INTEGRATION AND MEASURES ON THE SPACE OF COUNTABLE LABELLED GRAPHS

APOORVA KHARE AND BALA RAJARATNAM
STANFORD UNIVERSITY

Abstract. In this paper we develop a rigorous foundation for the study of integration and measures on the space $\mathscr{G}(V)$ of all graphs defined on a countable labelled vertex set V. We first study several interrelated σ-algebras and a large family of probability measures on graph space. We then focus on a "dyadic" Hamming distance function $\|\cdot\|_{\psi,2}$, which was very useful in the study of differentiation on $\mathscr{G}(V)$. The function $\|\cdot\|_{\psi,2}$ is shown to be a Haar measure-preserving bijection from the subset of infinite graphs to the circle (with the Haar/Lebesgue measure), thereby naturally identifying the two spaces. As a consequence, we establish a "change of variables" formula that enables the transfer of the Riemann-Lebesgue theory on $\mathbb{R}$ to graph space $\mathscr{G}(V)$. This also complements previous work in which a theory of Newton-Leibnitz differentiation was transferred from the real line to $\mathscr{G}(V)$ for countable V. Finally, we identify the Pontryagin dual of $\mathscr{G}(V)$, and characterize the positive definite functions on $\mathscr{G}(V)$.

1. Introduction and main results

The study of very large graphs and their limits has recently been the focus of tremendous interest, given its importance in a variety of scientific disciplines including probability and statistics, combinatorics, computer science, machine learning, and network analysis in various applied fields. In this regard several limiting theories have been developed in the literature. Prominent among these is the comprehensive theory of graphons, which are limits of (dense) unlabelled graphs (see [Lo] and the references therein).

In the present paper, we work in the parallel setting of labelled graphs and their limits. Our motivation comes from the fact that often graphs in real-world situations and observed network data are labelled, and each vertex has a specific meaning. Similarly in theoretical probability such as Markov random fields and their applications, nodes in graphs represent variables that are not exchangeable owing to the dependencies in the underlying model. This provides motivation to study the space of labelled graphs and their limits.

In [KR1] a framework was introduced in which to study all finite labelled graphs at once; namely, the space $\mathscr{G}(V)$ of graphs with a fixed countable, labelled vertex set V. The algebraic and topological properties of $\mathscr{G}(V)$, as well as continuous functions on $\mathscr{G}(V)$, were studied in [KR1]. Moreover, a theory of differentiation on graph space was developed in [KR1]; see also [DGKR] for differentiation in the unlabelled setting in graphon space. Note also that the space of graphons is naturally equipped with a large family of measures that arise from sampling. We now explore the parallel setting of labelled graph space, with the aim of studying measures on $\mathscr{G}(V)$ and developing a theory of integration. This is the goal of the present paper.

The space $\mathscr{G}(V)$ is a compact abelian group; hence the associated Haar measure naturally gives rise to a theory of integration. Our first objective is to identify and study the Haar measure. The next goal of this work is to explore the connections between Haar integration on graph space and the Riemann-Lebesgue theory on $\mathbb{R}$. As a consequence of our investigations, we show below that

Date: June 9, 2015.
Key words and phrases. Countable graphs, Haar measure, Pontryagin dual.

integration on $\mathscr{G}(V)$ can be reduced to that on the unit interval. This is akin to [KR1], in which differentiation on $\mathscr{G}(V)$ was shown to be closely related to the one-variable Newton-Leibnitz theory on $\mathbb{R}$.

In this section we will state the main results of the paper. We begin by setting some notation.

Definition 1.1. Given a fixed labelled set V, define the corresponding (labelled) graph space $\mathscr{G}(V)$ to be the set of all graphs with vertex set V. In other words, $\mathscr{G}(V) = \{0,1\}^{K_V} = (\mathbb{Z}/2\mathbb{Z})^{K_V}$, where K_V is the complete graph on V. Also define $\mathscr{G}_0(V)$ to be the set of all graphs with finitely many edges, and $\mathscr{G}_1(V)$ to be the set of all *co-finite* graphs – i.e., the complements in K_V of finite graphs.

Henceforth every labelled graph with vertex set V will be identified with its edge set, which is a subset of K_V. Note that $\mathscr{G}(V)$ is the set of all functions $f : K_V \to \mathbb{Z}/2\mathbb{Z}$, the discrete field with two elements. This makes $\mathscr{G}(V)$ a commutative topological $\mathbb{Z}/2\mathbb{Z}$-algebra under pointwise addition and multiplication. In particular, the binary operation $G + G' := G\Delta G'$ makes $\mathscr{G}(V)$ into an abelian topological group, where Δ denotes the symmetric difference and the zero element $\mathbf{0} \in \mathscr{G}(V)$ is given by the empty graph on V. Note also that $\mathscr{G}(V)$ is 2-torsion, i.e., $G+G = \mathbf{0}$ for all $G \in \mathscr{G}(V)$.

We next discuss the topological structure of labelled graph space $\mathscr{G}(V)$ for a countable vertex set V, as studied in [KR1]. The following family of metrics on $\mathscr{G}(V)$ was crucially used in [KR1] in developing differential calculus in $\mathscr{G}(V)$, and is also important for the purposes of the present paper.

Definition 1.2. Suppose V is countable and $\psi : K_V \to \mathbb{N}$ is a fixed bijection. Given $a > 1$, define $d_{\varphi_a} : \mathscr{G}(V) \times \mathscr{G}(V) \to [0,\infty)$ and $\|\cdot\|_{\psi,a} : \mathscr{G}(V) \to [0,\infty)$ via:

$$d_{\varphi_a}(G, G') := \sum_{e \in G\Delta G'} a^{-\psi(e)}, \qquad \|G\|_{\psi,a} := d_{\varphi_a}(\mathbf{0}, G). \tag{1.3}$$

Next, a sequence $\{G_n : n \in \mathbb{N}\}$ in $\mathscr{G}(V)$ is said to *converge (to $G \in \mathscr{G}(V)$)* if the indicator sequences $\{\mathbf{1}_{e\in G_n} : n \in \mathbb{N}\}$ each converge (to $\mathbf{1}_{e\in G}$) for each edge $e \in K_V$.

The following theorem collects together some of the topological results in [KR1] on labelled graph space $\mathscr{G}(V)$, which are needed for the purposes of this paper.

Theorem 1.4 ([KR1])**.** *For a fixed labelled set V, the set $\mathscr{G}(V)$ of graphs is a commutative, totally disconnected, compact Hausdorff topological $\mathbb{Z}/2\mathbb{Z}$-algebra.*

Now suppose that V is countable and $\psi : K_V \to \mathbb{N}$ is a fixed bijection.

1. *The maps $\{d_{\varphi_a} : a \geq 1\}$ are translation-invariant metrics on $\mathscr{G}(V)$, which are all topologically equivalent and metrize the above notion of graph convergence (i.e., they generate the product topology on $\mathscr{G}(V) = \{0,1\}^{K_V}$).*
2. *The sets $\mathscr{G}_0(V), \mathscr{G}_1(V)$ are dense in $\mathscr{G}(V)$.*
3. *The map $2d_{\varphi_3}(\mathbf{0}, -) = 2\|\cdot\|_{\psi,3} : \mathscr{G}(V) \to [0,1]$ is a homeomorphism onto the Cantor set. Thus $\mathscr{G}(V)$ is a compact metric space.*
4. *The map $d_{\varphi_2}(\mathbf{0}, -) = \|\cdot\|_{\psi,2} : \mathscr{G}(V) \to [0,1]$ is a surjection, which is a bijection outside $\mathscr{G}_0(V)$. For every finite nonempty graph G, there exists a unique co-finite graph G' such that $\|G\|_{\psi,2} = \|G'\|_{\psi,2}$.*

We now present the main results in this paper. Since $\mathscr{G}(V)$ is a compact abelian group, it is natural to seek out its associated Haar measure. We identify this measure in our first main result. We also show that the Haar measure is intimately connected to the distinguished metric $\|\cdot\|_{\psi,2}$ (for any ψ) that was used in [KR1] to develop a differential calculus on $\mathscr{G}(V)$. More precisely, the following holds.

Theorem A. *Fix a countably infinite set V. The Haar measure $\mu_{\rm Haar}$ on $\mathscr{G}(V)$ is the unique probability measure $\mu_{1/2}$ induced from the Bernoulli($\frac{1}{2}$)-measure on each factor $\{0,1\}$ of $\mathscr{G}(V)$. Now given any bijection $\psi : K_V \to \mathbb{N}$, the Haar measure of any open or closed $\|\cdot\|_{\psi,2}$-ball (in $\mathscr{G}(V)$) of radius $\epsilon \in [0,1]$ is ϵ.*

Note here that the Borel σ-algebra $\mathscr{B}_{\mathscr{G}(V)}$ of $\mathscr{G}(V)$, as well as the Haar measure $\mu_{1/2}$, do not depend on the choice of labelling $\psi : K_V \to \mathbb{N}$.

