6(1) gives that $\mathbf{x} \mapsto f(\mathbf{x}) - f(\mathbf{0})$ is absolutely monotone and we claim that $f(\mathbf{0}) \geq 0$, so that f is itself absolutely monotone. To see this, let $N = 1 + k_{\max}$ and $t_0 \in (0, \rho)$, and set

$$B^{(p)} := \mathbf{0}_{(N-k_p)\times(N-k_p)} \oplus -t_0 \mathrm{Id}_{k_p} \in \mathcal{S}_N^{(k_p)}(I) \qquad \text{for all } p \in [1:m]. \tag{4.5}$$

By hypothesis, $f[\mathbf{B}] \in \mathcal{S}_N^{(0)}$ is positive semidefinite. In particular, its $(1,1)$ entry $f(\mathbf{0})$ is non-negative.

To complete the first part of the proof, we now show that the power series that represents f contains no monomials involving any of $x_{m_0+1}, \ldots, x_m$. We suppose without loss of generality that there is a monomial containing x_m. Let a, b, $2t_0 \in (0, \rho)$, with $a < b$, let $N \geq 2 + k_{\max}$, and set

$$B_0^{(m)} := \begin{bmatrix} a & b \\ b & a \end{bmatrix} \oplus (2t_0 \mathbf{1}_{k_m \times k_m} - t_0 \mathrm{Id}_{k_m}) \oplus t_0 \mathrm{Id}_{N-k_m-2} \in \mathcal{S}_N^{(k_m)},$$

since the first matrix has one positive and one negative eigenvalue and $2t_0 \mathbf{1}_{k_m \times k_m}$ has eigenvalue 0 with multiplicity $k_m - 1$ and eigenvalue $2k_m t_0$ with multiplicity 1.

Similarly, for $p \in [1 : m-1]$, let

$$B'_p := (2t_0 \mathbf{1}_{(k_p+1)\times(k_p+1)} - t_0 \mathrm{Id}_{k_p+1}) \oplus t_0 \mathrm{Id}_{N-k_p-2} \in \mathcal{S}_{N-1}^{(k_p)}.$$

By the continuity of eigenvalues, we can now take ϵ_0 positive but sufficiently small so that, for all $p \in [1 : m-1]$,

$$B'_p + \epsilon_0 \mathbf{1}_{(N-1)\times(N-1)} \in \mathcal{S}_{N-1}^{(k_p)}\big((0,\rho)\big) \quad \text{and} \quad B_0^{(m)} + \epsilon_0 \mathbf{1}_{N \times N} \in \mathcal{S}_N^{(k_m)}\big((0,\rho)\big). \tag{4.6}$$

After this preamble, we can now define the test matrices we need. We take the partition $\pi = \big\{\{1,2\},\{3\},\ldots,\{N\}\big\}$ of $[1:N]$ into $N-1$ subsets and then use the inflation operator $\Sigma_\pi^\uparrow$ from Definition 3.8 to produce

$$B^{(p)} := \begin{cases} \Sigma_\pi^\uparrow(B'_p + \epsilon_0 \mathbf{1}_{(N-1)\times(N-1)}) & \text{if } p \in [1:m-1], \\ B_0^{(m)} + \epsilon_0 \mathbf{1}_{N\times N} & \text{if } p = m. \end{cases} \tag{4.7}$$

By Lemma 3.9, we have that $B^{(p)} \in \mathcal{S}_N^{k_p}\big((0,\rho)\big)$ for all $p \in [1:m]$. Hence, by hypothesis, the matrix $f[\mathbf{B}] \in \mathcal{S}_N^{(0)}$. In particular, its leading principal 2×2 submatrix is positive semidefinite, but this submatrix M equals

$$\begin{bmatrix} F(a) & F(b) \\ F(b) & F(a) \end{bmatrix}, \quad \text{where } F(x_m) := f(t_0+\epsilon_0,\ldots,t_0+\epsilon_0,x_m) \text{ for all } x_m \in (0,\rho).$$

As f is absolutely monotone and we have assumed that its power-series representation contains a monomial involving x_m, the function F is strictly increasing. This implies that the matrix M has negative determinant, contradicting the fact that it is positive semidefinite. Hence f cannot depend on x_m, and similarly not on x_p for any $p > m_0$. Thus (2) implies (3).

Finally, suppose $\mathbf{k} = \mathbf{0}$ and $l \geq 1$. The forward implication is a special case of Proposition 4.6(1). For the converse, note that if $g : \mathbf{x} \mapsto f(\mathbf{x}) - f(\mathbf{0})$ is absolutely monotone then the entrywise transform $g[-]$ sends $\mathcal{S}_n^{(\mathbf{0})}(I)$ to $\overline{\mathcal{S}_n^{(0)}}$, by Theorem 1.5. Hence, by Lemma 2.2, the entrywise map $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(1)}} \subseteq \overline{\mathcal{S}_n^{(l)}}$, as claimed. □

The remainder of this section is occupied by the following proof.

Proof of Theorem 4.2 for $I = (-\rho,\rho)$. We first show that (2) implies (1). Suppose f has the form (4.1) and let $G(\mathbf{x}') = F(\mathbf{x}') - F(\mathbf{0})$ for all $\mathbf{x}' := (x_1,\ldots,x_{m_0})$. If the matrix $B^{(p)} \in \mathcal{S}_n^{(k_p)}(I)$ for all $p \in [1:m]$ then

$$f[\mathbf{B}] = F(\mathbf{0})\mathbf{1}_{n\times n} + G[\mathbf{B}_{[1:m_0]}] + cB^{(p_0)}.$$

Since G is absolutely monotone and $B^{(1)}, \ldots, B^{(m_0)} \in \mathcal{S}_n^{(0)}$, the matrix $G[\mathbf{B}_{[1:m_0]}]$ is positive semidefinite. Hence, by repeated applications of Lemma 2.2, the number of negative eigenvalues that the matrix $f[\mathbf{B}]$ has is bounded above by that number for

$$M := F(\mathbf{0})\mathbf{1}_{n\times n} + cB^{(p_0)}.$$

If $c = 0$, then $M \in \overline{\mathcal{S}_n^{(1)}}$, so $f[\mathbf{B}] \in \overline{\mathcal{S}_n^{(1)}} \subseteq \overline{\mathcal{S}_n^{(l)}}$. If, instead, $c > 0$ (so $l \geq k_{p_0}$) and $F(\mathbf{0}) \geq 0$ then $M \in \overline{\mathcal{S}_n^{(k_{p_0})}}$, so $f[\mathbf{B}] \in \overline{\mathcal{S}_n^{(l)}}$, as desired. Finally, if $c > 0 > F(\mathbf{0})$ (and so $l > k_{p_0}$) then, by Lemma 2.2, the matrix M and hence $f[\mathbf{B}]$ lies in $\overline{\mathcal{S}_n^{(k_{p_0}+1)}} \subseteq \overline{\mathcal{S}_n^{(l)}}$.

The proof that (1) implies (2) is divided into several steps. We commence by noting that Proposition 4.6(1) gives that $\mathbf{x} \mapsto f(\mathbf{x}) - f(\mathbf{0})$ has a power-series representation with non-negative coefficients.

