ON FRACTIONAL HADAMARD POWERS OF POSITIVE BLOCK MATRICES

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Abstract. Entrywise powers of matrices have been well-studied in the literature, and have recently received renewed attention due to their application in the regularization of high-dimensional correlation matrices. In this paper, we study powers of positive semidefinite block matrices $(H_{st})_{s,t=1}^n$ where each block H_{st} is a complex $m \times m$ matrix. We first characterize the powers $\alpha \in \mathbb{R}$ such that the blockwise power map $(H_{st}) \mapsto (H_{st}^\alpha)$ preserves Loewner positivity. The characterization is obtained by exploiting connections with the theory of matrix monotone functions which was developed by C. Loewner. Second, we revisit previous work by D. Choudhury [Proc. Amer. Math. Soc. 108] who had provided a lower bound on α for preserving positivity when the blocks H_{st} pairwise commute. We completely settle this problem by characterizing the full set of powers preserving positivity in this setting. Our characterizations generalize previous results by FitzGerald-Horn, Bhatia-Elsner, and Hiai from scalars to arbitrary block size, and in particular, generalize the Schur Product Theorem. Finally, a natural and unifying framework for studying the cases where the blocks H_{st} are diagonalizable consists of replacing real powers by general characters of the complex plane. We thus classify such characters, and generalize our results to this more general setting. In the course of our work, given $\beta \in \mathbb{Z}$, we provide lower and upper bounds for the threshold power $\alpha > 0$ above which the complex characters $z = re^{i\theta} \mapsto r^\alpha e^{i\beta\theta}$ preserve positivity when applied entrywise to Hermitian positive semidefinite matrices. In particular, we completely resolve the $n = 3$ case of a question raised in 2001 by Xingzhi Zhan. As an application of our results, we also extend previous work by de Pillis [Duke Math. J. 36] by classifying the characters K of the complex plane for which the map $(H_{st})_{s,t=1}^n \mapsto (K(\mathrm{tr}(H_{st})))_{s,t=1}^n$ preserves Loewner positivity.

1. Introduction

The study of positive definite matrices and of functions that preserve them arises naturally in many branches of mathematics and other disciplines. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a matrix $A = (a_{st})$, the matrix $f[A] := (f(a_{st}))$ is obtained by applying f to the entries of A. Such mappings are called *entrywise* or *Hadamard functions* (see [22, §6.3]). Entrywise functions preserving Loewner positivity have been widely studied in the literature (see e.g. Schoenberg [33], Rudin [32], Herz [19], Horn [21], Christensen and Ressel [5], Vasudeva [36], FitzGerald, Micchelli, and Pinkus [9], Hiai [20]). The subject has recently received renewed attention due to its importance in the regularization of high-dimensional covariance/correlation matrices [13, 12, 17, 18, 27, 38]. An important family of functions is the set of power functions $f(x) = x^\alpha$ for $\alpha > 0$. Characterizing the entrywise powers that preserve positivity is a classical problem that has been well-studied in the literature and is now completely resolved (see [8, 3, 20, 10]). A natural generalization of this problem consists of studying powers preserving positivity when applied to block matrices (see e.g. [4, 14, 28]). More precisely, let $H := (H_{st})_{s,t=1}^n$ be an $mn \times mn$ Hermitian positive semidefinite matrix, where each block H_{st} is an $m \times m$ Hermitian positive semidefinite matrix. Our first main result in this paper is a complete characterization of the powers α such that the matrix $(H_{st}^\alpha)_{s,t=1}^n$ is always positive semidefinite. Here, the power H_{st}^α is computed using the spectral decomposition of H_{st}. Note that when each block of H is 1×1, the problem reduces

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to the classical problem of characterizing entrywise powers preserving positivity. In contrast, when H consists of only one block, every power trivially preserves positive semidefiniteness. Surprisingly, we demonstrate that except in trivial cases, powers do not preserve positivity when the block size is 2 or more. This sharply contrasts the classical case where all powers preserve positivity beyond a certain *critical exponent* (see e.g. [8, 24]).

In a previous paper, Choudhury [4] has studied powers $\alpha > 0$ such that the map $(H_{st}) \mapsto (H_{st}^{\alpha})$ preserves Loewner positivity, under the additional assumption that the blocks H_{st} pairwise commute. She demonstrates that every power $\alpha \in \mathbb{N} \cup [mn-2, \infty)$ preserves Loewner positivity. However, it is not clear if the bound $mn - 2$ is sharp, nor which smaller non-integer powers preserve positivity. In our second main result, we completely answer these questions by showing that the set of powers preserving positivity when the blocks commute is exactly $\mathbb{N} \cup [n-2)$. In contrast to previous results, the answer turns out to be independent of the block size m. Our result therefore shows that positivity is actually retained at a much lower threshold (critical exponent) than was previously thought. We then extend this characterization to commuting Hermitian blocks that are not necessarily positive semidefinite, by considering the odd and even extensions of the power functions. Our characterization extends previous work by FitzGerald and Horn [8], Bhatia and Elsner [3], Hiai [20], and Guillot, Khare, and Rajaratnam [10].

When studying powers of block matrices, one has to assume the blocks H_{st} are positive semidefinite for the powers H_{st}^{α} to be well-defined. When the blocks are only Hermitian, it is natural to replace the power functions by their odd or even extensions to $\mathbb{R}$ (see Hiai [20]). Note that these functions are precisely the Lebesgue measurable multiplicative functions on $\mathbb{R}$ (see e.g. [11]). More generally, when the blocks H_{st} are only diagonalizable, it is natural to replace the power functions by general Lebesgue measurable multiplicative functions on $\mathbb{C}$. Considering such multiplicative functions provides a general and systematic framework in which to study powers preserving Loewner positivity, either in the block case, the commuting block case, or the traditional scalar setting studied by FitzGerald and Horn, Bhatia and Elsner, and Hiai. Thus, in Section 3, we classify all measurable multiplicative functions on $\mathbb{C}$ that preserve $[0, \infty)$, and identify a natural two-parameter family of functions $\{\Psi_{\alpha,\beta} : \alpha \in \mathbb{R}, \beta \in \mathbb{Z}\}$ that is used throughout the paper to generalize the power functions. Next, in Section 4 we characterize which of these functions preserve Loewner positivity when applied blockwise to Hermitian positive semidefinite matrices $(H_{st})_{s,t=1}^n$. In Section 5, we consider the case where the blocks H_{st} pairwise commute, and complete the characterization initiated by D. Choudhury in [4]. We also demonstrate how our work can be used to generalize previous work by de Pillis [6], by characterizing the functions $\Psi_{\alpha,\beta}$ for which the map $(H_{st})_{s,t=1}^n \mapsto (\Psi_{\alpha,\beta}(\mathrm{tr}(H_{st})))_{s,t=1}^n$ preserves Loewner positivity.