It is natural to ask if the measure space $\mathscr{G}(V)$ with its Borel σ-algebra, can be modelled by a more familiar probability space. (This is akin to Theorem 1.4, which provided familiar topological models for graph space.) Note moreover that the last assertion in Theorem A has an obvious analogue for the usual Lebesgue measure, which is in fact the Haar measure on the real line. It is now natural to ask if the two Haar measures are related. The following result answers both of these questions, and shows how to transfer integration from $\mathscr{G}(V)$ to $\mathbb{R}$.

Theorem B. *Fix a bijection $\psi : K_V \to \mathbb{N}$.*

1. *The map $\|\cdot\|_{\psi,2} : \mathscr{G}(V) \to [0,1]$ – or to the circle $S^1 = \mathbb{R}/\mathbb{Z}$ – is a measurable, Haar measure-preserving map that is a bijection outside the countable (measure zero) sets $\mathscr{G}_0(V), \mathscr{G}_1(V)$.*
2. *Suppose $f : [0,1] \to [-\infty, \infty]$ is Lebesgue integrable. Then,*
$$\mathbb{E}_{\mu_{\rm Haar}}[f(\|\cdot\|_{\psi,2})] = \int_0^1 f(x)\ dx.$$
Conversely, for all integrable $g : \mathscr{G}(V) \to [-\infty, \infty]$, we have
$$\mathbb{E}_{\mu_{\rm Haar}}[g] = \int_0^1 g((\|\cdot\|_{\psi,2})^{-1}(x))\ dx.$$

Thus, Haar integration can be carried out on labelled graph space by transferring the classical Lebesgue theory from the unit interval (or the circle) to $\mathscr{G}(V)$.

The remaining sections are devoted to proving the above results. We add moreover that additional results concerning Fourier analysis, the Pontryagin dual, and positive definite functions for $\mathscr{G}(V)$ are shown in Section 3.2 below.

2. Measures on graph space

In this section we develop the necessary tools required to show Theorem A. We begin by studying several σ-algebras on graph space and showing how they are related. We then study probability measures on $\mathscr{G}(V)$ and prove Theorem A.

2.1. **σ-algebras on graph space.** We begin with an arbitrary (fixed) labelled index set V of vertices. Let $V(e)$ denote the vertices attached to an edge $e \in K_V$; then $\mathscr{G}(V) = \times_{e \in K_V} \mathcal{P}(K_{V(e)})$ is the Cartesian product of power sets, and each set is a σ-algebra of size 2. Define the product σ-algebra $\Sigma_{\rm meas}$ on $\mathscr{G}(V)$ to be the σ-algebra generated by the cylinder sets
$$S_{e_0} := \times_{e \neq e_0} \{\emptyset, \{e\}\} \times \{e_0\}, \qquad e_0 \in K_V. \tag{2.1}$$

We now define several other σ-algebras on $\mathscr{G}(V)$, as well as a closely related family of sets.

Definition 2.2. Given disjoint subsets $I_0, I_1 \subset K_V$, define
$$\mathscr{E}(I_0, I_1) := \{G \in \mathscr{G}(V) : I_1 \subset G,\ I_0 \subset K_V \setminus G\}. \tag{2.3}$$

Now define the following σ-algebras on $\mathscr{G}(V)$:

- $\mathscr{B}_{\mathscr{G}(V)}$ is the *Borel σ-algebra*, generated by all open sets.
- Σ_0 is the σ-algebra generated by all compact sets.
- $\Sigma_{\mathscr{E}}$ is the σ-algebra generated by all sets $\mathscr{E}(I_0, I_1)$ for disjoint $I_0, I_1 \subset K_V$.

- $\Sigma_{\mathscr{E},0}$ is the σ-algebra generated by all $\mathscr{E}(I_0, I_1)$ for finite (or countable) disjoint $I_0, I_1 \subset K_V$.

Consider the case when V, and hence K_V, is countable. In this case, as shown in [KR1], the product topology on the compact space $\mathscr{G}(V)$ can be metrized; this yields another candidate σ-algebra, as described presently. The main result of this subsection relates all of the above σ-algebras.

Theorem 2.4. *Suppose V is countable, and d is a metric on $\mathscr{G}(V)$ that metrizes the product topology. Define $\Sigma_{d,ball}$ to be the σ-algebra generated by the open (or closed) d-balls in $\mathscr{G}(V)$. Then,*

$$\Sigma_{\text{meas}} = \Sigma_{\mathscr{E},0} = \Sigma_{\mathscr{E}} = \Sigma_{d,ball} = \mathscr{B}_{\mathscr{G}(V)} = \Sigma_0.$$

In order to prove Theorem 2.4, some preliminary results are needed. The first result collects some basic facts about the sets $\mathscr{E}(I_0, I_1)$.

Lemma 2.5. *Suppose V is arbitrary and $I_0, I_1, J_0, J_1 \subset K_V$ are all disjoint.*

(1) *Then one has:*
$$\mathscr{E}(I_0 \cup J_0, I_1 \cup J_1) = \mathscr{E}(I_0, J_0) \cap \mathscr{E}(I_1, J_1). \tag{2.6}$$
In particular, $\mathscr{E}(-,-)$ is inclusion-reversing in each argument.

(2) *Given $S \subset \mathscr{G}(V) \ni G$, define $S + G := \{G' + G : G' \in S\}$. Then,*
$$\mathscr{E}(I_0, I_1) = \mathscr{E}(I_0 \cup I_1, \emptyset) + I_1, \tag{2.7}$$
where for every $I \subset K_V$, $\mathscr{E}(I, \emptyset)$ is an ideal of (the $\mathbb{Z}/2\mathbb{Z}$-algebra) $\mathscr{G}(V)$.

(3) *Given any $G \in \mathscr{G}(V)$,*
$$\mathscr{E}(I_0, I_1) + G = \mathscr{E}\left((I_0 \setminus G) \coprod (I_1 \cap G), (I_1 \setminus G) \coprod (I_0 \cap G)\right). \tag{2.8}$$

(4) *Given disjoint sets $I_0, I_1 \subset K_V$, the set $\mathscr{E}(I_0, I_1)$ is closed in K_V. It is open if and only if $I_0 \coprod I_1$ is finite.*

Proof. All but the last part are easy to prove using the definitions. For the last part, note that for finite disjoint $I_0, I_1 \subset K_V$, $\mathscr{E}(I_0, I_1)$ is closed as well as open in $\mathscr{G}(V)$. Hence $\mathscr{E}(I_0, I_1) \subset \mathscr{G}(V)$ is closed for all disjoint $I_0, I_1 \subset K_V$, by using Equation (2.6). Finally, suppose that $I_0 \cup I_1$ is infinite; the goal is now to prove that its complement in $(\mathbb{Z}/2\mathbb{Z})^{K_V}$ is not closed in the product topology. To do so, it suffices to produce a sequence $G_n \notin \mathscr{E}(I_0, I_1)$, that converges to a graph $G_0 \in \mathscr{E}(I_0, I_1)$. Thus, fix a countable subset $\{i_n : n \in \mathbb{N}\} \subset I_0 \cup I_1$, and define $G_n := I_1 \Delta \{i_n\}$, $G_0 := I_1 \in \mathscr{E}(I_0, I_1)$. It is easy to check that $G_n \to G_0$ in $\mathscr{G}(V)$, and that this sequence satisfies the desired properties. $\square$

In order to state and prove the next result, the following notation is required.

Definition 2.9. Suppose V is countable. Fix a bijection $\psi : K_V \to \mathbb{N}$ and define $E_n(\psi) := \{e \in K_V : \psi(e) \leq n\}$. Given $a > 1, \epsilon \geq 0$, and $G \in \mathscr{G}(V)$, define $B(G, \epsilon, \|\cdot\|_{\psi,a})$ to be the open ball in $(\mathscr{G}(V), d_{\varphi_a})$ with center G and radius ϵ, and $\overline{B}(G, \epsilon, \|\cdot\|_{\psi,2})$ to be its closure in $(\mathscr{G}(V), d_{\varphi_a})$.

The last preliminary result shows that the sets $\mathscr{E}(I_0, I_1)$ lie in the Borel σ-algebra for finite I_0, I_1.