Step 1: The only powers of x_p with $p \in [m_0+1:m]$ that may occur in any monomial in the power-series representation of f are x_p^0 and x_p^1.

Suppose otherwise, for contradiction; without loss of generality, we assume that x_m^r occurs in a monomial in the power-series representation of f, with $r \geq 2$.

Our strategy is to apply Corollary 4.7 with $P = \{m\}$, to reduce to the one-variable setting, and then use the two examples from Step 4 in the proof of Theorem A. As was necessary there, we proceed differently depending on the value of k_m.

(1) Suppose $k_m \geq 2$ and note that $l \leq 2k_m - 2$, even if $l = 1$. Fix an integer N such that that $N \geq k_1, \ldots, k_{m-1}, 2k_m - 1$, choose $t_0 \in (0, \rho/2)$ and let

$$B^{(p)} := \begin{cases} \mathbf{0}_{N\times N} & \text{if } 1 \leq p \leq m_0, \\ t_0 \mathrm{Id}_{k_p - 1} \oplus t_0 \mathbf{1}_{(N-k_p+1)\times(N-k_p+1)} & \text{if } m_0 < p < m. \end{cases}$$

Let $B_0^{(m)} \in \mathcal{S}_{2k_m-1}^{(0)}(I)$ be the matrix given by (2.3) and let $B^{(m)} := \Sigma_\pi^\uparrow(B_0^{(m)})$, where the partition

$$\pi := \big\{\{1\}, \{2\}, \ldots, \{2k_m - 2\}, \{2k_m - 1, \ldots, N\}\big\}.$$

Choose $\epsilon_p \in (0, \rho/2)$ for all $p \in [1 : m-1]$ and let $\epsilon_m = 0$. The matrix $B^{(p)}$ is positive semidefinite for all p, by inspection and Lemma 3.9. Hence we may apply Corollary 4.7 with $P = \{m\}$ to see that the matrix

$$\sum_{n=1}^{\infty} c_n(\boldsymbol{\epsilon}_{[1:m-1]})(B^{(m)})^{\circ n}$$

is positive semidefinite and has rank at most l, where $c_n(\boldsymbol{\epsilon}_{[1:m-1]}) \geq 0$ for all n and this inequality is strict for $n = r$. It follows that the positive semidefinite matrix $(B^{(m)})^{\circ r}$ also has rank at most l, and so does the leading principal submatrix $(B_0^{(m)})^{\circ r}$. This is the desired contradiction, by the discussion that follows (2.3).

(2) Alternatively, we have that $k_m = 1$ and $l = 1$. Fix an integer $N \geq 2 + k_{\max}$, choose $t_0 \in (0, \rho/2)$, let $A^{(p)} = \mathbf{0}_{N\times N}$ for all $p \in [1 : m-1]$ and let

$$B^{(p)} = \begin{cases} \mathbf{0}_{N\times N} & \text{if } 1 \leq p \leq m_0, \\ \mathbf{0}_{2\times 2} \oplus t_0 \mathrm{Id}_{k_p - 1} \otimes t_0 \mathbf{1}_{(N-k_p-1)\times(N-k_p-1)} & \text{if } m_0 < p < m. \end{cases}$$

Let the 2×2 matrices $A_0^{(m)}$ and $B_0^{(m)}$ be as in (2.4) and let $A^{(m)} = \Sigma_\pi^\uparrow(A_0^{(m)})$ and $B^{(m)} = \Sigma_\pi^\uparrow(B_0^{(m)})$, where the partition

$$\pi := \big\{\{1\}, \{2, \ldots, N\}\big\}.$$

Note that $A^{(p)} - B^{(p)} \in \mathcal{S}_N^{(k_p)}$ for all p, where we use Lemma 3.9 when $p = m$. Choose $\epsilon_p \in (0, \rho/2)$ for each $p \in [1 : m-1]$ and let $\epsilon_m = 0$, then apply Proposition 4.6(2) to see that

$$C := f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$$

is positive semidefinite and has rank at most 1.

We conclude by showing that the leading principal 2×2 submatrix D of C is positive definite, which provides the necessary contradiction. This submatrix equals $h[B_0^{(m)}] - h[A_0^{(m)}]$, where

$$h : I \to R;\ x \mapsto f(\epsilon_1, \ldots, \epsilon_{m-1}, x) - f(\epsilon_1, \ldots, \epsilon_{m-1}, 0)$$

is absolutely monotone. We may write

$$h(x) = \sum_{n=1}^{\infty} h_n x^n \qquad \text{for all } x \in I,$$

where $h_n \geq 0$ for all n and $h_r > 0$. The working following (2.5) now shows that the matrix D is positive definite, as claimed.

Step 2: Let $P = [1 : m_0]$. There is at most one $q \in P'$ such that a monomial of the form $\mathbf{x}_P^{\boldsymbol{\alpha}_P} x_q$ appears in the power-series representation of f.

If $m = m_0 + 1$ then the assertion is immediate, so we suppose $m > m_0 + 1$. We may write

$$f(\mathbf{x}) = \sum_{\alpha_{m-1}=0}^{1} \sum_{\alpha_m=0}^{1} c_{(\alpha_{m-1},\alpha_m)}(\mathbf{x}_{[1:x_{m-2}]}) x_{m-1}^{\alpha_{m-1}} x_m^{\alpha_m},$$

and if the monomials x_{m-1} and x_m both appear then the absolutely monotone functions $c_{(1,0)}$ and $c_{(0,1)}$ are both non-zero. Let $N := \sum_{p=1}^{m} k_p$ and $t_0 \in (0, \rho/2)$, and set

$$B^{(p)} := \begin{cases} \mathbf{0}_{N \times N} & \text{if } p \in [1 : m_0], \\ C_{m_0+1}^{(p)} \oplus \cdots \oplus C_m^{(p)} & \text{if } p \in [m_0 + 1 : m], \end{cases}$$

where $C_p^{(p)} = t_0 \mathrm{Id}_{k_p}$ and $C_q^{(p)} = \mathbf{0}_{k_q \times k_q}$ for all $q \neq p$. Let $\epsilon_p \in (0, \rho/2)$ for $p \in [1 : m]$ and apply Corollary 4.7 with $P = \{m - 1, m\}$ to see that the matrix

$$M = c_{(1,0)}(\boldsymbol{\epsilon}_{[1:m-2]}) B^{(m-1)} + c_{(0,1)}(\boldsymbol{\epsilon}_{[1:m-2]}) B^{(m)} + c_{(1,1)}(\boldsymbol{\epsilon}_{[1:m-2]}) B^{(m-1)} \circ B^{(m)}$$

has rank at most l. The coefficients $c_{(1,0)}(\boldsymbol{\epsilon}_{[1:m-2]})$ and $c_{(0,1)}(\boldsymbol{\epsilon}_{[1:m-2]})$ are positive and $B^{(m-1)} \circ B^{(m)} = \mathbf{0}_{N \times N}$, so

$$M = \mathbf{0}_{(N-k_{m-1}-k_m)\times(N-k_{m-1}-k_m)} \oplus c_{(1,0)}(\boldsymbol{\epsilon}_{[1:m-2]}) t_0 \mathrm{Id}_{k_{m-1}} \oplus c_{(0,1)}(\boldsymbol{\epsilon}_{[1:m-2]}) t_0 \mathrm{Id}_{k_m},$$

which has rank $k_{m-1} + k_m > 2K - 2 \geq l$ (by the constraints on l in the statement of the theorem). This contradiction shows that $c_{(1,0)}$ and $c_{(0,1)}$ cannot both be non-zero.