Finally, in Section 6, we consider the traditional setting where each block is 1×1. For all integers $\beta \in \mathbb{Z}$ and $n \in \mathbb{N}$, we provide lower and upper bounds for the threshold power $\alpha > 0$ above which $\Psi_{\alpha,\beta}[-]$ preserves Loewner positivity on $n \times n$ Hermitian positive semidefinite matrices. In particular, when $\beta = 1$, we completely resolve the $n = 3$ case of a question raised in 2001 by Xingzhi Zhan [20, Acknowledgment Section], concerning the powers $\alpha > 0$ for which $\Psi_{\alpha,1}[-]$ preserves Loewner positivity. Moreover, we study the same problem for arbitrary β, which had not been previously done in the literature.

Notation: Given a subset $S \subset \mathbb{C}$, denote by $\mathbb{P}_n(S)$ the set of $n \times n$ Hermitian positive semidefinite matrices with entries in S. We denote the complex disc centered at $a \in \mathbb{C}$ and of radius $R > 0$ by $D(a, R)$. We write $A \geq 0$ to denote that $A \in \mathbb{P}_n(\mathbb{C})$, and write $A \geq B$ when $A - B \in \mathbb{P}_n(\mathbb{C})$. We denote by I_n the $n \times n$ identity matrix, and by $\mathbf{0}_{n\times n}$ and $\mathbf{1}_{n\times n}$ the $n \times n$ matrices with every entry equal to 0 and 1 respectively. Finally, we denote the conjugate transpose of a vector or matrix A by A^*.

2. Literature review

Entrywise powers and their properties have been studied by many authors including Horn and FitzGerald [8], Bhatia and Elsner [3], Hiai [20], and Guillot, Khare, and Rajaratnam [10]. Most of the known results concern matrices with blocks of dimension 1×1. We now review two of the most important results in the area.

Theorem 2.1 (FitzGerald and Horn, [8, Theorem 2.2])**.** *Suppose $A = (a_{st}) \in \mathbb{P}_n((0,\infty))$ for some $n \geq 2$. Then $A^{\circ\alpha} := (a_{st}^\alpha) \in \mathbb{P}_n$ for all $\alpha \in \mathbb{N} \cup [n-2,\infty)$. If $\alpha \in (0, n-2)$ is not an integer, then there exists $A \in \mathbb{P}_n((0,\infty))$ such that $A^{\circ\alpha} \notin \mathbb{P}_n$. More precisely, Loewner positivity is not preserved for $A = ((1+\epsilon st))_{s,t=1}^n$, for all sufficiently small $\epsilon = \epsilon(\alpha, n) > 0$ for $\alpha \in (0, n-2) \setminus \mathbb{N}$.*

Note that in Theorem 2.1, the entries of the matrix A are assumed to be positive for the power x^α to be well-defined. In practice, one also commonly encounters matrices with negative and complex entries. In order to work with matrices with real entries, the papers [3, 20] considered the odd and even extensions of the power functions to the real line.

Definition 2.2. Let $\alpha \in \mathbb{R}$. We define the even and odd extensions to $\mathbb{R}$ of the power function $x \mapsto x^\alpha$ via:

$$\phi_\alpha(x) := |x|^\alpha, \qquad \psi_\alpha(x) := \operatorname{sgn}(x)|x|^\alpha, \qquad \forall x \neq 0, \tag{2.1}$$

and $\phi_\alpha(0) = \psi_\alpha(0) := 0$. Also define $f_\alpha(x) := x^\alpha$ for $x > 0$, and $f_\alpha(0) := 0$.

Note that the definitions of ϕ_α, ψ_α given above are natural, as they yield the unique even and odd multiplicative extensions to $\mathbb{R}$ of the standard power functions. The following result completely characterizes the powers α such that ϕ_α or ψ_α preserves Loewner positivity when applied entrywise. The reader is referred to [10] for a proof and history of this result.

Theorem 2.3 (Bhatia and Elsner [3], Hiai [20], Guillot, Khare, and Rajaratnam [10])**.** *Let $\alpha \in \mathbb{R}$ and let $n \geq 2$. Then*

1. *$\phi_\alpha[A] \in \mathbb{P}_n(\mathbb{R})$ for all $A \in \mathbb{P}_n(\mathbb{R})$ if and only if $\alpha \in 2\mathbb{N} \cup [n-2,\infty)$.*
2. *$\psi_\alpha[A] \in \mathbb{P}_n(\mathbb{R})$ for all $A \in \mathbb{P}_n(\mathbb{R})$ if and only if $\alpha \in (-1+2\mathbb{N}) \cup [n-2,\infty)$.*

Moreover, if $f = \phi_\alpha$ or $f = \psi_\alpha$ does not preserve positivity on $\mathbb{P}_n(\mathbb{R})$ for some $\alpha \in \mathbb{R}$, there exists a rank 2 matrix $A \in \mathbb{P}_n(\mathbb{R})$ such that $f[A] \notin \mathbb{P}_n(\mathbb{R})$.

Blockwise powers yield a generalization of the entrywise powers analysis studied above. We now recall a sufficient condition for preserving positivity that was shown in [4] in the case where $H = (H_{st})$ is a block matrix with commuting blocks H_{st}.

Theorem 2.4 (Choudhury, [4, Theorem 5])**.** *Let $H = (H_{st})$ be a given positive semidefinite $mn \times mn$ matrix, where $\{H_{st} : 1 \leq s,t \leq n\}$ are a commuting family of normal $m \times m$ matrices. If H is positive semidefinite, then so is (H_{st}^α) for all $\alpha \in \mathbb{N}$. If in addition each H_{st} is positive semidefinite, then (H_{st}^α) is positive semidefinite for all real $\alpha \geq mn-2$.*

In Section 4 we completely characterize the powers α that preserve positivity when the blocks do not necessarily commute. We then show in Section 5 that the bound $\alpha \geq mn-2$ in Theorem 2.4 is not sharp and that the optimal bound is $\alpha \geq n-2$. Moreover, we will demonstrate how Theorem 2.4 can be naturally extended to blocks H_{st} that are diagonalizable.