Proposition 2.10. *Suppose V is countable. Fix a bijection $\psi : K_V \to \mathbb{N}$, and given disjoint subsets $I_0, I_1 \subset \mathbb{N}$, define $\mathscr{E}(I_0, I_1) := \mathscr{E}(\psi^{-1}(I_0), \psi^{-1}(I_1))$.*

(1) *If $I_0 \coprod I_1 = \{1, \dots, n\}$ for some n, then*

$$\begin{aligned}
\mathscr{E}(I_0, I_1) &= B(\psi^{-1}(I_1), 2^{-n}, \|\cdot\|_{\psi,2}) \coprod \{K_V \setminus \psi^{-1}(I_0)\} \\
&= B(\psi^{-1}(I_1), 2^{-n}, \|\cdot\|_{\psi,2}) \bigcup B(K_V \setminus \psi^{-1}(I_0), 2^{-n}, \|\cdot\|_{\psi,2}) \\
&= \overline{B}(\psi^{-1}(I_1), 2^{-n-1}, \|\cdot\|_{\psi,2}) \bigcup \overline{B}(K_V \setminus \psi^{-1}(I_0), 2^{-n-1}, \|\cdot\|_{\psi,2}).
\end{aligned}$$

(2) *For all finite disjoint I_0, I_1, $\mathscr{E}(I_0, I_1)$ is a finite union of open balls, as well as closed balls. Alternatively, it can be partitioned into finitely many open balls and a finite set.*

Proof.

(1) For the first equality, it is clear that the left-hand side is contained in the right. To show the reverse inclusion, if G is in the right-hand side, then $G\Delta\psi^{-1}(I_1)$ cannot intersect $E_n(\psi)$ (which was defined in Definition 2.9), so it must be disjoint from $\psi^{-1}(I_0)$, and must contain $\psi^{-1}(I_1)$. The second equality is now easy to show, and one inclusion in the third equality as well. For the converse, if $G \in \mathscr{E}(I_0, I_1)$, then either $\psi^{-1}(n+1) \in G$ – whence G is in the second closed ball – otherwise G is in the first closed ball.

(2) Suppose $\max(I_0 \coprod I_1) = n$, and $\{1, \dots, n\} \setminus (I_0 \cup I_1) = \{m_1 < \cdots < m_l\}$. Then:
$$\mathscr{E}(I_0, I_1) = \coprod_{J_0 \subset \{m_1, \dots, m_l\}} \mathscr{E}(I_0 \cup J_0, I_1 \cup \{m_1, \dots, m_l\} \setminus J_0).$$
The result now follows from the previous part.

□

Finally, we use the above results to prove the main result in this subsection.

Proof of Theorem 2.4. It is clear by Definition 1.2 that every $G \in \mathscr{G}(V)$ is the limit of a sequence G_n of finite graphs: set $G_n := G \cap E_n(\psi)$. Now $\mathscr{B}_{\mathscr{G}(V)} \subset \Sigma_{d,ball}$; the reverse inclusion is obvious. Also note that the σ-algebras generated by the open and closed d-balls are both equal.

We now claim that some of the inclusion relations hold among the σ-algebras defined above, for arbitrary vertex sets V. Namely, we claim for all sets V:
$$\Sigma_{\text{meas}} = \Sigma_{\mathscr{E},0} \subset \Sigma_{\mathscr{E}} \subset \mathscr{B}_{\mathscr{G}(V)} = \Sigma_0. \tag{2.11}$$
To show Equation (2.11), note that since $\mathscr{G}(V)$ is a compact Hausdorff topological space by Theorem 1.4, hence $K \subset \mathscr{G}(V)$ is compact if and only if $\mathscr{G}(V) \setminus K$ is open. This proves that $\mathscr{B}_{\mathscr{G}(V)} = \Sigma_0$. Also note that $\Sigma_{\mathscr{E},0}$ is generated by all sets $\mathscr{E}(I_0, I_1)$, where we may assume I_0, I_1 to be either finite or countable – that both of these choices yield equivalent σ-algebras follows from repeated applications of Equation (2.6). Next, note that $S_{i_0} = \mathscr{E}(\emptyset, \{i_0\})$ for all $i_0 \in I$. Hence if I_0, I_1 are finite disjoint subsets of K_V, then
$$\mathscr{E}(I_0, I_1) = \bigcap_{i \in I_0} (\mathscr{G}(V) \setminus S_i) \cap \bigcap_{i \in I_1} S_i.$$
This proves that $\Sigma_{\text{meas}} = \Sigma_{\mathscr{E},0}$. Next, that $\Sigma_{\mathscr{E},0} \subset \Sigma_{\mathscr{E}}$ is obvious. Finally, $\Sigma_{\mathscr{E}} \subset \mathscr{B}_{\mathscr{G}(V)}$ by Lemma 2.5, since the sets $\mathscr{E}(I_0, I_1)$ are closed for disjoint $I_0, I_1 \subset K_V$.

Given Equation (2.11), it remains to prove that every open set $\mathcal{U} \subset \mathscr{G}(V)$ is in $\Sigma_{\mathscr{E}}$. First, fix any $G \in \mathcal{U}$ that is not cofinite, i.e., $G \notin \mathscr{G}_1(V)$. Since $\mathcal{U}$ is open, $B(G, \epsilon, \|\cdot\|_{\psi,2}) \subset \mathcal{U}$ for some $\epsilon > 0$. Choose $N > 0$ such that $2^{-N} < \epsilon$, and fix $n > N$ such that $\psi^{-1}(n) \notin G$ (since $G \notin \mathscr{G}_1(V)$). Define the finite graph $G_0 = E_{n-1}(\psi) \cap G \in \mathscr{G}_0(V)$. Then since $n \notin \psi(G)$ and $G \notin \mathscr{G}_1(V)$, hence $\|G - G_0\|_{\psi,2} < 2^{-n} < 2^{-N} < \epsilon$. Thus, $2^{-n} \leq 2^{-N-1} < \epsilon/2$.

Now use the first part of Proposition 2.10 with $I_1 := \{\psi(G_0)\} \subset \{1, \dots, n-1\}$. It follows that $B(G_0, 2^{-n}, \|\cdot\|_{\psi,2}) \in \Sigma_{\mathscr{E}}$. Moreover, $2^{-n} < \epsilon/2$, so
$$G \in B(G_0, 2^{-n}, \|\cdot\|_{\psi,2}) \subset B(G_0, \epsilon/2, \|\cdot\|_{\psi,2}) \subset B(G, \epsilon, \|\cdot\|_{\psi,2}) \subset \mathcal{U}.$$
But now we are done: $\mathcal{U}$ is the union of the countable set $\mathcal{U} \cap \mathscr{G}_1(V)$, and for each $G \in \mathcal{U} \setminus \mathscr{G}_1(V)$, the open ball $B(G_0, 2^{-n}, \|\cdot\|_{\psi,2})$ as above. Since each of these sets is in $\Sigma_{\mathscr{E}}$, and there are only countably many such sets (since they are in bijection with a subset of $\mathscr{G}_0(V) \times \mathbb{N}$), hence $\mathcal{U}$ is a countable union of elements of $\Sigma_{\mathscr{E}}$. Thus, $\mathcal{U} \in \Sigma_{\mathscr{E}}$, whence $\mathscr{B}_{\mathscr{G}(V)} \subset \Sigma_{\mathscr{E}}$, as desired. □

2.2. **Haar measure.** We now define and study a large family of measures μ_P on the space $\mathscr{G}(V)$, eventually focussing on the Haar measure and the proof of the first main result, Theorem A. Given any labelled set V and any function $P : K_V \to [0,1]$, the map $\mu_{P,e}$ assigning $P(e)$ to $\{e\}$ and $1 - P(e)$ to $\emptyset$ is a Bernoulli probability measure on the Bernoulli space $K_{V(e)}$. Now recall the σ-algebra $\Sigma_{\rm meas}$ (2.1), and define the product measure μ_P on finite intersections of these sets via: $\mu_P(S_{e_1} \cap \cdots \cap S_{e_n}) := \prod_{i=1}^n P(e_i)$ for distinct $e_i \in K_V$. One can ask if this information is sufficient to determine μ_P on $(\mathscr{G}(V), \Sigma_{\rm meas})$. To answer this question, recall the following results.

Proposition 2.12 ([JP, Theorem 6.1 and Corollary 6.1])**.** *Suppose a σ-algebra $(\Omega, \mathscr{A})$ is generated by a subset $\mathcal{C} \subset \mathscr{A}$ that is closed under finite intersections.*

1. *Two probability measures on $\mathscr{A}$ are equal if and only if they agree on $\mathcal{C}$.*
2. *Suppose $\mathcal{C}$ is an algebra, and $\mu' : \mathcal{C} \to [0,1]$ is a probability measure (satisfying countable additivity as well as that $\mu'(\Omega) = 1$). Then μ' extends uniquely to a measure on all of $\mathscr{A}$.*

The proof uses the Monotone Class Theorem [JP, Theorem 6.2]. In particular, the result affirmatively answers the above question. Thus μ_P satisfies the following properties when V is countable:

- μ_P is determined uniquely by its restriction to finite intersections of the sets S_{e_0}. In particular, μ_P is a probability measure on $\mathscr{G}(V)$, and one writes: $\mu_P := \prod_{e \in K_V} \mu_{P,e}$.
- For all disjoint I_0, I_1, the sets $\mathscr{E}(I_0, I_1)$ and all Borel sets are μ_P-measurable.
- In particular, every locally constant function on $\mathscr{G}(V) \setminus C$ (where C is a countable set) is μ_P-measurable for all $P : K_V \to [0,1]$.