Step 3: In the notation of Step 2, suppose without loss of generality that $q = m$. The function f has the form

$$f(\mathbf{x}) = c_{(0,0)}(x_1, \ldots, x_{m_0}) + c_{(0,1)}(x_1, \ldots, x_{m_0}) x_m.$$

Suppose for contradiction that this fails to hold, so the power-series representation of f has a monomial containing $x_p x_q$ for distinct p and $q \in [m_0 + 1 : m]$. We assume without loss of generality that $p = m_0 + 1$ and $q \geq m_0 + 2$. Let $N := k_p + k_q + k_{\max}$, fix $t_0 \in (0, \rho/3)$, and define $N \times N$ positive-semidefinite test matrices by setting

$$A^{(r)} = t_0 \big(\mathbf{1}_{(k_p+k_q)\times(k_p+k_q)} \oplus \mathbf{0}_{k_{\max}\times k_{\max}} \big)$$

$$\text{and} \quad B^{(r)} = t_0 \big(\mathbf{1}_{(k_p+k_q)\times(k_p+k_q)} \oplus \mathrm{Id}_{k_r} \oplus \mathbf{0}_{(k_{\max}-k_r)\times(k_{\max}-k_r)} \big)$$

for all $r \in [1 : m] \setminus \{p, q\}$,

$$A^{(p)} = t_0 \big(\mathrm{Id}_{k_p} \oplus 2\mathrm{Id}_{k_q} \oplus \mathbf{0}_{k_{\max}\times k_{\max}} \big),$$

$$A^{(q)} = t_0 \big(2\mathrm{Id}_{k_p} \oplus \mathrm{Id}_{k_q} \oplus \mathbf{0}_{k_{\max}\times k_{\max}} \big)$$

$$\text{and} \quad B^{(p)} = B^{(q)} = t_0 \big(2\mathrm{Id}_{k_p+k_q} \oplus \mathbf{0}_{k_{\max}\times k_{\max}} \big).$$

Choose a constant $\epsilon \in (0, \rho/3)$, let $\epsilon_1 = \cdots = \epsilon_m = \epsilon$, and note that we may apply Proposition 4.6(2) to conclude that $f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$ is positive semidefinite and has rank at most l.

Furthermore, writing $g(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0})$, the matrix $g[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - g[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$ is positive semidefinite and has rank at most l. As this is a block-diagonal matrix, its leading principal $(k_p + k_q) \times (k_p + k_q)$ submatrix M is also positive semidefinite and has rank at most l. We will now show that this is impossible.

As in the previous step, and letting $\mathbf{x}'$ be the reduced $m-2$-tuple obtained from $\mathbf{x}$ by deleting x_p and x_q, we may write

$$g(\mathbf{x}) = g(\mathbf{x}', x_p, x_q) = d_{(0,0)}(\mathbf{x}') + d_{((0,1)}(\mathbf{x}')x_q + d_{(1,1)}(\mathbf{x}')x_p x_q,$$

where the functions $d_{(0,0)}$, $d_{(0,1)}$ and $d_{(1,1)}$ are absolutely monotone, with $d_{(1,1)}$ non-zero, and $d_{(0,1)} = 0$ unless $q = m$.

As the leading principal $(k_p + k_q) \times (k_p + k_q)$ submatrices of $A^{(r)}$ and $B^{(r)}$ are equal when $r \neq p$, q, we see that the contributions of the $d_{(0,0)}[-]$ terms to M cancel and

$$M = t_0 d_{(0,1)}\big((t_0 + \epsilon)\mathbf{1}_{m-2}\big)\big(\mathbf{0}_{k_p \times k_p} \oplus \mathrm{Id}_{k_q}\big) + t_0(2t_0 + \epsilon)d_{(1,1)}\big((t_0 + \epsilon)\mathbf{1}_{m-2}\big)\mathrm{Id}_{k_p + k_q}.$$

As $d_{(0,1)}\big((t_0 + \epsilon)\mathbf{1}_{m-2}\big) \geq 0$ and $d_{(1,1)}\big((t_0 + \epsilon)\mathbf{1}_{m-2}\big) > 0$, it follows that

$$M \geq t_0(2t_0 + \epsilon)d_{(1,1)}\big((t_0 + \epsilon)\mathbf{1}_{m-2}\big)\mathrm{Id}_{k_p + k_q},$$

which is a positive semidefinite matrix having rank $k_p + k_q > l$ (by the theorem's restrictions on l). Thus M has rank greater than l, and this contradiction finishes the proof of Step 3.

Step 4: There exists a function F defined on I^{m_0}, such that $\mathbf{x}' \mapsto F(\mathbf{x}') - F(\mathbf{0})$ is absolutely monotone, and a non-negative constant c such that

$$f(\mathbf{x}) = F(x_1, \ldots, x_{m_0}) + c x_m \qquad \text{for all } \mathbf{x} \in I^m.$$

It follows from Step 3 that we need only show that the function $c_{(0,1)}$ obtained there is constant. If $m_0 = 0$ then this is immediate, so for this step we suppose that $m_0 \geq 1$. We will again apply Proposition 4.6(2).

For this, let $N := 2k_m + k_{\max}$, fix $t_0 \in (0, \rho/6)$ and $A_0 := t_0 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{\oplus k_m}$. We set

$$A^{(p)} = \begin{cases} A_0 \oplus \mathbf{0}_{k_{\max} \times k_{\max}} & \text{if } 1 \leq p \leq m_0, \\ t_0 \mathbf{1}_{N \times N} & \text{if } m_0 < p < m, \\ t_0 \mathbf{1}_{2\times 2}^{\oplus k_m} \oplus \mathbf{0}_{k_{\max} \times k_{\max}} & \text{if } p = m \end{cases}$$

and

$$B^{(p)} = \begin{cases} A^{(p)} & \text{if } 1 \leq p \leq m_0, \\ A^{(p)} + \big(t_0 \mathrm{Id}_{k_p} \oplus \mathbf{0}_{(N-k_p)\times(N-k_p)}\big) & \text{if } m_0 < p < m, \\ 2A^{(p)} & \text{if } p = m. \end{cases}$$

Choose any $\epsilon \in (0, \rho/6)$, let $\epsilon_1 = \cdots = \epsilon_m = \epsilon$, and apply Proposition 4.6(2) to see that $f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] - f[\mathbf{A} + \boldsymbol{\epsilon}\mathbf{1}]$ is positive semidefinite and has rank at most l. As this is a block-diagonal matrix, the same holds for its $2k_m \times 2k_m$ leading principal submatrix M.