3. Preliminaries and main results

Before we proceed to characterize functions preserving Loewner positivity for block matrices, we provide a framework in which to work with powers of complex matrices. In order to do so, first note that the functions ϕ_α and ψ_α defined in Section 2 are in fact the unique non-constant Lebesgue measurable multiplicative functions on $\mathbb{R}$ (see e.g. [11]). Since we work with complex matrices in the present paper, it is natural to first classify the multiplicative maps on the complex plane under mild measurability assumptions. Such a classification has been achieved in related work [11].

3.1. **Multiplicative maps on the complex plane.** Given $\alpha, \beta \in \mathbb{R}$, define $\Psi_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$ by:

$$\Psi_{\alpha,\beta}(r\exp(i\theta)) := r^\alpha \exp(i\beta\theta) \ \forall r > 0, \theta \in (-\pi, \pi], \qquad \Psi_{\alpha,\beta}(0) := 0. \tag{3.1}$$

When $\beta \in \mathbb{Z}$, the maps $\Psi_{\alpha,\beta}$ are multiplicative on $\mathbb{C}$ and continuous on the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Moreover, $(\alpha, \beta) \mapsto \Psi_{\alpha,\beta}$ is a monoid homomorphism from the additive group $(\mathbb{R} \times \mathbb{Z}, +)$ to the monoid of multiplicative maps on $\mathbb{C}$ (under pointwise multiplication). The following lemma shows that the functions $\Psi_{\alpha,\beta}$ for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$ are in fact the only non-constant multiplicative functions from $\mathbb{C}$ to $\mathbb{C}$ that 1) are continuous on S^1, 2) map the positive real axis into itself (needed to preserve Loewner positivity), and 3) satisfy natural measurability conditions.

Lemma 3.1. *Given $R \in (1, \infty]$ and $K : D(0, R) \to \mathbb{C}$, the following are equivalent.*

1. *K is multiplicative on $D(0, R)$, continuous on $S^1 \subset D(0, R)$, sends $\widetilde{I} := (0, R)$ to $\mathbb{R}$, and is Lebesgue measurable on some subinterval $I \subset \widetilde{I}$ which contains 1.*
2. *Either $K \equiv 0$ or $K \equiv 1$ on $D(0, R)$, or there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}$ such that $K \equiv \Psi_{\alpha,\beta}$.*

Moreover, the maps $\{\Psi_{\alpha,\beta} : \alpha \in \mathbb{R}, \beta \in \mathbb{Z}\} \cup \{K \equiv 1\}$ are linearly independent as functions on $D(0, r)$ for any $0 < r \leq \infty$.

Proof of Lemma 3.1. Note that $K : S^1 \to \mathbb{C}$ is multiplicative and continuous, hence a character. Therefore $K : D(0, R) \to \mathbb{C}$ is multiplicative and conjugation-equivariant. The result now follows from [11, Theorem 8]. $\square$

3.2. **Main results.** Before stating the main results of the paper, we introduce some notation. Let $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$. Given a complex diagonalizable matrix A with eigen-decomposition $A = P^{-1}DP$ and spectrum contained in S, we denote by $f(A)$ the matrix $f(A) = P^{-1}f(D)P$ where $f(D)$ denotes the diagonal matrix with diagonal $f(d_{11}), \dots, f(d_{nn})$. We denote by $\mathbb{P}_{mn}^{[m]}(S)$ the subset of block matrices $H = (H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}(\mathbb{C})$ where each block H_{st} is an $m \times m$ diagonalizable matrix with spectrum contained in S. Note that when $m = 1$, the set $\mathbb{P}_{mn}^{[m]}(S)$ reduces to $\mathbb{P}_n(S)$. Given $H = (H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}^{[m]}(S)$, we define

$$f^{[m]}[H] := (f(H_{st}))_{s,t=1}^n. \tag{3.2}$$

When $m = 1$, $f^{[m]}[A]$ reduces to $f[A]$. Using this notation, we can now state the main results of the paper.

Recall that by Theorem 2.1, a power function x^α preserves positivity when applied entrywise to all $n \times n$ symmetric positive semidefinite matrices with positive entries, if and only if $\alpha \geq n-2$ or $\alpha \in \mathbb{N}$. Our first main result shows that, surprisingly, the situation is radically different when the blocks have size greater than 1.

Theorem A. *Let $\beta \in \mathbb{Z}$ and let $m, n \geq 2$.*

1. *Given $\alpha > 0$, the matrix $f_\alpha^{[m]}[(H_{st})] = (H_{st}^\alpha) \in \mathbb{P}_{mn}(\mathbb{C})$ for all $(H_{st}) \in \mathbb{P}_{mn}^{[m]}([0, \infty))$, if and only if $\alpha = 1$. If $\alpha \leq 0$, then $f_0^{[m]}[-]$ preserves positivity on $\mathbb{P}_{mn}^{[m]}((0, \infty))$ if and only if $\alpha = 0$.*
2. *The functions $\phi_\alpha^{[m]}[-]$ do not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$ for any $\alpha \in \mathbb{R}$.*
3. *For $\alpha \in \mathbb{R}$, the functions $\psi_\alpha^{[m]}[-]$ preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$ if and only if $\alpha = 1$.*
4. *For $\alpha \in \mathbb{R}$, the functions $\Psi_{\alpha,\beta}^{[m]}[-]$ preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ if and only if $\alpha = 1$ and $\beta = \pm 1$ – i.e., $\Psi_{\alpha,\beta}(z) \equiv z$ or $\overline{z}$.*

A natural relaxation of the hypothesis in Theorem A is to assume that the blocks H_{st} all commute with each other. Powers preserving positivity when applied to block matrices where the blocks commute have been studied by D. Choudhury in [4]. It is natural to ask if the lower bound $\alpha \geq mn - 2$ in Theorem 2.4 is sharp, or if other powers preserve positivity. We completely settle this question in our second main result, Theorem B, by showing that the *critical exponent* is in fact $\alpha = n - 2$ and that smaller non-integer powers do not preserve Loewner positivity. In Section 5 we also consider the analogue of Theorem B where the blocks are complex diagonalizable.