Remark 2.13. A special case is the measure μ_p for $p \in [0,1]$, given by $P(e) = p \ \forall e$. This is precisely the **Erdös-Rényi model** for $\mathscr{G}(V)$. When V is countable, this construction generalizes the analysis in [Cam], where $\mathscr{G}(V)$ is identified (via a bijection $\psi : K_V \to \mathbb{N}$) with $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$, the space of binary sequences, as well as with $[0,1]$ via the binary expansion of any real number $x \in [0,1]$. Note that this map is precisely the function $\|\cdot\|_{\psi,2}$. Cameron [Cam] also informally writes down a measure on $\mathscr{G}(V)$ as being induced from countably many independent tosses of a fair coin; this is precisely the Erdös-Rényi measure $\mu_{1/2}$ above.

We now outline the contents in the remainder of the paper. The immediate task is to prove the first main result in the paper (Theorem A) using the above preliminary results. Following the proof, in Section 2.3 we compute expectations of several real-valued functions on graph space with respect to the Erdös-Rényi measures μ_p, as an illustration of how to work with these measures. Having computed the Haar-expectations of specific functions, we then prove the other main result, Theorem B (in the following section); this result deals with transporting the Haar-expectation of arbitrary functions between graph space and the real line. The paper concludes with a study of Fourier analysis on $\mathscr{G}(V)$.

Proof of Theorem A. We make the following claim:

For any V (and up to scaling), when restricted to $\Sigma_{\rm meas} = \Sigma_{\mathscr{E},0}$, the Haar measure necessarily equals μ_P with $P(e) = 1/2$ for all $e \in K_V$. In other words, $\mu_{\rm Haar} \equiv \mu_{1/2}$ on $\Sigma_{\rm meas}$.

That $\mu_{\rm Haar}$ exists and is the unique translation-invariant probability measure on $\mathscr{G}(V)$ follows by a classical result of Weil [We] (also proved by Cartan), since $\mathscr{G}(V)$ is a compact topological group. Now suppose $I \subset K_V$ is finite. Then for all partitions $I = I_0 \coprod I_1$, one computes using Equation (2.7):

$$\mu_{\rm Haar}(\mathscr{E}(I_0, I_1)) = \mu_{\rm Haar}(\mathscr{E}(I_0, I_1) + I_1) = \mu_{\rm Haar}(\mathscr{E}(I, \emptyset)),$$

by translation-invariance. Now since $\mathscr{G}(V) = \displaystyle\coprod_{I_0 \subset I} \mathscr{E}(I_0, I \setminus I_0)$, hence

$$\mu_{\rm Haar}(\mathscr{E}(I_0, I_1)) = 2^{-|I|} = \mu_{1/2}(\mathscr{E}(I_0, I_1)). \tag{2.14}$$

Note that the sets $\mathscr{E}(I_0, I_1)$ generate $\Sigma_{\mathscr{E},0}$. Hence $\mu_{\text{Haar}} \equiv \mu_{1/2}$ on $\Sigma_{\mathscr{E},0}$ by Proposition 2.12, since the collection of sets $\mathscr{E}(I_0, I_1)$ (for finite disjoint $I_0, I_1 \subset K_V$) is closed under finite intersections by Equation (2.6).

We now prove the main theorem using the claim. Note that computing μ_{Haar} on Σ_{meas} uniquely determines the Haar measure when V is countable, since $\Sigma_{\mathscr{E},0} = \mathscr{B}_{\mathscr{G}(V)}$ when V is countable (by Theorem 2.4). Thus $\mu_{\text{Haar}} \equiv \mu_{1/2}$.

Next, we assert that the Haar measure of a $\|\cdot\|_{\psi,2}$-ball of radius $\epsilon \in [0,1]$ is ϵ. The assertion is clear for $\epsilon = 0$ (i.e. for the empty set) and $\epsilon = 1$ (which yields the entire space $\mathscr{G}(V)$ except one point), since points must have Haar measure zero or else $\mu_{\text{Haar}}(\mathscr{G}(V)) = \infty$. Now assume $\epsilon \in (0,1)$ and $G = \mathbf{0}$ (by the translation-invariance of $\mu_{1/2}$). It is enough to prove the assertion for ϵ of the form $2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k}$ with $n_1 < n_2 < \cdots < n_k \in \mathbb{N}$, because then given any $\epsilon > 0$, approximate it from below by a nondecreasing sequence $\epsilon_n \to \epsilon^-$ (and from above by a nonincreasing sequence $\epsilon'_n \to \epsilon^+$), with each ϵ_n, ϵ'_n a finite sum of the above form. (For instance, take ϵ_n to be the truncated binary expansions of ϵ.) Then,

$$\begin{aligned} \mu_{1/2}(B(\mathbf{0}, \epsilon, \|\cdot\|_{\psi,2})) \geq \mu_{1/2}\left(\bigcup_{n=1}^{\infty} B(\mathbf{0}, \epsilon_n, \|\cdot\|_{\psi,2})\right) &= \lim_{n\to\infty} \mu_{1/2}(B(\mathbf{0}, \epsilon_n, \|\cdot\|_{\psi,2})) \\ &= \lim_{n\to\infty} \epsilon_n = \epsilon, \end{aligned}$$

and similarly, $\mu_{1/2}(B(\mathbf{0}, \epsilon, \|\cdot\|_{\psi,2})) \leq \lim_{n\to\infty} \epsilon'_n = \epsilon$.

Thus it remains to prove the assertion for $\epsilon = 2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k}$; we do so by induction on $k \geq 0$. For $k = 0$ the result was proved earlier in this proof; from this the result follows for $k = 1$ by using Proposition 2.10 and Equation (2.14). Now given the result for $k - 1 \geq 0$, set $\epsilon' = \sum_{0<i<k} 2^{-n_i}$ and $\epsilon = \epsilon' + 2^{-n_k}$. The graphs in $B(\mathbf{0}, \epsilon, \|\cdot\|_{\psi,2})$ have possible $\|\cdot\|_{\psi,2}$-values in $[0,\epsilon) = [0,\epsilon') \coprod \{\epsilon'\} \coprod (\epsilon', \epsilon)$. Using that the binary expansion is a bijection from $[0,1]$ to binary sequences (except on a countable set described in Theorem 1.4), one notes that the graphs corresponding to the first two sets of $\|\cdot\|_{\psi,2}$-values above are, respectively, $S_{\epsilon'} := B(\mathbf{0}, \epsilon', \|\cdot\|_{\psi,2})$ and the doubleton set $S' := \{T_{k-1}, T_{k-2} \coprod \{n_{k-1}+1, n_{k-1}+2, \dots\}\}$, where $T_k := \{n_1, \dots, n_k\}$ for all k. Moreover, if $\|G\|_{\psi,2} \in (\epsilon', \epsilon)$ for some $G \in \mathscr{G}(V)$, then it is not too hard to show (again using binary expansions, via Theorem 1.4) that from among the integers $1, \dots, n_k$, the only ones in $\psi(G)$ are precisely $n_1, \dots, n_{k-1}$. Thus, the graphs whose $\|\cdot\|_{\psi,2}$-values lie in (ϵ', ϵ) form the set

$$S'' := \mathscr{E}(\{1, \dots, n_k\} \setminus T_{k-1}, T_{k-1}) \setminus \{\{n_1, \dots, n_{k-1}, n_k+1, n_k+2, \dots\}\}.$$

Since points have zero measure, $\mu_{1/2}(S'') = 2^{-n_k}$ by Equation (2.14). Hence:

$$\mu_{1/2}(B(\mathbf{0}, \epsilon, \|\cdot\|_{\psi,2})) = \mu_{1/2}\left(S_{\epsilon'} \coprod S' \coprod S''\right) = \epsilon' + 0 + 2^{-n_k} = \epsilon,$$

where the penultimate equality follows from the induction hypothesis and previous results. This completes the proof for open balls, by induction.