We now obtain a contradiction if the absolutely monotone function $c_{(0,1)}$ is non-zero. First note that

$$M = c_{(0,1)}[B_0, \ldots, B_0] \circ \big(t_0 \mathbf{1}_{2\times 2}^{\oplus k_m}\big), \quad \text{where } B_0 := A_0 + \epsilon \mathbf{1}_{2k_m \times 2k_m}.$$

Now suppose $c_{(0,1)}$ contains a non-trivial monomial, say $b\mathbf{x}_{[1:m_0]}^{\boldsymbol{\alpha}'}$, where $b > 0$ and $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{m_0}) \neq \mathbf{0}$. Letting $|\boldsymbol{\alpha}'| := \sum_{p=1}^{m_0} \alpha_p$, we have that

$$M \geq b\mathbf{B}_0^{\circ\boldsymbol{\alpha}'} \circ \big(t_0 \mathbf{1}_{2\times 2}^{\oplus k_m}\big) = b t_0^{1+|\boldsymbol{\alpha}'|} \begin{bmatrix} (1+\epsilon')^{|\boldsymbol{\alpha}'|} & (2+\epsilon')^{|\boldsymbol{\alpha}'|} \\ (2+\epsilon')^{|\boldsymbol{\alpha}'|} & (5+\epsilon')^{|\boldsymbol{\alpha}'|} \end{bmatrix}^{\oplus k_m},$$

where $\epsilon' = \epsilon / t_0$. This matrix has rank $2k_m > l$, by the initial hypotheses of the theorem, and therefore so does M. This is the desired contradiction.

Step 5: Completing the proof.

To conclude, we assume f has the form claimed, with $p_0 = m$ and $c > 0$. We need to rule out the possibilities (a) $l < k_m$ and (b) $l = k_m$ and $F(\mathbf{0}) < 0$.

To do this, we let $N := 1 + k_{\max}$, choose $t_0 \in (0, \rho)$, and set

$$B^{(p)} = \begin{cases} \mathbf{0}_{N\times N} & \text{if } 1 \leq p \leq m_0, \\ -t_0 \mathrm{Id}_{k_p} \oplus \mathbf{0}_{(N-k_p)\times(N-k_p)} & \text{if } m_0 < p < m, \\ \Sigma_\pi^{\uparrow}(A) & p = m, \end{cases}$$

where A is as in (2.6) with $k = k_m$ and $\pi := \big\{\{1\}, \ldots, \{k_m\}, \{k_m + 1, \ldots, N\}\big\}$. The matrix $B^{(m)} \in \mathcal{S}_N^{(k_m)}(I)$, by Lemma 3.9, so $\mathbf{B} \in \mathcal{S}_N^{(\mathbf{k})}$ and therefore $f[\mathbf{B}] \in \overline{\mathcal{S}_N^{(l)}}$. As we see directly that $f[\mathbf{B}] = F(\mathbf{0})\mathbf{1}_{N\times N} + cB^{(m)}$ and $B^{(m)}$ has eigenvector $\mathbf{1}_N$, by construction, the reasoning of Step 5 in the proof of Theorem A now provides the desired contradiction in each case. $\square$

5. Multi-variable transforms for matrices with positive entries

In this section, we provide proofs for results stated above where matrices have entries in the one-sided intervals $I = (0, \rho)$ and $I = [0, \rho)$, where $0 < \rho \leq \infty$, and functions have domains of the form $(0, \rho)^m$ and $[0, \rho)^m$, where $m \geq 1$.

Remark 5.1. As with Remark 4.3, such classification results imply their counterparts for functions with domains of the form

$$(0, \rho_1) \times \cdots \times (0, \rho_m) \qquad \text{or} \qquad [0, \rho_1) \times \cdots \times [0, \rho_m),$$

where $0 < \rho_1, \ldots, \rho_m \leq \infty$.

Having explored the one-variable and multi-variable situations separately in the two-sided case, where $I = (-\rho, \rho)$, in this section we adopt a unified approach.

The case of $I = [0, \rho)$ will follow from the other two cases. Thus, we first let $I = (0, \rho)$ and defer the proofs for $I = [0, \rho)$ to Section 5.1. While the classification results for $I = (-\rho, \rho)$ above hold verbatim (except for the change of domain) when $I = (0, \rho)$, the proofs need to be modified in several places. We describe these modifications in what follows, and skip lightly over the remaining arguments, which are essentially the same as those in Sections 2 and 4.

The first step is to prove Proposition 4.6 when $I = (0, \rho)$. For this, we require the enhanced set of test matrices given by the following lemma.

Lemma 5.2. *Given constants a and b, with $0 \leq a < b$, a non-negative integer k and a non-negative constant ϵ, the map*

$$\Psi = \Psi(a,b,k,\epsilon) : \mathcal{S}_n^{(0)} \to \mathcal{S}_{k+1+n}^{(k)};\ B \mapsto (M_{k+1}(a,b) \oplus B) + \epsilon \mathbf{1}_{(k+1+n)\times(k+1+n)}$$

is well defined, where

$$M_{k+1}(a,b) := \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix} = (a-b)\mathrm{Id}_{k+1} + b\mathbf{1}_{(k+1)\times(k+1)}.$$

Furthermore, the k negative eigenvalues of $\Psi(B)$ are all equal to $a-b$ for any choice of B.

Proof. The $k=0$ case is immediate, so we assume henceforth that $k \geq 1$.

The matrix $b\mathbf{1}_{(k+1)\times(k+1)}$ has rank one and eigenvector $\mathbf{1}_{k+1}$ with eigenvalue $(k+1)b$, so $M_{k+1}(a,b)$ has eigenvalue $a+kb$ with multiplicity 1 and eigenvalue $a-b$ with multiplicity k.

Let $P = (k+1)^{-1}\mathbf{1}_{k+1}\mathbf{1}_{k+1}^T$ and $P^\perp = \mathrm{Id}_{k+1} - P$, so that P and $P^\perp$ are spectral projections such that $M_{k+1}(a,b) = (a+kb)P + (a-b)P^\perp$ and

$$\Psi(B) = \big((a+kb)P + (a-b)P^\perp\big) \oplus B + \epsilon\mathbf{1}_{(k+1+n)\times(k+1+n)}.$$

If $\mathbf{v}$ lies in the range of $P^\perp$, which has dimension k, then $P\mathbf{v} = \mathbf{0}_{k+1}$ and $\mathbf{1}_{k+1}^T\mathbf{v} = 0$, so

$$\Psi(B)\begin{bmatrix}\mathbf{v} \\ \mathbf{0}_n\end{bmatrix} = (a-b)\begin{bmatrix}\mathbf{v} \\ \mathbf{0}_n\end{bmatrix}.$$

This shows that $a-b$ is an eigenvalue of $\Psi(B)$ with multiplicity at least k. As the matrix $M_{k+1}(a,b)\oplus B$ has $k+1$ non-negative eigenvalues, so does $\Psi(B)$, by [11, Corollary 4.3.9] and the previous observation. This completes the proof. $\square$

Proof of Proposition 4.6 and Corollary 4.7 for $I=(0,\rho)$. The reader can verify that the above proofs of Proposition 4.6(2) and Corollary 4.7 with $I=(-\rho,\rho)$ go through verbatim for $I=(0,\rho)$, since all of the test matrices used therein have all their entries in $(0,\rho)$. It remains to show that the first part of Proposition 4.6 holds in this setting.