Theorem B. *Let $\alpha > 0$ and $m, n \geq 2$. Then $(H_{st}^{\alpha}) \in \mathbb{P}_{mn}(\mathbb{C})$ for all $(H_{st})_{s,t=1}^{n} \in \mathbb{P}_{mn}(\mathbb{C})$ such that $H_{st} \in \mathbb{P}_m(\mathbb{C})$ and the blocks H_{st} commute, if $\alpha \in \mathbb{N} \cup [n-2, \infty)$. If $\alpha \notin \mathbb{N} \cup [n-2, \infty)$, there exist matrices $H_{st} \in \mathbb{P}_m(\mathbb{C})$ such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$, the blocks H_{st} commute, but (H_{st}^{α}) is not positive semidefinite. Moreover, if $\alpha < 0$, there exist real symmetric positive definite matrices H_{st}, $s, t = 1, \dots, n$ such that $(H_{st})_{s,t=1}^{n} \in \mathbb{P}_{mn}(\mathbb{R})$, but (H_{st}^{α}) is not positive semidefinite.*

In our third main result, we consider an interesting question raised by X. Zhan in 2001 (see [20, Acknowledgments]). Zhan asked if Theorem 2.1 can be generalized to matrices with complex entries when the power functions x^{α} are replaced by the functions $z = re^{i\theta} \mapsto r^{\alpha} e^{i\theta}$. This is precisely the power function $\Psi_{\alpha,1}$. More generally, in the framework developed in Section 3.1, it is natural to generalize Zhan's question by asking for which values of α, β does $\Psi_{\alpha,\beta}$ preserve positivity when applied entrywise. Our third result, Theorem C, provides bounds on α, β which guarantee that $\Psi_{\alpha,\beta}$ preserves or does not preserve Loewner positivity.

Theorem C. *Let $n \geq 3$.*

(1) *The entrywise function $\Psi_{\alpha,\beta}$ preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ if $\beta \in \mathbb{Z}$, $(\alpha, \beta) \neq (0, 0)$, and either $\alpha \in |\beta| - 2 + 2\mathbb{N}$ or $\alpha \geq \max(n-2, |\beta| + 2n - 6)$.*
(2) *The entrywise function $\Psi_{\alpha,\beta}$ fails to preserve positivity if either:*
   (a) *$\beta \notin \mathbb{Z}$, or*
   (b) *$\alpha < 1$, or*
   (c) *$1 \leq \alpha < \max(n-2, |\beta| + 2\lfloor(\sqrt{8n+1} - 5)/2\rfloor)$ and $\alpha \notin |\beta| - 2 + 2\mathbb{N}$.*

Thus for $n \geq 3$, $\beta \in \mathbb{Z}$, and $\alpha \notin |\beta| - 2 + 2\mathbb{N}$, we see that $\Psi_{\alpha,\beta}$ preserves Loewner positivity for $\alpha \geq \max(n-2, |\beta| + 2n - 6)$, but not for $\alpha < \max(n-2, |\beta| + 2\lfloor(\sqrt{8n+1} - 5)/2\rfloor)$. Note that if $n = 3$, these two quantities coincide and equal $\max(1, |\beta|)$. We therefore have the following corollary, which completely answers Zhan's question for the $n = 3$ case.

Corollary 3.2. *For $n = 3$, the entrywise power function $\Psi_{\alpha,\beta}$ preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ if and only if $\beta \in \mathbb{Z}$ and $\alpha \geq \max(1, |\beta|)$.*

A consequence of Theorem C is that complex critical exponents exist for the power functions $\Psi_{\alpha,\beta}$:

Corollary 3.3. *For every $n \geq 3$ and $\beta \in \mathbb{Z}$, there exists a smallest real number $\alpha_{\min}$ such that $\Psi_{\alpha,\beta}[-]$ preserves $\mathbb{P}_n(\mathbb{C})$ for all $\alpha \geq \alpha_{\min}$. Moreover, $\alpha_{\min} = \max(1, |\beta|)$ for $n = 3$, while for $n \geq 4$,*

$$\max(n-2, |\beta| + 2\lfloor(\sqrt{8n+1} - 5)/2\rfloor) \leq \alpha_{\min} \leq |\beta| + 2n - 6.$$

Note that Theorem 2.1 and an application of the Schur product theorem imply that $n - 2 \leq \alpha_{\min} \leq |\beta| + 2n - 4$. Corollary 3.3 thus greatly improves this lower bound for the critical exponent $\alpha_{\min}$.

4. Powers preserving positivity: the block case

We now characterize powers preserving positivity when applied blockwise. To prove Theorem A we need some preliminaries. First recall the notion of an m-matrix monotone function.

Definition 4.1. Let $I \subset \mathbb{R}$ be an interval and let $m \geq 1$. A function $f : I \to \mathbb{R}$ is said to be *m-matrix monotone* (or *m-monotone*) if given $m \times m$ Hermitian matrices A, B with spectrum in I,

$$A \geq B \implies f(A) \geq f(B).$$

The following lemma reformulates m-monotonicity of power functions in terms of block matrices, and will be crucial in proving Theorem A.

Lemma 4.2. *Given an integer $m \in \mathbb{N}$, define the subset $\mathcal{P}_m \subset \mathbb{P}_{2m}(\mathbb{C})$ via:*

$$\mathcal{P}_m := \{\begin{pmatrix} A & B \\ B & C \end{pmatrix} \in \mathbb{P}_{2m}^{[m]}([0,\infty)) : \det C \neq 0, BC = CB\}.$$

Also fix $\alpha \in \mathbb{R}$. Then the following are equivalent:

1. *The blockwise power function $f_\alpha^{[m]}[-]$ sends $\mathcal{P}_m$ to $\mathbb{P}_{2m}(\mathbb{C})$.*
2. *The function f_α is m-monotone on $(0,\infty)$.*

In particular, if $f_\alpha^{[m]}[-]$ preserves Loewner positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ for some $n \geq 2$, then it is m-monotone.