For the closed ball $\overline{B} := \overline{B}(G, \epsilon, \|\cdot\|_{\psi,2})$, if $\epsilon = 1$ then $\mu_{1/2}(\overline{B}) = \mu_{1/2}(\mathscr{G}(V)) = 1$, while if $\epsilon < 1$,

$$B(G, \epsilon, \|\cdot\|_{\psi,2}) \subset \overline{B}(G, \epsilon, \|\cdot\|_{\psi,2}) = \bigcap_{n\in\mathbb{N}} B(G, \epsilon + n^{-1}, \|\cdot\|_{\psi,2}).$$

Thus the result for closed balls follows from the result for open balls, since we have:

$$\epsilon \leq \mu_{1/2}(\overline{B}(G, \epsilon, \|\cdot\|_{\psi,2})) \leq \inf_{(1-\epsilon)^{-1} \leq n \in \mathbb{N}} \epsilon + n^{-1} = \epsilon.$$

□

2.3. **Examples: computing expectations.** We now work out some examples of computing expectations with respect to the probability measures on $\mathscr{G}(V)$ that were introduced in Section 2.2. In the following results, V is assumed to be countable.

Proposition 2.15. *Fix $P : K_V \to [0,1]$ and a bijection $\psi : K_V \to \mathbb{N}$. Then every countable set has μ_P-measure zero if and only if $\prod_{e \in K_V} \max(P(e), 1-P(e)) = 0$. In particular, this holds if there exists $\epsilon > 0$ such that the set $\{e \in K_V : P(e) \in (\epsilon, 1-\epsilon)\}$ is infinite (e.g., if $\mu_P \equiv \mu_p$ for $p \in (0,1)$).*

Proof. Given $G \in \mathscr{G}(V)$ and $n \in \mathbb{N}$, first define

$$p_n(G) := \prod_{e \in E_n(\psi) \cap G} P(e) \prod_{e \in E_n(\psi) \setminus G} (1 - P(e)), \quad \pi_n := \prod_{e \in E_n(\psi)} \max(P(e), 1-P(e)).$$

Note that both $p_n(G)$ and π_n are non-increasing nonnegative sequences, with $p_0(G) = \pi_0 = 1$. In particular, they are both convergent. Moreover, note that

$$\{G\} = \bigcap_{n \in \mathbb{N}} \mathscr{E}(\{1, \dots, n\} \setminus G, G \cap \{1, \dots, n\}) \ \forall G \in \mathscr{G}(V).$$

Now since $\mu_P(\mathscr{E}(I_0, I_1)) = \prod_{i \in I_0} (1 - P(\psi^{-1}(i))) \prod_{i \in I_1} P(\psi^{-1}(i))$ for finite disjoint subsets $I_0, I_1 \subset \mathbb{N}$, we compute:

$$\begin{aligned} 0 \leq \mu_P(G) &= \lim_{n \to \infty} \mu_P(\mathscr{E}(\{1, \dots, n\} \setminus G, G \cap \{1, \dots, n\})) = \lim_{n \to \infty} p_n(G) \\ &\leq \lim_{n \to \infty} \pi_n = \mu_P(G_P), \end{aligned}$$

where G_P (or its set of edges) equals $\{e \in K_V : P(e) \geq 1/2\}$. Thus, every countable set has μ_P-measure zero, if and only if $\mu_P(G) = 0 \ \forall G$, if and only if $\mu_P(G_P) = 0$.

Finally, for the second sub-part, we simply note that infinitely many of the terms in the product are less than $\epsilon < 1$, so $\mu_P(G_P) = 0$. ☐

In order to state the next result, first recall some notation from [KR1].

Definition 2.16 ([KR1])**.** Let V be a countable labelled set.

(1) Define $\ell^1_+(K_V) := \left\{ \varphi : K_V \to (0, \infty) \mid \sum_{e \in K_V} \varphi(e) < \infty \right\}$.
(2) Also define $\ell^\infty_+(K_V)$ to be the set of functions $\zeta : K_V \to (0, \infty)$ such that $\zeta(K_V)$ has precisely one accumulation point: 0 (and ∞ is not an accumulation point, i.e., ζ is bounded).
(3) Further define $\ell^1_\times(K_V) := \{\phi : K_V \to (1, \infty), \ \prod_{e \in K_V} \phi(e) < \infty\}$.
(4) Given $\mathbf{0} \neq G \in \mathscr{G}(V)$, $\varphi \in \ell^1_+(K_V)$, $\zeta \in \ell^\infty_+(K_V)$, and $\phi \in \ell^1_\times(K_V)$, define:

$$\|G\|^1_\varphi := \sum_{e \in G} \varphi(e), \quad \|G\|^\infty_\zeta := \max\{\zeta(e) : e \in G\}, \quad \|\mathbf{0}\|^1_\varphi = \|\mathbf{0}\|^\infty_\zeta = \|\mathbf{0}\|^\times_\phi := 0,$$

and $\|G\|^\times_\phi := \left(\prod_{e \in G} \phi(e) - 1 \right)^{1/n}$, with the smallest $n \in \mathbb{N}$ such that $2^n \geq 1 + \prod_{e \in K_V} \phi(e)$.

It was shown in [KR1] that the maps $\|.\|^1_\varphi, \|.\|^\infty_\zeta$ induce topologically equivalent translation-invariant metrics on graph space $\mathscr{G}(V)$, which metrize its product topology. One can similarly show the following fact.

Lemma 2.17. *For all $\phi \in \ell^1_\times(K_V)$, the maps $\|.\|^\times_\phi$ induce topologically equivalent translation-invariant metrics on $\mathscr{G}(V)$, which metrize its product topology.*

The following result provides examples of computing the expectations of the aforementioned functions (and others) with respect to the Erdös-Rényi type product measures μ_p defined in Section 2.2.

Proposition 2.18. *For all $p \in (0,1)$, the expectation $\mathbb{E}_{\mu_p}$ is a linear functional on the space of measurable functions $h : (\mathscr{G}(V), \mathscr{B}_{\mathscr{G}(V)}) \to \mathbb{R}$.*

1. *Given $k > 0$ and a graph $G \in \mathscr{G}(V)$ with at least k edges, labelled by $\psi(G) = \{n_1 < n_2 < \cdots < n_k < \cdots\} \subset \mathbb{N}$, define the kth minimum edge number of G to be $\Psi_k(G) := n_k$. Then $\mathbb{E}_{\mu_p}[\Psi_k] = k/p$.*
2. *Given $\zeta \in \ell^\infty_+(K_V)$, choose any bijection $\psi_\zeta : K_V \to \mathbb{N}$ as in Lemma 2.19 (below). Then for all $f : \operatorname{im}(\zeta) \to \mathbb{R}$,*
$$\mathbb{E}_{\mu_p}[f(\|.\|^\infty_\zeta)] = \sum_{n \in \mathbb{N}} p(1-p)^{n-1} f(\zeta(\psi_\zeta^{-1}(n))).$$
3. *For all $\varphi \in \ell^1_+(K_V)$ and $G \in \mathscr{G}(V)$, $\|G\|^1_\varphi = \sum_{e \in K_V} \varphi(e) \mathbf{1}_{G \in \mathscr{E}(\emptyset, \psi(e))}$. Moreover,*
$$\mathbb{E}_{\mu_p}[\|.\|^1_\varphi] = p\,\|K_V\|^1_\varphi, \qquad \mathbb{E}_{\mu_p}[(\|.\|^1_\varphi)^2] = p(1-p)\|K_V\|^1_{\varphi^2} + p^2(\|K_V\|^1_\varphi)^2.$$
4. *If $\|K_V\|^\times_\phi \leq \sqrt[n]{2^n - 2}$ for some $\phi \in \ell^1_\times$ and $n > 0$, then*
$$\mathbb{E}_{\mu_p}[(\|.\|^\times_\phi)^n] = -1 + \prod_{e \in K_V} (1 - p + p\phi(e)) \leq (\|K_V\|^\times_\phi)^n.$$

In particular, if $X : \mathscr{G}(V) \to \mathbb{R}$ denotes the random variable $X(G) := \|G\|^1_\varphi$, then X has μ_p-mean $p\,\|K_V\|^1_\varphi$, and variance $p(1-p)\|K_V\|^1_{\varphi^2}$. More generally, one can imitate the proof below to show that for all $n \in \mathbb{N}$, $\mathbb{E}_{\mu_p}[(\|.\|^1_\varphi)^n] = \mathbb{E}_{\mu_p}[X^n]$ equals some "homogeneous" polynomial in $\{\|K_V\|^1_{\varphi^r} = \|K_V\|^r_\varphi : 0 \leq r \leq n\}$, with coefficients that are polynomials in p. (Here, "homogeneous" means that every monomial has the same total degree, with $\|K_V\|^1_{\varphi^r}$ having degree r.)