We begin by fixing $\epsilon \in I = (0,\rho)$ and a, $b \in (0,\rho-\epsilon)$ with $a<b$. Given a positive integer n and an m-tuple of matrices $\mathbf{A} = (A^{(1)},\ldots,A^{(m)}) \in \mathcal{S}_n^{(\mathbf{0})}\big((0,\rho-\epsilon)\big)$, the matrix

$$B_0^{(p)} := \Psi(a,b,k_p,0)\big((A^{(p)})^{\oplus(l+2)}\big) = M_{k_p+1}(a,b) \oplus (A^{(p)})^{\oplus(l+2)}$$

is an $N_p\times N_p$ block-diagonal matrix with entries in $[0,\rho-\epsilon)$, where $N_p := k_p+1+(l+2)n$, for all $p \in [1:m]$. By Lemma 5.2, the matrix $B_0^{(p)} + \epsilon\mathbf{1}_{N_p\times N_p} \in \mathcal{S}_{N_p}^{(k_p)}(I)$. We apply the inflation operator $\Sigma_{\pi_p}^{\uparrow}$ to this matrix, where

$$\pi_p := \big\{\{1,\ldots,k_{\max}-k_p+1\},\{k_{\max}-k_p+2\},\ldots,\{k_{\max}-k_p+N_p\}\big\}.$$

By Lemma 3.9, the matrix $B^{(p)} := \Sigma_{\pi_p}^{\uparrow}(B_0^{(p)})$ is such that $B^{(p)} + \epsilon\mathbf{1}_{N\times N} \in \mathcal{S}_N^{(k_p)}(I)$, where $N := k_{\max}+1+(l+2)n$.

It now follows from the hypotheses of the theorem that $f[\mathbf{B} + \boldsymbol{\epsilon}\mathbf{1}] \in \overline{\mathcal{S}_N^{(l)}}$, where $\epsilon_1 = \cdots = \epsilon_m = \epsilon$. Thus, by Lemma 2.2, we have that

$$g_\epsilon[\mathbf{B}] \in \overline{\mathcal{S}_N^{(l+1)}}, \qquad \text{where } g_\epsilon(\mathbf{x}) := f(\mathbf{x} + \epsilon\mathbf{1}_m) - f(\epsilon\mathbf{1}_m).$$

The matrix $g_\epsilon[\mathbf{B}]$ is block diagonal of the form

$$g_\epsilon[\Sigma^{\uparrow}_{\pi_{1'}}\big(M_{k_1+1}(a,b)\big), \dots, \Sigma^{\uparrow}_{\pi_{m'}}\big(M_{k_m+1}(a,b)\big)] \oplus g_\epsilon[\mathbf{A}]^{\oplus(l+2)},$$

where

$$\pi_{p'} := \{A \cap \{1, \dots, k_{\max} + 1\} : A \in \pi_p\} \qquad \text{for all } p \in [1:m].$$

Thus $g_\epsilon[\mathbf{A}]$ cannot have any negative eigenvalues, for any m-tuple $\mathbf{A} \in \mathcal{S}_n^{(\mathbf{0})}\big((0, \rho - \epsilon)\big)$ and all $n \geq 1$. It now follows from Theorem 1.5 that

$$f(\mathbf{x} + \epsilon\mathbf{1}_m) - f(\epsilon\mathbf{1}_m) = g_\epsilon(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha},\epsilon}\mathbf{x}^{\boldsymbol{\alpha}} \quad \text{for all } \mathbf{x} \in (0, \rho - \epsilon)^m,$$

where $c_{\boldsymbol{\alpha},\epsilon} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$.

As this holds for all $\epsilon \in (0, \rho)$, we see that f is smooth on $(0, \rho)^m$ and $(\partial^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\boldsymbol{\alpha} \neq \mathbf{0}$ and all $\mathbf{x} \in (0, \rho)^m$. In particular, $\partial_{x_p} f$ is absolutely monotone on $(0, \rho)^m$ for any $p \in [1:m]$ and so, by Theorem A.1, it has there a power-series representation with non-negative Maclaurin coefficients: we can write

$$(\partial_{x_p} f)(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} (\alpha_p + 1) c^{(p)}_{\boldsymbol{\alpha} + \mathbf{e}_p^T} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{x} \in (0, \rho)^m,$$

where $c^{(p)}_{\boldsymbol{\alpha} + \mathbf{e}_p^T} \geq 0$ for all $\boldsymbol{\alpha}$. Since f is smooth, the mixed partial derivatives $\partial_{x_p}\partial_{x_q} f$ and $\partial_{x_q}\partial_{x_p} f$ are equal for any distinct p and q, whence $c^{(p)}_{\boldsymbol{\alpha}} = c^{(q)}_{\boldsymbol{\alpha}}$ whenever $\alpha_p > 0$ and $\alpha_q > 0$. Hence setting $c_{\boldsymbol{\alpha}} := c^{(p)}_{\boldsymbol{\alpha}}$ makes $c_{\boldsymbol{\alpha}}$ well defined whenever $\boldsymbol{\alpha} \neq \mathbf{0}$. We let

$$\widetilde{f}(\mathbf{x}) := \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{x} \in (0, \rho)^m,$$

which is convergent because

$$\sum_{\boldsymbol{\alpha} \neq \mathbf{0}} |c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}| \leq \sum_{p=1}^m x_p \sum_{\boldsymbol{\alpha} : \alpha_p > 0} (\alpha_p + 1) c_{\boldsymbol{\alpha} + \mathbf{e}_p^T} \mathbf{x}^{\alpha}.$$

Note that $\partial_{x_p} \widetilde{f} = \partial_{x_p} f$ on $(0, \rho)^m$ for any $[1:m]$ and therefore $f = \widetilde{f} + c_{\mathbf{0}}$ if we let $c_{\mathbf{0}} := f(\epsilon\mathbf{1}_m) - \widetilde{f}(\epsilon\mathbf{1}_m)$. This shows that $x \mapsto f(\mathbf{x})$ has a power-series representation on $(0, \rho)^m$ of the form claimed and so completes the proof of Proposition 4.6(1) when $I = (0, \rho)$. $\square$

With Proposition 4.6 and Corollary 4.7 now established for $I = (0, \rho)$, we next show that the two main theorems hold in this context.

We note first that if f has a power-series representation on $(0, \rho)^m$ as in the conclusion of Proposition 4.6(1) then the unique extension of the function f to $(-\rho, \rho)^m$ will also be denoted by f and will be used without further comment.