Proof. First suppose $f_\alpha^{[m]}[-]$ preserves Loewner positivity on $\mathcal{P}_m$, and assume $A \geq B > 0$. Let $X \in \mathbb{P}_m(\mathbb{C})$ denote the principal square root of B. Then the block matrix $M := \begin{pmatrix} A & X \\ X & I_m \end{pmatrix} \in \mathbb{P}_{2m}^{[m]}([0,\infty))$, by computing the Schur complement of I_m in M. Therefore by hypothesis, the matrix $f_\alpha^{[m]}[M] = \begin{pmatrix} A^\alpha & X^\alpha \\ X^\alpha & I_m \end{pmatrix}$ is also positive semidefinite. Using Schur complements again, we conclude that $A^\alpha - (X^\alpha)^2 = A^\alpha - B^\alpha \geq 0$. Thus $A \geq B > 0 \Rightarrow A^\alpha \geq B^\alpha$ and so f_α is m-monotone on $(0,\infty)$.

Conversely, suppose f_α is m-monotone on $(0,\infty)$, and suppose $\begin{pmatrix} A & B \\ B & C \end{pmatrix} \in \mathcal{P}_m$. Then $A \geq BC^{-1}B$ (by taking Schur complements). Moreover, B, C are simultaneously diagonalizable, whence $B, C^{\pm 1}$ commute. It is now easy to verify that $(BC^{-1}B)^\alpha = B^\alpha (C^\alpha)^{-1} B^\alpha$. Now using the m-monotonicity of f_α, we compute:

$$C^\alpha \geq 0^\alpha = 0, \qquad A^\alpha = f_\alpha(A) \geq f_\alpha(BC^{-1}B) = B^\alpha (C^\alpha)^{-1} B^\alpha.$$

In turn, this implies that the matrix $\begin{pmatrix} A^\alpha & B^\alpha \\ B^\alpha & C^\alpha \end{pmatrix}$ is positive semidefinite, proving (1). The final assertion is also clear since $\mathcal{P}_m \oplus \mathbf{0}_{m(n-2)\times m(n-2)} \subset \mathbb{P}_{mn}^{[m]}(\mathbb{C})$ (via padding by zeros). □

Matrix monotone functions have been the subject of a detailed analysis by Loewner [29] and many others including Wigner and von Neumann [37], Bendat and Sherman [2], Korányi [25], Donoghue [7], Sparr [34], Hansen and Petersen [16], Ameur [1], and more recently by Hansen [15] - also see [15] for a history of the problem. We now state an important and interesting characterization of matrix monotone functions using Loewner matrices. This result was shown by Hansen [15] and plays an essential role in proving Theorem A.

Definition 4.3. Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ be differentiable. The *first divided difference* of f for $\lambda_1, \lambda_2 \in I$, denoted by $[\lambda_1, \lambda_2]_f$ is given by

$$[\lambda_1, \lambda_2]_f := \begin{cases} \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ f'(\lambda_1) & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Now given $m \geq 2$ and $\lambda_1, \dots, \lambda_m \in I$, define the *Loewner matrix* $L_f(\lambda_1, \dots, \lambda_m)$ of f at the points λ_j to be

$$L_f(\lambda_1, \dots, \lambda_m) := ([\lambda_s, \lambda_t]_f)_{s,t=1}^m. \tag{4.1}$$

Theorem 4.4 (Hansen [15, Theorem 3.2])**.** *Let $m \in \mathbb{N}$ and f be a real function in $C^1(I)$, where $I \subset \mathbb{R}$ is an open interval. Then f is m-monotone if and only if the Loewner matrix $L_f(\lambda_1, \dots, \lambda_m)$ is positive semidefinite for all sequences $\lambda_1, \dots, \lambda_m \in I$.*

We now have all the ingredients for proving Theorem A.

Proof of Theorem A.
Proof of (1). Clearly, $f_1^{[m]}[-]$ preserves positivity on $\mathbb{P}_{mn}^{[m]}([0,\infty))$. Next, if $\alpha = 0$ and the blocks H_{st} are positive definite, then $f_0^{[m]}[(H_{st})] = \mathbf{1}_{n\times n} \otimes I_m$, where $\otimes$ denotes the Kronecker product, and so $f_0^{[m]}[(H_{st})] \in \mathbb{P}_{mn}(\mathbb{C})$. Now assume $\alpha \in \mathbb{R}$ and $\alpha \neq 0, 1$. We claim that the function $f_\alpha^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}([0,\infty))$. It suffices to prove the claim for $m = n = 2$ (the general case follows by padding with zeros).

Thus, suppose $f_\alpha^{[2]}[-]$ preserves positivity on $\mathbb{P}_4^{[2]}((0,\infty))$. By Lemma 4.2, the function $f_\alpha(x) = x^\alpha$ is 2-monotone on $(0,\infty)$. By Theorem 4.4, this is possible if and only if the Loewner matrix $L_{f_\alpha}(\lambda_1, \lambda_2)$ is positive semidefinite for all $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \neq \lambda_2$. Thus, the $(1,1)$-entry of $L_{f_\alpha}(\lambda_1, \lambda_2)$ has to be nonnegative and so $\alpha \geq 0$. Computing the determinant of $L_{f_\alpha}(\lambda_1, \lambda_2)$, we obtain:

$$\det L_{f_\alpha}(\lambda_1, \lambda_2) = \alpha\lambda_1^{\alpha-1} \cdot \alpha\lambda_2^{\alpha-1} - \left(\frac{\lambda_1^\alpha - \lambda_2^\alpha}{\lambda_1 - \lambda_2}\right)^2 \geq 0 \qquad \forall \lambda_1, \lambda_2 > 0, \lambda_1 \neq \lambda_2. \tag{4.2}$$

Now fix $\lambda_2 > 0$. If $\alpha > 1$, then $\det L_{f_\alpha}(\lambda_1, \lambda_2) \to -\infty$ as $\lambda_1 \to \infty$ since $\alpha \neq 0, 1$. Thus, $\det L_{f_\alpha}(\lambda_1, \lambda_2) < 0$ for λ_1 large enough. This proves that $f_\alpha(x) = x^\alpha$ is not 2-monotone, and hence $f_\alpha^{[m]}[-]$ does not preserve positivity if $\alpha > 1$ or $\alpha < 0$.