Also note that some of these results can be shown more generally for all μ_P (with $P : K_V \to [0,1]$ as in Section 2.2). For example, if $\|K_V\|^\times_\phi \leq \sqrt[n]{2^n - 2}$, then
$$\mathbb{E}_{\mu_P}[\|.\|^1_\varphi] = \sum_{e \in K_V} P(e)\varphi(e), \qquad \mathbb{E}_{\mu_P}[(\|.\|^\times_\phi)^n] = -1 + \prod_{e \in K_V} (1 - P(e) + P(e)\phi(e)).$$

The following observation will be used to prove Proposition 2.18.

Lemma 2.19. *For all $\zeta \in \ell^\infty_+(K_V)$, there exists a bijection $\psi = \psi_\zeta : K_V \to \mathbb{N}$, such that $\zeta(\psi^{-1}(1)) \geq \zeta(\psi^{-1}(2)) \geq \cdots$.*

For instance if $\zeta(e) = \|e\|_{\psi,a}$ for fixed $a > 1$ and all $e \in K_V$, then $\zeta(\psi^{-1}(n)) = a^{-n}$, so $\psi_\zeta = \psi$.

Proof. Since the only accumulation point of the image set $\zeta(K_V)$ is 0, it follows that for every $e \in K_V$, there are only finitely many values above $\zeta(e)$ – and they can all be totally ordered. In other words, every subset of $\zeta(K_V)$ has a maximum element. Thus, define $\beta : \mathbb{N} \to K_V$ inductively: $\beta(1)$ is any element of $\arg\max_{e \in K_V} \zeta(e)$, and given $\beta(1), \dots, \beta(k-1)$, define $\beta(k)$ to be any element of the set $\arg\max_{e \in S_k} \zeta(e)$, where $S_k := K_V \setminus \{\beta(1), \dots, \beta(k-1)\}$. It is clear that this inductively covers all $e \in K_V$, by the previous paragraph. Hence $\beta : \mathbb{N} \to K_V$ is a bijection such that $\zeta(\beta(1)) \geq \zeta(\beta(2)) \geq \cdots$. Now define $\psi = \psi_\zeta := \beta^{-1}$. □

Proof of Proposition 2.18.

(1) Note that Ψ_k is defined at all but countably many graphs in $\mathscr{G}(V)$. Moreover, the kth minimum edge of $G \in \mathscr{G}(V)$ is n if and only if $n \geq k$ and $\psi(G) \cap \{1, \dots, n-1\}$ has size exactly $k-1$. This means that $G \in \mathscr{E}(\{1, \dots, n-1\} \setminus S, S \coprod \{n\})$, where $S \subset \{1, \dots, n-1\}$ has size precisely $k-1$. Now there are precisely $\binom{n-1}{k-1}$ such sets S, and by Proposition 2.15, each corresponding set $\mathscr{E}(\{1, \dots, n-1\} \setminus S, S \coprod \{n\})$ has measure $(1-p)^{n-k} p^k$. Hence we exclude the countable (measure zero) set of graphs with fewer than k edges, and compute:

$$\mathbb{E}_{\mu_p}[\Psi_k] = \sum_{n=k}^{\infty} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} = kp^k \sum_{n=k}^{\infty} \binom{n}{k} (1-p)^{n-k}.$$

On the other hand, the Binomial Formula easily yields:

$$\begin{aligned} p^{-(k+1)} &= \sum_{l=0}^{\infty} \binom{-(k+1)}{l} (-(1-p))^l \\ &= \sum_{l=0}^{\infty} (-1)^l \binom{(k+1)+l-1}{l} (-(1-p))^l = \sum_{l=0}^{\infty} \binom{k+l}{k} (1-p)^l. \end{aligned}$$

Setting $l = n-k$, the expected value above equals $kp^k \cdot p^{-(k+1)} = k/p$.

(2) Note that $\zeta(\psi_\zeta^{-1}(1)) \geq \zeta(\psi_\zeta^{-1}(2)) \geq \cdots$ by choice of ψ_ζ. It is clear that $\|G\|_\zeta^\infty = \zeta(\psi_\zeta^{-1}(n))$ if $G \in \mathscr{E}_\zeta(\{1, \dots, n-1\}, \{n\})$, with $\mathscr{E}_\zeta$ denoting the $\mathscr{E}$-set corresponding to ψ_ζ. Since these sets partition $\mathscr{G}(V) \setminus \{\mathbf{0}\}$, use Theorem A to compute:

$$\begin{aligned} \mathbb{E}_{\mu_p}[f(\|.\|_\zeta^\infty)] &= \sum_{n \in \mathbb{N}} f(\zeta(\psi_\zeta^{-1}(n))) \mu_p(\mathscr{E}_\zeta(\{1, \dots, n-1\}, \{n\})) \\ &= \sum_{n \in \mathbb{N}} p(1-p)^{n-1} f(\zeta(\psi_\zeta^{-1}(n))). \end{aligned}$$

(3) Define $f_N(G) := \sum_{n=1}^N \varphi(\psi^{-1}(n)) \mathbf{1}_{G \in \mathscr{E}(\emptyset, \{n\})}$. Thus, $\{f_N\}$ is a nondecreasing sequence of $[0, \infty)$ valued μ-measurable functions on $\mathscr{G}(V)$. It is not hard to show that their pointwise limit at any $G \in \mathscr{G}(V)$ is

$$\sum_{n \in \mathbb{N}} \varphi(\psi^{-1}(n)) \mathbf{1}_{G \in \mathscr{E}(\emptyset, \{n\})} = \sum_{e \in K_V} \varphi(e) \mathbf{1}_{G \in \mathscr{E}(\emptyset, \{\psi(e)\})} \leq \sum_{e \in K_V} \varphi(e) = \|K_V\|_\varphi^1 < \infty.$$

Moreover, in computing $\|G\|_\varphi^1$, $\varphi(e)$ is a summand if and only if $e \in G$, i.e., $G \in \mathscr{E}(\emptyset, \{\psi(e)\})$. This proves the first statement. Now use Proposition 2.15 and the Monotone Convergence Theorem to compute:

$$\begin{aligned} \mathbb{E}_{\mu_p}[\|G\|_\varphi^1] &= \lim_{N \to \infty} \mathbb{E}_{\mu_p}[f_N] = \lim_{N \to \infty} \sum_{n=1}^{N} \varphi(\psi^{-1}(n)) \mu_p(\mathscr{E}(\emptyset, \{n\})) \\ &= \sum_{n=1}^{\infty} \varphi(\psi^{-1}(n)) \cdot p = p \, \|K_V\|_\varphi^1 , \end{aligned}$$

which proves the first part of the second statement. For the second part, note that $(\|G\|^1_\varphi)^2 = \lim_{N\to\infty} f_N^2$. Now write out the summand:

$$f_N^2(G) = \sum_{n=1}^N \varphi(\psi^{-1}(n))^2 \mathbf{1}_{G\in\mathscr{E}(\emptyset,\{n\})} + 2\sum_{n=1}^N \sum_{m=1}^{n-1} \varphi(\psi^{-1}(m))\varphi(\psi^{-1}(n))\mathbf{1}_{G\in\mathscr{E}(\emptyset,\{m\})}\mathbf{1}_{G\in\mathscr{E}(\emptyset,\{n\})}.$$

Taking expectations yields:

$$\begin{aligned}\mathbb{E}_{\mu_p}[f_N^2(G)] &= p\sum_{n=1}^N \varphi(\psi^{-1}(n))^2 + 2p^2 \sum_{n=1}^N \sum_{m=1}^{n-1} \varphi(\psi^{-1}(m))\varphi(\psi^{-1}(n)) \\ &= p(1-p)\sum_{n=1}^N \varphi(\psi^{-1}(n))^2 + \left(p\sum_{n=1}^N \varphi(\psi^{-1}(n))\right)^2.\end{aligned}$$

From above computations, the second term is just $\mathbb{E}_{\mu_p}[f_N]^2$, which converges to $(p\,\|K_V\|^1_\varphi)^2$ by the first part. Hence as above, using Proposition 2.15 and the Monotone Convergence Theorem, $\mathbb{E}_{\mu_p}[(\|G\|^1_\varphi)^2]$ equals $\lim_{N\to\infty} \mathbb{E}_{\mu_p}[f_N^2(G)] = p(1-p)\|K_V\|^1_{\varphi^2} + p^2(\|K_V\|^1_\varphi)^2$.

(4) This is similar to the previous part: define

$$f_N(G) := \prod_{n=1}^N \left(1 + (\phi(\psi^{-1}(n)) - 1)\mathbf{1}_{G\in\mathscr{E}(\emptyset,\{n\})}\right).$$

Once again, $0 \leq f_N(G) \leq f_{N+1}(G) \leq \prod_{e\in K_V} \phi(e) < \infty$ for all G, so we can apply the Monotone Convergence Theorem. Moreover, the pointwise limit of the f_N is precisely $1+(\|.\|^\times_\phi)^n$, and one easily checks that the expectation of any product of k distinct indicators as above is p^k. This proves that $\mathbb{E}_{\mu_p}[1 + (\|.\|^\times_\phi)^n] = \lim_{N\to\infty} \prod_{n=1}^N (1 + p(\phi(\psi^{-1}(n)) - 1))$, and the result follows.