Proof of Theorem 4.1 for $I = (0, \rho)$. The proof of Theorem 4.1 for $I = (-\rho, \rho)$ goes through verbatim, with one exception: the test matrix in (4.5) must be changed. We let $N := k_{\max} + 1$ and

$$\pi_p := \big\{\{1\}, \{2\}, \dots, \{k_p\}, \{k_p + 1, \dots, N\}\big\}.$$

Given a, $b \in (0, \rho)$, with $a < b$, the matrix

$$B^{(p)} := \Sigma^{\uparrow}_{\pi_p}\big(M_{k_p+1}(a, b)\big) \in \mathcal{S}^{(k_p)}_N(I) \qquad \text{for all } p \in [1:m],$$

by Lemma 5.2 and Lemma 3.9. It follows that the matrix $f[\mathbf{B}]$ is positive semidefinite, and so its $(1,1)$ entry $f(a\mathbf{1}_m) \geq 0$. Since f is continuous on $(-\rho, \rho)^m$, letting $a \to 0^+$ give that $f(\mathbf{0}) \geq 0$, as required. □

Proof of Theorem 4.2 for $I = (0, \rho)$. The proof that (2) implies (1) goes through as it does for $I = (-\rho, \rho)$. In the converse direction, Steps 1 to 4 of the proof for $I = (-\rho, \rho)$ require no changes. The only modification to be made is in Step 5, since the test matrices $B^{(1)}$, …, $B^{(m-1)}$ used there have non-positive entries.

Instead, with $N := 1 + k_{\max}$, $\epsilon \in (0, \rho)$ and a, $b \in (0, \rho - \epsilon)$ with $a < b$, we let

$$B^{(p)} = \begin{cases} b\mathbf{1}_{N\times N} & \text{if } 1 \leq p \leq m_0, \\ \big(M_{k_p+1}(a,b) \oplus \mathbf{0}_{(N-k_p-1)\times(N-k_p-1)}\big) + \epsilon\mathbf{1}_{N\times N} & \text{if } m_0 < p < m, \\ \Sigma^{\uparrow}_{\pi}(A) & \text{if } p = m, \end{cases}$$

where A is as in (2.6) and

$$\pi := \big\{\{1\}, \dots, \{k_m\}, \{k_m + 1, \dots, N\}\big\}.$$

It follows from Lemma 5.2 and Lemma 3.9 that $\mathbf{B} \in \mathcal{S}^{(\mathbf{k})}_N$. Hence

$$f[\mathbf{B}] = F[B^{(1)}, \dots, B^{(m_0)}] + cB^{(m)} \in \overline{\mathcal{S}^{(l)}_N}.$$

Since F is continuous, letting $b \to 0^+$ gives that

$$\begin{aligned} F[\mathbf{0}_{N\times N}, \dots, \mathbf{0}_{N\times N}] + cB^{(m)} &= F(\mathbf{0}_{m_0})\mathbf{1}_{N\times N} + cB^{(m)} \\ &= \Sigma^{\uparrow}_{\pi}(F(\mathbf{0}_{m_0})\mathbf{1}_{(k_m+1)\times(k_m+1)} + cA) \in \overline{\mathcal{S}^{(l)}_N}, \end{aligned}$$

since $\overline{\mathcal{S}^{(l)}_N}$ is closed for the topology of entrywise convergence. By Lemma 3.9, it follows that the matrix $F(\mathbf{0}_{m_0})\mathbf{1}_{(k_m+1)\times(k_m+1)} + cA \in \overline{\mathcal{S}^{(l)}_{k_m+1}}$, but this is false, according to the reasoning in Step 5 of the proof of Theorem A. □

We now use Theorem 4.2 to establish two results for preservers when $I = (0, \rho)$, including a version of Corollary 4.4. The proofs require another carefully chosen set of test matrices.

Lemma 5.3. *There exists an invertible matrix $A \in \mathcal{S}^{(1)}_3\big((0,5)\big)$ such that, for each positive integer k, the matrix*

$$A^{\oplus k} + t\mathbf{1}_{3k\times 3k} \in \mathcal{S}^{(k-1)}_{3k} \qquad \textit{whenever } t > 1.$$

Proof. Note first that the matrix

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

has characteristic polynomial

$$\det(x\mathrm{Id}_3 - A) = (x-1)(x^2 - 8x - 1)$$

and eigenvalues 1, $4+\sqrt{17}$ and $4-\sqrt{17}$, whence $A \in \mathcal{S}_3^{(1)}\big((0,5)\big)$.

To proceed, we recall the matrix determinant lemma, that

$$\det(B + \mathbf{u}\mathbf{v}^T) = \det B + \mathbf{v}^T (\operatorname{adj} B)\, \mathbf{u},$$

where $\operatorname{adj} B$ denotes the adjugate of the matrix B, and the identity

$$\operatorname{adj}(B \oplus C) = (\det C) \operatorname{adj} B \oplus (\det B) \operatorname{adj} C.$$

It follow from these identities that, for any $t \in \mathbb{R}$,

$$\begin{aligned} &\det(x\mathrm{Id}_{3k} - A^{\oplus k} - t\mathbf{1}_{3k\times 3k}) \\ &\qquad = \det(x\mathrm{Id}_3 - A)^{k-1}(\det(x\mathrm{Id}_3 - A) - tk\mathbf{1}_3^T \operatorname{adj}(x\mathrm{Id}_3 - A)\mathbf{1}_3). \end{aligned}$$

A straightforward computation reveals that

$$\mathbf{1}_3^T \operatorname{adj}(x\mathrm{Id}_3 - A)\mathbf{1}_3 = (3x-1)(x-1)$$

and so the characteristic polynomial of the linear pencil $A^{\oplus k} + t\mathbf{1}_{3k\times 3k}$ is

$$\det(x\mathrm{Id}_3 - A)^{k-1}(x-1)(x^2 - (8+3tk)x - 1 + tk).$$

Thus $A^{\oplus k} + t\mathbf{1}_{3k\times 3k} \in \mathcal{S}_{3k}^{(k-1)}$ if both roots of the polynomial $x^2 - (8+3tk)x - 1 + tk$ are positive, but this holds for all $t > 1$. $\square$

Now we have:

Proof of Corollary 4.4 for $I = (0,\rho)$. The proof with $I = (-\rho,\rho)$ of the second part goes through verbatim over $(0,\rho)$, as does the proof of the first part until the final sentence, which uses the matrices $B^{(1)} = \cdots = B^{(m)} = -\epsilon\mathrm{Id}_k$ having no entries in $(0,\rho)$. Thus, it remains to show the following holds.

Suppose $I := (0,\rho)$, where $0 < \rho \leq \infty$, and let $f(\mathbf{x}) = cx_{p_0} + d$ for all $\mathbf{x} \in I^m$, where $c > 0$, $p_0 \in [1:m]$ and $d \geq 0$. If $k \geq 1$ and $f[-]$ sends $\mathcal{S}_n^{(k\mathbf{1}_m^T)}(I)$ to $\mathcal{S}_n^{(k)}$ for all $n \geq k$ then $d = 0$.