Finally, suppose $\alpha \in (0,1)$. We first claim that there exists a real matrix $\begin{pmatrix} A & X \\ X & N \end{pmatrix} \in \mathbb{P}_4^{[2]}((0,\infty))$ such that the matrix $\begin{pmatrix} A^\alpha & X^\alpha \\ X^\alpha & N^\alpha \end{pmatrix}$ is not positive semidefinite. To prove the claim, consider the matrix

$$M := \begin{pmatrix} 3/2 & 0 & 1 & 1/2 \\ 0 & 2 & 1/2 & 1 \\ 1 & 1/2 & 1 & 4/5 \\ 1/2 & 1 & 4/5 & 223/250 \end{pmatrix} = \begin{pmatrix} A & X \\ X & N \end{pmatrix}, \tag{4.3}$$

where $A, X, N \in \mathbb{P}_2(\mathbb{R})$. It can be verified that $\det(\lambda I_4 - M)$ is a fourth-degree polynomial which is positive for $|\lambda|$ large and at $\lambda = 1, 4$; zero at $\lambda = 0$; and negative at $1/5, 2$. Therefore 0 is an eigenvalue of M, and the other three eigenvalues of M lie in $(1/5, 1), (1, 2), (2, 4)$. It is now easily verified that $M \in \mathbb{P}_4^{[2]}((0,\infty))$. We next claim that $f_\alpha^{[2]}[M] \notin \mathbb{P}_4$ for small $\alpha > 0$ close enough to zero. To verify the claim, we will compute explicitly the determinant of $f_\alpha^{[2]}[M]$, and show that it is negative close to $\alpha = 0$. We begin by computing the powers of the 2×2 blocks A, X, N of M. The block A is diagonal, while the powers of the off-diagonal block X are computed using its spectral decomposition:

$$X = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = U \operatorname{diag}(\frac{1}{2}, \frac{3}{2}) U^T, \; U := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

from which it follows that

$$X^\alpha = \frac{1}{2} \begin{pmatrix} (3/2)^\alpha + (1/2)^\alpha & (3/2)^\alpha - (1/2)^\alpha \\ (3/2)^\alpha - (1/2)^\alpha & (3/2)^\alpha + (1/2)^\alpha \end{pmatrix}.$$

To compute the spectral powers of the last remaining block $N := \begin{pmatrix} 1 & 4/5 \\ 4/5 & 223/250 \end{pmatrix}$, we define $x_\pm := 27 \pm \sqrt{160729} = 27 \pm \sqrt{27^2 + 400^2}$ for convenience. Then N has spectral decomposition

$N = VDV^{-1}$, where

$$V := \begin{pmatrix} x_-/400 & x_+/400 \\ 1 & 1 \end{pmatrix}, \quad D := \operatorname{diag}(1 - \frac{x_+}{500}, 1 - \frac{x_-}{500}), \quad V^{-1} = \frac{1}{2\sqrt{160729}} \begin{pmatrix} -400 & x_+ \\ 400 & -x_- \end{pmatrix}.$$

Let $\lambda_\pm := 1 - \frac{x_\pm}{500}$ be the eigenvalues of N. Since $V = UD'$ with U unitary and D' diagonal, we obtain:

$$N^\alpha := VD^\alpha V^{-1} = \frac{1}{2\sqrt{160729}} \begin{pmatrix} x_+\lambda_-^\alpha - x_-\lambda_+^\alpha & 400(\lambda_-^\alpha - \lambda_+^\alpha) \\ 400(\lambda_-^\alpha - \lambda_+^\alpha) & x_+\lambda_+^\alpha - x_-\lambda_-^\alpha \end{pmatrix}.$$

Therefore if we define $g_M(\alpha) := \det f_\alpha^{[2]}[M]$, then

$$\begin{aligned} \frac{4}{2^\alpha} g_M(\alpha) = 2^{2-\alpha} \det \begin{pmatrix} A^\alpha & X^\alpha \\ X^\alpha & N^\alpha \end{pmatrix} &= 4a^2b^3 + 4aL_-L_+ + \frac{54}{\sqrt{160729}} ab(1-ab)(L_- - L_+) \\ &+ (ab+1)\left((2L-1)(L_-a^2 + L_+b^2) - (2L+1)(L_+a^2 + L_-b^2)\right), \end{aligned}$$

where $L_\pm := \lambda_\pm^\alpha, a := (3/2)^\alpha, b := (1/2)^\alpha$, and $L := 200/\sqrt{160729}$. Note that $g_M(0) = \det f_0^{[2]}[M] = 0$. Moreover, using the explicit form of the function $g_M(\alpha)$, it can be verified that $g'_M(0) = 0$ and $g''_M(0) < 0$. This shows that $g_M(\alpha) < 0$ for all $0 < |\alpha| < \epsilon_M$ for some $\epsilon_M > 0$.

Now suppose $f_\alpha^{[2]}[-]$ preserves positivity on $\mathbb{P}_4^{[2]}((0,\infty))$ for some $\alpha \in (0,1)$. Choose $k \in \mathbb{N}$ such that $\alpha^k \in (0, \epsilon_M)$, with ϵ_M as above. Then $(f_\alpha^{[2]})^{\circ k}[M] = f_{\alpha^k}^{[2]}[M] \in \mathbb{P}_4^{[2]}((0,\infty)) \subset \mathbb{P}_4(\mathbb{C})$, which contradicts the previous paragraph. This proves that $f_\alpha^{[2]}[-]$ does not preserve positivity for $\alpha \in (0,1)$.

Proof of (2). The first part shows that $\phi_\alpha^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$ for $\alpha \neq 0, 1$. We now prove that $\phi_\alpha^{[m]}[-]$ also does not preserve positivity for $\alpha = 0$ and $\alpha = 1$. Suppose first $\alpha = 0$. Fix $B := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, and for $c \in \mathbb{R}$, define the matrix

$$A(c) := \begin{pmatrix} cI_2 & B \\ B^T & cI_2 \end{pmatrix}. \tag{4.4}$$

Note that $A(c)$ has eigenvalues $c, c, c \pm \sqrt{2}$. Moreover, B is diagonalizable and has eigenvalues 0 and 1. As a consequence, $\phi_0(B) = B$. Therefore the matrix $A(\sqrt{2}) \in \mathbb{P}_4^{[2]}(\mathbb{R})$, but $\phi_0^{[2]}[A(\sqrt{2})] = A(1) \notin \mathbb{P}_4$. This proves $\phi_0^{[2]}[-]$ does not preserve positivity on $\mathbb{P}_4^{[2]}(\mathbb{R})$. The case of general $m, n \geq 2$ follows by padding $A(\sqrt{2})$ with zeros. To prove that $\phi_1^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$, consider the matrix

$$M := \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

It is not difficult to verify that $M \in \mathbb{P}_4^{[2]}(\mathbb{R})$, but $\det \phi_1^{[2]}[M] = -4/5$. This proves that $\phi_1^{[2]}[-]$ does not preserve positivity on $\mathbb{P}_4^{[2]}(\mathbb{R})$. It follows that $\phi_1^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$ for $m, n \geq 2$.