□

3. Haar integration and Fourier analysis

We now study graph space $\mathscr{G}(V)$ in further detail. Recall that $\mathscr{G}(V)$ is a compact topological group; these are objects for which a comprehensive theory of analysis and probability has been systematically developed in the literature – see e.g. [Gre, Pa, Ru]. In this section we further explore two aspects of the theory: first, we find a more familiar model for graph space as a compact group with Haar measure. Second, we study Fourier analysis on $\mathscr{G}(V)$. This includes classifying the Pontryagin dual, as well as all positive definite functions on $\mathscr{G}(V)$.

3.1. **Haar integration on graph space.** In this part we study the relationship between the Haar measure on $\mathscr{G}(V)$ and the Lebesgue measure on $\mathbb{R}$. As seen above, the Haar measure of an ϵ-ball is ϵ. This property also holds (up to scaling by 2) for the usual Lebesgue measure $\mu_{\mathbb{R}}$ on the real line. Thus, it is natural to ask if the two measure spaces $(\mathscr{G}(V), \mu_{\text{Haar}} = \mu_{1/2})$, and $(\mathbb{R}, \mu_{\mathbb{R}})$ – or more precisely, the circle group S^1 with its Haar measure – are related. If so, how does one account for the "Jacobian" in transforming Haar integration from $\mathscr{G}(V)$ into the usual Lebesgue theory on $\mathbb{R}$? These questions are the focus of the next main result in the paper.

Proof of Theorem B. Note that $\|\cdot\|_{\psi,2}$ is a continuous map $: \mathscr{G}(V) \to \mathbb{R}$, hence (Borel) measurable with respect to the respective Borel σ-algebras. Thus, consider two measures on the Borel σ-algebra $\mathscr{B}_{\mathscr{G}(V)}$, given by $A \mapsto \mu_{1/2}(A)$ and $A \mapsto \mu_{\mathbb{R}}(\|A\|_{\psi,2})$. The latter is indeed a measure that satisfies countable additivity because $\|\cdot\|_{\psi,2}$ is a bijection except on a countable set, and countable sets have measure zero in either measure.

We now claim that the measures $\mu_{1/2}$ and $\mu_{\mathbb{R}} \circ \|\cdot\|_{\psi,2}$ on $(\mathscr{G}(V), \mathscr{B}_{\mathscr{G}(V)})$ agree on all sets $\mathscr{E}(I_0, I_1)$ for finite disjoint $I_0, I_1 \subset \mathbb{N}$. If this holds, then since these sets are closed under finite intersections, Proposition 2.12 implies that these measures are identical (using Theorem 2.4). This proves the first assertion.

To prove the claim, recall from [Cam] that $([0,1], \mu_{\mathbb{R}})$ is equivalent – via binary expansion – to the countable sequences of independent tosses of a fair coin. Moreover, countable sets have probability zero in all cases, and the set $\mathscr{E}(I_0, I_1)$ corresponds precisely to all sequences where the ith coin toss is a tail if $i \in I_0$, and a head if $i \in I_1$. In turn, these correspond to the set of all $x \in [0,1]$ whose ith digit in the binary expansion is 0 if $i \in I_0$ and 1 if $i \in I_1$ – and these sets are measurable because they are unions of intervals. It is now clear, by partitioning $[0,1]$ into $2^{\max(I_0 \cup I_1)}$-many intervals of equal length, that each of these sets has measure $2^{-|I_0 \cup I_1|}$, which proves the claim.

We now show the second assertion. Note that $f(x) = \pm\infty$ only on a set of measure zero, since f is Lebesgue integrable. Next, the functions $f^{\pm} := \max(\pm f, 0)$ are also measurable (and integrable) if f is; hence by linearity it suffices to prove the result for each of them. Thus, suppose without loss of generality that $0 \leq f < \infty$. We carry out a standard construction to approximate f by a sequence of nonnegative simple functions $0 \leq f_1 \leq f_2 \leq \cdots$ on $[0,1]$, which converge pointwise to f almost everywhere. Given $n \in \mathbb{N}$, define $I_{n,k} := \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ for $1 \leq k \leq 2^{2n}$, and $I_{n,2^{2n}+1} := [2^n, \infty)$. Now define $A_{n,k} := f^{-1}(I_{n,k}) \subset [0,1]$, and $B_{n,k} := \{G \in \mathscr{G}(V) : f(\|G\|_{\psi,2}) \in I_{n,k}\}$. Thus by Theorem B, both $A_{n,k}$ and $B_{n,k}$ are measurable and of equal measures. Now define the functions

$$f_n := \sum_{k=1}^{2^{2n}+1} \frac{k-1}{2^n} \mathbf{1}_{A_{n,k}}, \qquad g_n := \sum_{k=1}^{2^{2n}+1} \frac{k-1}{2^n} \mathbf{1}_{B_{n,k}}.$$

It is then standard that $0 \leq f_n \leq f_{n+1} \leq f$ at each point, and $f_n(x) \to f(x)$ for all x. The same facts also hold for g_n and g, where we define: $g(G) := f(\|G\|_{\psi,2})$ and $g_n(G) := f_n(\|G\|_{\psi,2})$. Moreover, since $\|\cdot\|_{\psi,2}$ is measure-preserving, the simple functions f_n, g_n are pullbacks of each other (via the invertible map $\|\cdot\|_{\psi,2}$ and its inverse – outside the countable set $\mathscr{G}_0(V)$, say). Hence,

$$\mathbb{E}_\mu[g_n] = \int_{\mathscr{G}(V)} g_n(G)\ d\mu_{1/2} = \int_0^1 f_n(x)\ dx = \int_0^1 f_n\ d\mu_{\mathbb{R}}\ \forall n \in \mathbb{N}.$$

Now use the Monotone Convergence Theorem twice:

$$\mathbb{E}_{\mu_{1/2}}[f(\|\cdot\|_{\psi,2})] = \mathbb{E}_{\mu_{1/2}}[g] = \lim_{n\to\infty} \mathbb{E}_{\mu_{1/2}}[g_n] = \lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \int_0^1 f(x)\ dx.$$

□

Remark 3.1. Note that $\|\cdot\|_{\psi,2}$ is not a bijection on $\mathscr{G}(V)$; thus to make sense of $(\|\cdot\|_{\psi,2})^{-1}$, ignore all finite graphs $\mathscr{G}_0(V)$, and/or cofinite graphs $\mathscr{G}_1(V)$ (since all countable sets have measure zero). Then $\|\cdot\|_{\psi,2}$ is a measure-preserving bijection on the complement, whence a random variable $T|_{\mathscr{G}(V)\setminus\mathscr{G}_0(V)}$ is measurable if and only if $T \circ (\|\cdot\|_{\psi,2})^{-1} : [0,1] \setminus \|\mathscr{G}_0(V)\|_{\psi,2} \to \mathscr{G}(V) \setminus \mathscr{G}_0(V) \to X$ is measurable.

Remark 3.2. We now illustrate an application of Theorem B. Recall the definition of the "kth minimum edge number" as defined in Proposition 2.18(1). One can now show that

$$\Psi_1(G) = \min \psi(G) := -\lfloor \log_2 \|G\|_{\psi,2} \rfloor.$$

More generally, $\Psi_k(G)$ can be inductively defined as $f_k(\|G\|_{\psi,2})$, where:

$$f_k(x) := f_1\left(x - \sum_{i=1}^{k-1} 2^{-f_i(x)}\right), \qquad f_1(x) := -\lfloor \log_2 x \rfloor.$$

In particular, one can compute the expected value of Ψ_1 (with respect to $\mu_{1/2}$) using Theorem B, to be $-\int_0^1 \lfloor \log_2(x) \rfloor \, dx$, which can be shown to converge to 2. Recall that this expectation was also computed in Proposition 2.18(1).

3.2. **Pontryagin duality and Walsh-Rademacher functions.** We conclude the paper with a discussion of Fourier analysis in graph space. We require the following terminology.

Definition 3.3. Suppose V is any fixed vertex set.