To show this, suppose $d > 0$ and let $A \in \mathcal{S}_3^{(1)}\big((0,5)\big)$ be as in Lemma 5.3. Choose $\delta \in (0, d/c)$ such that $\delta < \rho/5$, so that $(\delta A)^{\oplus k} \in \mathcal{S}_{3k}^{(k)}\big([0,\rho)\big)$. This matrix is invertible, so $(\delta A)^{\oplus k} + \epsilon\mathbf{1}_{3k\times 3k} \in \mathcal{S}_{3k}^{(k)}\big((0,\rho)\big)$ for sufficiently small $\epsilon > 0$, by the continuity of eigenvalues. Then, by the assumption on f,

$$B := f[(\delta A)^{\oplus k} + \epsilon\mathbf{1}_{3k\times 3k}, \ldots, (\delta A)^{\oplus k} + \epsilon\mathbf{1}_{3k\times 3k}] = c(\delta A)^{\oplus k} + (c\epsilon + d)\mathbf{1}_{3k\times 3k} \in \mathcal{S}_{3k}^{(k)}.$$

Furthermore, we can write

$$B = c\delta(A^{\oplus k} + (d/c\delta)\mathbf{1}_{3k\times 3k}) + c\epsilon\mathbf{1}_{3k\times 3k},$$

and Lemma 5.3 implies that $A^{\oplus k} + t\mathbf{1}_{3k\times 3k} \in \mathcal{S}_{3k}^{(k-1)}$ for all $t > 1$, so $B \in \overline{\mathcal{S}_{3k}^{(k-1)}}$, by Lemma 2.2.

This contradiction shows that $d = 0$, as claimed. $\square$

We end by returning full circle to the first result we stated in the one-variable setting: the classification of inertia preservers for matrices with positive or non-negative entries.

Corollary 5.4. *Let $I := (0, \rho)$ or $[0, \rho)$, where $0 < \rho \leq \infty$, and let k be a non-negative integer. Given a function $f : I \to \mathbb{R}$, the following are equivalent.*

(1) *The entrywise transform $f[-]$ preserves the inertia of all matrices in $\mathcal{S}^{(k)}(I)$.*
(2) *The function is a positive homothety: $f(x) \equiv cx$ for some constant $c > 0$.*

In other words, Theorem 1.2 holds verbatim if $I = (-\rho, \rho)$ is replaced by $(0, \rho)$ or $[0, \rho)$.

Proof for $I = (0, \rho)$. Clearly, (2) implies (1). Conversely, if $k = 0$ then the proof of Theorem 1.2 goes through in this case, using Theorem 1.5 with $m = 1$ in place of Schoenberg's Theorem 1.1. Otherwise, we have $k \geq 1$ and $f[-]$ sends $\mathcal{S}_n^{(k)}\big((0, \rho)\big)$ into $\overline{\mathcal{S}_n^{(k)}}$ for all $n \geq k$. The $m = 1$ case of Theorem 4.2 with $k_1 = l = k$ gives that either $f(x) \equiv d$ for some $d \in \mathbb{R}$ or $f(x) \equiv cx + d$, with $c > 0$ and $d \geq 0$. Now we are done by following the proof of Theorem 1.2 to out the first possibility and using the italicized assertion in the proof of Corollary 4.4 to show that $d = 0$. $\square$

5.1. **Proofs for matrices with non-negative entries.** We conclude by explaining how to modify the proofs given above to obtain Theorems 4.1 and 4.2, together with their classification consequences in Corollaries 4.4 and 5.4, when $I = [0, \rho)$. We begin by establishing Proposition 4.6 and Corollary 4.7 in this context.

Proof of Proposition 4.6 and Corollary 4.7 for $I = [0, \rho)$. Proofs of the second parts of Proposition 4.6 and Corollary 4.7 when $I = (0, \rho)$ go through verbatim if $I = [0, \rho)$, since the test matrices used there have all their entries in $(0, \rho)$. It remains to prove the first part of Proposition 4.6.

This follows the same reasoning as the proof of Proposition 4.6(1) for $I = (-\rho, \rho)$, but with $B^{(p)}$ there replaced by

$$B^{(p)} := M_{k_p+1}(a, b) \oplus \mathbf{0}_{(k_{\max} - k_p) \times (k_{\max} - k_p)} \oplus A_p^{\oplus (l+2)} \in \mathcal{S}_N^{(k_p)}$$

for each $p \in [1 : m]$, where a, $b \in (0, \rho)$ with $a < b$ and $M_{k_p+1}(a, b)$ is as in Lemma 5.2. $\square$

With these results in hand, we can now provide the final proofs.

Proof of Theorems 4.1 and 4.2, and of Corollaries 4.4 and 5.4, for $I = [0, \rho)$. First, it is clear that (1) implies (2) in Theorem 4.1. If (2) holds then $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}\big((0, \rho)\big)$ to $\mathcal{S}_n^{(0)}$, for all $n \geq k_{\max}$, so the $I = (0, \rho)$ part of Theorem 4.1 implies that the restriction of $\mathbf{x} \to f(\mathbf{x})$ to $(0, \rho)^m$ is independent of $x_{m_0+1}, \ldots, x_m$. However, Proposition 4.6(1) implies that f is continuous on $[0, \rho)^m$, so this independence extends to the whole domain of f and therefore (2) implies (3). Finally, if (3) holds then f extends to a function of the same form on $(-\rho, \rho)^m$, so (1) holds for $I = (-\rho, \rho)$ by this case of Theorem 4.1, which implies (1) for $I = [0, \rho)$. The proof of the final part of Theorem 4.1 is identical to the $I = (-\rho, \rho)$ version.

For Theorem 4.2, if (1) holds then (2) holds for the restriction of f to $(0,\rho)^m$, and again Proposition 4.6(1) allows extension by continuity, so that (2) holds in general. The proof that (2) implies (1) is unchanged from the $I = (-\rho, \rho)$ case.

For both parts of Corollary 4.4, if f has the prescribed form then the entrywise transform maps $\mathcal{S}_n^{(k\mathbf{1}_m^T)}(I)$ or $\overline{\mathcal{S}_n^{(k\mathbf{1}_m^T)}}(I)$ to $\mathcal{S}_n^{(k)}$ or $\overline{\mathcal{S}_n^{(k)}}$ when $I = (-\rho, \rho)$, so the same holds when $I = [0, \rho)$. The converse uses the same restriction and continuity argument as before.

Finally, that (2) implies (1) in Corollary 5.4 is immediate. For the converse, note again that the conclusion holds for the restriction of f to $(0,\rho)$ and the continuity of f on $I = [0, \rho)$ follows from Proposition 4.6(1). $\square$

Appendix A. Absolutely monotone functions of several variables

The purpose of this Appendix is to provide the proof of a result on absolutely monotone functions of several variables that seems to not be readily available in the literature, but was used in previous work [2] and is relevant to the present paper.

Given an open interval $I \subseteq \mathbb{R}$ and a positive integer m, let $\mathbf{x} = (x_1, \ldots, x_m) \in I^m$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m$. Set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_m$.

Recall that a smooth function $f : I^m \to \mathbb{R}$ is *absolutely monotone* if

$$\partial^{\boldsymbol{\alpha}} f(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}(x_1, \ldots, x_m) \geq 0 \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{Z}_+^m \text{ and } \mathbf{x} \in I^m.$$

If, instead, the function f is such that

$$(-1)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} f(\mathbf{x}) \geq 0 \quad \text{for all } \boldsymbol{\alpha} \in \mathbb{Z}_+^m \text{ and } \mathbf{x} \in I^m$$

then f is said to be *completely monotone*.