Proof of (3). By part (1), the function $\psi_\alpha^{[m]}[-]$ does not preserve positivity if $\alpha \neq 0, 1$. Clearly, $\psi_1^{[m]}[-]$ preserves positivity since $\psi_1(x) = x$ for all $x \in \mathbb{R}$. That $\psi_0^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{R})$ follows by considering the matrix $A(c)$ in Equation (4.4).

Proof of (4). By part (1), $\Psi_{\alpha,\beta}^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ if $\alpha \neq 0, 1$. Moreover, the above analysis of the matrix $A(c)$ in Equation (4.4) shows that $\Psi_{0,\beta}^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ for any $\beta \in \mathbb{Z}$. Now suppose $\alpha = 1$. By the second part of the proof, $\Psi_{1,0}^{[m]} \equiv \phi_1^{[m]}$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$. Also, $\Psi_{1,1}^{[m]}$ clearly preserves positivity. Note that since a matrix A is positive semidefinite if and only if its complex conjugate $\overline{A}$ is positive semidefinite, $\Psi_{\alpha,\beta}^{[m]}[-]$ preserves positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ if and only if $\Psi_{\alpha,-\beta}^{[m]}[-]$ does so. To conclude the proof, it thus remains to prove that $\Psi_{1,\beta}^{[m]}[-]$ does not preserve positivity on $\mathbb{P}_{mn}^{[m]}(\mathbb{C})$ for $\beta \geq 2$. Without loss of generality, let $m = n = 2$, and define:

$$M(a,b,c) := \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & a \\ \overline{a} & \overline{c} & 1 & 0 \\ \overline{b} & \overline{a} & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{C}. \tag{4.6}$$

One verifies that the four eigenvalues of the matrix $M(a,a,0)$ are $1 \pm a(\sqrt{5} \pm 1)/2$. Therefore if we fix $a \in (0, (\sqrt{5}-1)/2)$, the matrix $M(a,a,0)$ is positive definite. Consequently, there exists $\epsilon > 0$ such that $M(a,a,c) \in \mathbb{P}_4^{[2]}((0,\infty))$ for $|c| < \epsilon$.

We now claim that $\Psi_{1,\beta}^{[2]}[M(a,a,c)] \notin \mathbb{P}_4(\mathbb{C})$ if c is negative and close enough to 0. To prove the claim, we first compute $\Psi_{1,\beta}^{[2]}[M(a,a,c)]$. Note that $\Psi_{1,\beta}(I_2) = I_2$; now set $B := \begin{pmatrix} a & a \\ c & a \end{pmatrix}$, with $c < 0$. The eigenvalues of B are $a \pm i\sqrt{a|c|}$, with corresponding eigenvectors $v_\pm := (\mp i\sqrt{a/|c|}, 1)^T$. As a consequence, defining $\lambda_\pm := \Psi_{1,\beta}(a \pm i\sqrt{a|c|})$, we obtain:

$$\begin{aligned}\Psi_{1,\beta}(B) &= \begin{pmatrix} -i\sqrt{a/|c|} & i\sqrt{a/|c|} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} -i\sqrt{a/|c|} & i\sqrt{a/|c|} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\lambda_+ + \lambda_-}{2} & \frac{-i\sqrt{a}}{\sqrt{|c|}} \cdot \frac{\lambda_+ - \lambda_-}{2} \\ \frac{i\sqrt{|c|}}{\sqrt{a}} \cdot \frac{\lambda_+ - \lambda_-}{2} & \frac{\lambda_+ + \lambda_-}{2} \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix},\end{aligned}$$

say. Thus $\Psi_{1,\beta}^{[2]}[M(a,a,c)] = M(a',b',c')$.

Now suppose $\beta \geq 2$. We will prove that there exists $a > 0$ such that the $(1,2)$-entry of the real matrix $\Psi_{1,\beta}(B)$ is greater than 1, if c is negative and close enough to 0. Indeed, note that

$$\lambda_\pm = \Psi_{1,\beta}(a \pm i\sqrt{a|c|}) = \sqrt{a^2 + a|c|}\, e^{i\beta \arctan(\pm\sqrt{|c|/a})}.$$

Thus,

$$\begin{aligned}\lim_{c \to 0^-} \Psi_{1,\beta}(B)_{12} &= \lim_{c \to 0^-} \frac{-i\sqrt{a}}{\sqrt{|c|}} \cdot \frac{\lambda_+ - \lambda_-}{2} \\ &= \lim_{c \to 0^-} \frac{-i\sqrt{a}}{\sqrt{|c|}} \sqrt{a^2 + a|c|} \frac{e^{i\beta \arctan(\sqrt{|c|/a})} - e^{i\beta \arctan(-\sqrt{|c|/a})}}{2} \\ &= -ia \lim_{c \to 0^-} \frac{e^{i\beta \arctan(\sqrt{|c|/a})} - e^{i\beta \arctan(-\sqrt{|c|/a})}}{2\sqrt{|c|/a}} \\ &= -ia \left.\frac{d}{dy} e^{i\beta \arctan(y)}\right|_{y=0} = -ia\, e^{i\beta \arctan(y)} i\beta \left.\frac{1}{1+y^2}\right|_{y=0} = a\beta.\end{aligned}$$

As a consequence, if $\beta \geq 2$ and $a \in (1/\beta, (\sqrt{5}-1)/2)$, then for $c < 0$ small enough, the $(1,2)$-entry of $\Psi_{1,\beta}(B)$ is greater than 1. But then the minor of $\Psi_{1,\beta}^{[2]}[M(a,a,c)]$ obtained by deleting the second row and column is negative, from which it follows that $\Psi_{1,\beta}^{[2]}[M(a,a,c)] \not\in \mathbb{P}_4(\mathbb{C})$. Therefore $\Psi_{1,\beta}^{[2]}[-]$ does not preserve positivity on $\mathbb{P}_4^{[2]}(\mathbb{C})$ if $\beta \neq \pm 1$. As before, the case of general $m, n \geq 2$ follows by padding with zeros. This concludes the proof. $\square$

Remark 4.5. In the proof of part (1) of Theorem A, we showed that $\det f_\alpha^{[2]}[M] < 0$ for all $\alpha \in (0, \epsilon_M)$ for some $\epsilon_M \in (0,1)$, with M as in Equation (4.3). In fact, numerical computations indicate that $\det f_\alpha^{[2]}[M] < 0$ for all $\alpha \in (0,1)$; this would provide a "universal" counterexample M for the proof of part (1).