(1) A function $f : \mathscr{G}(V) \to \mathbb{C}$ is *positive definite* if for all integers $n \in \mathbb{N}$ and $G_1, \dots, G_n \in \mathscr{G}(V)$, the matrix $(f(G_i \Delta G_j))_{1 \leq i,j \leq n}$ is positive semidefinite.
(2) Given a finite graph $E \in \mathscr{G}_0(V)$, the corresponding *Walsh function* $\chi_E : \mathscr{G}(V) \to \mathbb{C}$ is defined as follows:
$$\chi_E(G) := (-1)^{|E \cap G|} = \prod_{e \in E} \chi_{\{e\}}(G).$$

The Walsh functions turn out to be important for several reasons, including for Fourier analysis via Pontryagin duality. Recall the following terminology:

Definition 3.4. A unitary character of a group $\mathscr{G}$ is a group homomorphism $\chi : \mathscr{G} \to S^1$, the unit circle in $\mathbb{C}^\times$. The *Pontryagin dual* of a locally compact abelian group is simply the set of continuous unitary characters, which form a group under pointwise multiplication.

Note for $\mathscr{G} = \mathscr{G}(V)$ that all unitary characters have image in $\{\pm 1\}$, since $\mathscr{G}(V)$ is a group with exponent 2. Now the following result completely characterizes all positive definite functions on $\mathscr{G}(V)$, as well as its Pontryagin dual.

Theorem 3.5. *Suppose V is a countable set. Then the Pontryagin dual to $\mathscr{G}(V)$ is naturally identified with its subgroup $\mathscr{G}_0(V)$ of finite graphs, via Walsh functions. They also form an orthonormal basis of $L^2(\mathscr{G}(V), \mathbb{R})$.*

Moreover, a function $f : \mathscr{G}(V) \to \mathbb{C}$ is positive definite and satisfies $f(\mathbf{0}) = 1$, if and only if there exists a probability measure μ on $\mathscr{G}_0(V)$ (i.e., a countable set of nonnegative numbers $\mu(H)$ that add up to 1), such that
$$f(G) = \sum_{H \in \mathscr{G}_0(V)} (-1)^{|G \cap H|} \mu(H).$$
(In particular, f has image in $[-1, 1]$.)

Since $\mathscr{G}(V)$ is a compact abelian group, one can also apply the theory of Pontryagin duality to carry out Fourier analysis on it, or to state Parseval's identity and Plancherel's theorem (a useful reference is [Ru, Chapter 1]). We now write down some of the results in this setting.

Proposition 3.6. *Suppose V is an arbitrary set (of labelled vertices).*

(1) *The "group algebra" $L^1(\mathscr{G}(V), \mathbb{R})$ is a Banach algebra under convolution.*
(2) *The set of Walsh functions $\{\chi_E : E \in \mathscr{G}_0(V)\}$ is an orthonormal subset of $L^2(\mathscr{G}(V), \mathbb{R})$. When V is countable, the Walsh functions form a complete/Hilbert basis; moreover, they transform into the usual Walsh functions – i.e., products of* Rademacher functions *– via the Haar measure-preserving map $\|\cdot\|_{\psi,2}$.*

Note that the first part follows from [Ru, Theorem 1.1.7], and the second part follows from Theorem B, since the Walsh functions form a complete orthonormal system in $L^2([0,1],\mathbb{R})$. Moreover, they comprise the Pontryagin dual of $\mathscr{G}(V)$:

Theorem 3.7. *Suppose $\mathscr{G}^\wedge$ is the Pontryagin dual group to $\mathscr{G}(V)$ for a set V.*

(1) *For all finite sets $E \in \mathscr{G}_0(V)$, we have $\chi_E \in \mathscr{G}^\wedge$, with image in $\{\pm 1\}$.*
(2) *$\mathscr{G}^\wedge$ is a discrete (locally compact) abelian group, which is metrizable if V is countable.*
(3) (Plancherel's Theorem.) *The Fourier transform, when restricted to $(L^1 \cap L^2)(\mathscr{G}(V))$, is a linear isometry in the L^2-metric, onto a dense subset of $L^2(\mathscr{G}^\wedge)$. Hence it has a unique extension to a unitary operator from $L^2(\mathscr{G}(V))$ onto $L^2(\mathscr{G}^\wedge)$ (for some Haar measure $\mu^\wedge$ on $\mathscr{G}^\wedge$).*
(4) *When V is countable, $\mathscr{G}^\wedge$ is precisely the set of Walsh functions, and the assignment $\chi_E \mapsto E := \{e \in K_V : \chi_E(\{e\}) = -1\}$ is a group isomorphism onto $(\mathscr{G}_0(V), \Delta)$.*

Proof. To show (1), one shows that $\chi_{\{e\}}$ is continuous for each $e \in K_V$:

$$\chi_{\{e\}}^{-1}(-1) = \mathscr{E}(\emptyset, \{e\}), \qquad \chi_{\{e\}}^{-1}(1) = \mathscr{E}(\{e\}, \emptyset),$$

and these are both open by Proposition 2.10. Note that by [Ru, Theorem 1.2.5], $\mathscr{G}^\wedge$ is discrete since $\mathscr{G}(V)$ is compact. Similarly, $\mathscr{G}^\wedge$ is metrizable since $\mathscr{G}(V)$ is separable. This proves (2). Part (3) is shown (for more general $\mathscr{G}$) in [Ru, Theorem 1.6.1]. Part (4) is also not hard to show – see e.g. [Fi, Wa]. $\square$

Finally, we prove the remaining unproved result above.

Proof of Theorem 3.5. All but the last assertion follow from Proposition 3.6 and Theorem 3.7. To prove the last part, apply Bochner's Theorem [Ru, Theorem 1.4.3] to the compact abelian group $\mathscr{G}(V)$. Thus, every normalized function f is of the form $f(G) = \displaystyle\int_{\xi \in \mathscr{G}^\wedge} \xi(G)\ d\mu(\xi)$ for some probability measure μ on $\mathscr{G}^\wedge$. Since $\mathscr{G}^\wedge \cong \mathscr{G}_0(V)$ from above, every measure is a countable tuple as claimed. $\square$

Concluding remarks. In this paper we analyzed measures and integration on labelled graph space $\mathscr{G}(V)$. We showed that $\mathscr{G}(V)$ with its Haar measure is very closely related to the circle with its Haar measure, which allowed us to transport Haar-Lebesgue integration on $[0,1]$ over to graph space $\mathscr{G}(V)$.

A more involved task is to study random graphs – i.e., sequences of $\mathscr{G}(V)$-valued random variables. This involves the analysis of measurable functions from a probability space into $\mathscr{G}(V)$ (as opposed to real-valued functions of graphs studied in this paper). Note that graph space $\mathscr{G}(V)$ is a 2-torsion group, and hence does not embed as a group into a normed linear space. Thus the next step in the study of labelled graphs and their limits involves developing the foundations of probability theory on $\mathscr{G}(V)$, and studying probability inequalities and stochastic convergence on random graphs. The study of probability theory on graph space is addressed in recent work [KR2]. Such a formalism is essential in order to discuss issues like probability generating mechanisms for graphs, or to sample from graph space.

Acknowledgments. We thank David Montague and Doug Sparks for useful discussions.

References

[Cam] P.J. Cameron, *Cyclic automorphisms of a countable graph and random sum-free sets*, Graphs and Combinatorics **1** (1985), no. 1, 129–135.

[DGKR] P. Diao, D. Guillot, A. Khare, and B. Rajaratnam, *Differential calculus on graphon space*, Journal of Combinatorial Theory Series A **133** (2015), 183–227.

[Fi] N.J. Fine, *On the Walsh functions*, Transactions of the American Mathematical Society **65** (1949), no. 3, 372–414.

[Gre] U. Grenander, *Probabilities on algebraic structures*, Almqvist & Wiksell, and John Wiley & Sons Inc., Stockholm and New York, 1963.

[JP] J. Jacod and P. Protter, *Probability Essentials*, Universitext, Springer-Verlag, Berlin-New York, 2004.

[KR1] A. Khare and B. Rajaratnam, *Differential calculus on the space of countable labelled graphs*, Technical Report, Departments of Mathematics and Statistics, Stanford University, submitted (arXiv:1410.6214), 2014.

[KR2] A. Khare and B. Rajaratnam, *Probability inequalities and tail estimates for metric semigroups*, Technical Report, Departments of Mathematics and Statistics, Stanford University, submitted (arXiv:1506.02605), 2015.

[Lo] L. Lovász, *Large networks and graph limits*, Colloquium Publications **60**, American Mathematical Society, Providence, RI, 2012.

[Pa] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Probability and Mathematical Statistics series of Monographs and Textbooks, Academic Press, New York, 1967.

[Ru] W. Rudin, *Fourier analysis on groups*, Interscience, John Wiley & Sons, New York-London, 1962.

[Wa] J.L. Walsh, *A closed set of normal orthogonal functions*, American Journal of Mathematics **55** (1923), 5–24.

[We] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, Hermann, 1940.

Department of Mathematics, Stanford University, Stanford, CA - 94305, USA
E-mail address: A.K.: khare@stanford.edu; B.R.: brajarat@stanford.edu