A function $f : [0,\rho)^m \to \mathbb{R}$, where $0 < \rho \leq \infty$, is said to be *absolutely monotone* if the restriction of f to $(0,\rho)^m$ is absolutely monotone, as defined above, and f is continuous on $[0,\rho)^m$. Step II in the proof of [14, Theorem 5] shows that such a function has non-negative one-sided derivatives at the boundary points of its domain.

Above and in previous work [2], we use the fact that absolutely monotone functions have power-series representations with non-negative Maclaurin coefficients. This result is used for functions with domains of the form $(0,\rho)^m$, where $0 < \rho \leq \infty$ and $m \geq 1$. For $m = 1$, this is a special case of Bernstein's theorem [6], and for $m = 2$ the power-series representation is derived in Schoenberg's paper [16] using completely monotone functions.

However, we were unable to find in the literature a reference for the case $m > 2$. When $I = [0, \rho)$ and ρ is finite, this representation theorem, for all $m \geq 1$, is found in Ressel's work [14, Theorem 8]; the extension to the case $\rho = \infty$ is immediate, by the identity theorem.

To fill this gap, we state the following theorem and provide its proof.

Theorem A.1. *Let $I = (0, \rho)$, where $0 < \rho \leq \infty$, and let m be a positive integer. The smooth function $f : I^m \to \mathbb{R}$ is absolutely monotone if and only if f is represented on I^m by a power series with non-negative Maclaurin coefficients:*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{x} \in I^m, \text{ where } c_{\boldsymbol{\alpha}} \geq 0 \text{ for all } \boldsymbol{\alpha}.$$

Note that any function with such a representation extends to a real-analytic function on the domain $(-\rho, \rho)^m$.

Theorem A.1 was implicitly used to prove the multi-variable version of Schoenberg's Theorem 1.5 with $I = (0, \rho)^m$ in our previous work [2], where it is used in turn to show this theorem for $I = [0, \rho)^m$ and $I = (-\rho, \rho)^m$. The same result is used in the present work, to prove Proposition 4.6(1) for all three choices of I, leading to the characterization results of Theorems 4.1 and 4.2 and their corollaries.

We now turn to the proof of Theorem A.1. While it is likely that it would be possible to show this result for $m > 2$ by adapting the proof for the $m = 2$ case given by Schoenberg [16], we proceed differently here: we use Ressel's theorem and a natural group operation on convex cones, well known in Hardy-space theory.

Proof of Theorem A.1. The reverse implication is clear. To prove the other, we note that, from the previous remarks, we need only to show that an absolutely monotone function on $(0, \rho)^m$ has a continuous extension to $[0, \rho)^m$ when ρ is finite. Moreover, by scaling, we may assume that $\rho = 1$. We offer two different paths to show that such an extension exists.

Path 1: We note first that

$$g : (0, \infty)^m \to [0, \infty);\ \mathbf{x} \mapsto f(e^{-x_1}, \ldots, e^{-x_m})$$

is completely monotone, because an inductive argument shows that

$$(\partial^{\boldsymbol{\alpha}} g)(\mathbf{x}) = (-1)^{|\boldsymbol{\alpha}|} g(\mathbf{x}) p_{\boldsymbol{\alpha}}(e^{-x_1}, \ldots, e^{-x_m}) \qquad \text{for all } \boldsymbol{\alpha} \in \mathbb{Z}_+^m,$$

where $p_{\boldsymbol{\alpha}}$ is a polynomial with non-negative coefficients. Hence, by [9, Corollaries 2.1 and 2.2], the function g is real analytic on $(0, \infty)^m$. Composition with the change of coordinates

$$T : (0, 1)^m \to (0, \infty)^m;\ (x_1, \ldots, x_m) \mapsto (-\log x_1, \ldots, -\log x_m)$$

now shows that $f = g \circ T$ is real analytic on $(0, 1)^m$.

Now fix $\epsilon \in (0, 1/2)$ and consider the function.

$$h : (-\epsilon, 1 - \epsilon)^m \to \mathbb{R};\ \mathbf{x} \mapsto f(\mathbf{x} + \epsilon \mathbf{1}_m).$$

It is immediate that h is absolutely monotone on $[0, 1 - \epsilon)^m$, so it is represented there by a power series with non-negative Maclaurin coefficients, by [14, Theorem 8]:

$$f(\mathbf{x} + \epsilon \mathbf{1}_m) = h(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} d_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } \mathbf{x} \in [0, 1 - \epsilon)^m, \tag{A.1}$$

where $d_{\boldsymbol{\alpha}} \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$. Moreover, this series is absolutely convergent for all $\mathbf{x} \in [-\epsilon, 1 - \epsilon)^m$, since $\epsilon < 1/2$. Thus h admits a continuous extension to the boundary. Since f is real analytic and agrees with the power series (A.1) on an open set, it admits a continuous extension to $[0, 1)^m$, as desired. □

Path 2: The following approach was communicated to us by Paul Ressel. Given any point $\mathbf{x}_0 \in [0, 1)^m$, there exists $\epsilon_0 > 0$ such that $\mathbf{x}_0 + \epsilon_0 \mathbf{1}_m \in [0, 1)^m$. If

$$g_{\mathbf{x}_0} : (0, \epsilon_0] \to \mathbb{R};\ t \mapsto f(\mathbf{x}_0 + t\mathbf{1}_m)$$

then $g''_{\mathbf{x}_0} \geq 0$ and so $g_{\mathbf{x}_0}$ is convex on $(0, \epsilon_0]$. Hence $g_{\mathbf{x}_0}(0^+)$ exists, is finite and agrees with $g_{\mathbf{x}_0}(0) = f(\mathbf{x}_0)$ if $\mathbf{x}_0 \in (0, 1)^m$. We can now extend f to $[0, 1)^m$ by setting

$$\widetilde{f} : [0, 1)^m \to \mathbb{R};\ \mathbf{x}_0 \mapsto g_{\mathbf{x}_0}(0^+).$$

One can verify that this function satisfies the "forward difference" definition of absolute monotonicity, as given on [14, pp. 259–260]. Hence, by [14, Theorem 8], $\widetilde{f}$ is represented by a convergent series with non-negative Maclaurin coefficients, and therefore so is f. □

Acknowledgements. The authors hereby express their gratitude to the University of Delaware for its hospitality and stimulating working environment, and in particular to the Virden Center in Lewes, where an initial segment of this work was carried out. The authors also thank the Institute for Advanced Study, Princeton and the American Institute of Mathematics, Pasadena for their hospitality while this work was concluded. A.K. thanks Paul Ressel for useful discussions related to Theorem A.1.

A.B. was partially supported by Lancaster University while some of this work was undertaken.

D.G. was partially supported by a University of Delaware Research Foundation grant, by a Simons Foundation collaboration grant for mathematicians, and by a University of Delaware strategic initiative grant.

A.K. was partially supported by the Ramanujan Fellowship SB/S2/RJN-121/2017 and SwarnaJayanti Fellowship grants SB/SJF/2019-20/14 and DST/SJF/MS/2019/3 from SERB and DST (Govt. of India), a Shanti Swarup Bhatnagar Award from CSIR (Govt. of India), and the DST FIST program 2021 [TPN–700661].

M.P. was partially supported by a Simons Foundation collaboration grant.

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