5. Powers preserving positivity for commuting blocks

Recall that D. Choudhury [4] studied an interesting variant of the problem considered in Section 4 - namely, which blockwise powers $(H_{st})_{s,t=1}^n \mapsto (H_{st}^\alpha)$ preserve positivity when all the $m \times m$ blocks H_{st} commute and are positive semidefinite. It was shown in [4] that if $\alpha \in \mathbb{N} \cup [mn-2, \infty)$ then the corresponding blockwise power preserves positivity. We now demonstrate that the bound $mn-2$ can be significantly improved. More precisely, we completely characterize the powers preserving Loewner positivity in that setting.

Proof of Theorem B. The proof is a refinement of the argument in [4, Theorem 5]. Let $H = (H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$ be as given. Since the blocks H_{st} commute, they are simultaneously diagonalizable, i.e., there exists a $m \times m$ unitary matrix U and diagonal matrices Λ_{st} such that $H_{st} = U \Lambda_{st} U^* \ \forall s,t$. Letting $T := U^{\oplus n}$ and $\Lambda := (\Lambda_{st})$, we obtain $H = T \Lambda T^{-1}$. Let P be the permutation matrix such that

$$P^{-1} \Lambda P = A_1 \oplus \cdots \oplus A_m, \tag{5.1}$$

where $(A_k)_{st} := (\Lambda_{st})_{kk}$ with $1 \leq k \leq m$ and $1 \leq s,t \leq n$. Then $H = (TP)(A_1 \oplus \cdots \oplus A_m)(TP)^{-1}$. By assumption, $A_k \in \mathbb{P}_n([0,\infty)) \ \forall k$. Moreover, since the entries of the matrices A_k are the eigenvalues of the blocks H_{st}, we have $(H_{st}^\alpha) = (TP)(A_1^{\circ\alpha} \oplus \cdots \oplus A_m^{\circ\alpha})(TP)^{-1}$. Here $A^{\circ\alpha} := (a_{st}^\alpha)$ denotes the entrywise power of $A = (a_{st})$. Since A_k are $n \times n$ matrices, it follows immediately by Theorem 2.1 that $(H_{st}^\alpha) \in \mathbb{P}_{mn}(\mathbb{C})$ if $\alpha \in \mathbb{N} \cup [n-2, \infty)$.

Now suppose $\alpha \in (0, n-2) \setminus \mathbb{N}$. Choose $\epsilon > 0$ such that the matrix $A := (1 + \epsilon st)_{s,t=1}^n$ satisfies $A^{\circ\alpha} \not\in \mathbb{P}_n$ (see Theorem 2.1). Let

$$\Lambda = (\Lambda_{st})_{s,t=1}^n := P A^{\oplus m} P^{-1}, \tag{5.2}$$

where P is the permutation matrix given in Equation (5.1) and Λ_{st} are $m \times m$ diagonal matrices. Define $H_{st} := \Lambda_{st}$. Then the matrices H_{st} are Hermitian positive semidefinite, as is the matrix $H = (H_{st})$, but $(H_{st}^\alpha) = P(A^{\circ\alpha} \oplus \cdots \oplus A^{\circ\alpha})P^{-1}$ is not positive semidefinite by construction of A. This shows that the powers $\alpha \in (0, n-2) \setminus \mathbb{N}$ do not preserve positivity when applied blockwise.

Finally, suppose $\alpha < 0$. Let $A := I_{m \times m} + \mathbf{1}_{m \times m} \in \mathbb{P}_m([1,2])$. Examining the leading principal 2×2 block of A, it follows that $A^{\circ\alpha} \not\in \mathbb{P}_m$. Repeating the same construction as in Equation (5.2), we conclude that there exist commuting blocks $H_{st} := \Lambda_{st} \in \mathbb{P}_m(\mathbb{C})$ such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$, but $(H_{st}^\alpha) \not\in \mathbb{P}_m(\mathbb{C})$ if $\alpha < 0$. This concludes the proof. $\square$

In Theorem B, we assumed each block H_{st} to be positive semidefinite. This assumption was necessary for the powers H_{st}^α to be well-defined. We now consider the case where the blocks are not positive semidefinite. Using the functions ϕ_α and ψ_α, it is natural to extend the characterization provided by Theorem B to Hermitian blocks with arbitrary eigenvalues. Using Theorem 2.3, we can now characterize the powers α such that $\phi_\alpha^{[m]}$ and $\psi_\alpha^{[m]}$ preserve positivity when the blocks commute.

Theorem 5.1. *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $m, n \geq 2$. Then*

(1) *$\phi_\alpha^{[m]}[H] \in \mathbb{P}_{mn}(\mathbb{C})$ for all $m \times m$ Hermitian matrices H_{st} such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$ and the blocks H_{st} commute if $\alpha \in 2\mathbb{N} \cup [n-2, \infty)$. If $\alpha \notin 2\mathbb{N} \cup [n-2,\infty)$, there exist real symmetric matrices H_{st} such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{R})$, the blocks H_{st} commute, but $\phi_\alpha^{[m]}[H] \notin \mathbb{P}_{mn}(\mathbb{R})$.*
(2) *$\psi_\alpha^{[m]}[H] \in \mathbb{P}_{mn}(\mathbb{C})$ for all Hermitian $m \times m$ matrices H_{st} such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$ and the blocks H_{st} commute if $\alpha \in (-1+2\mathbb{N}) \cup [n-2,\infty)$. If $\alpha \notin (-1+2\mathbb{N}) \cup [n-2,\infty)$, there exist real symmetric matrices H_{st} such that $(H_{st}) \in \mathbb{P}_{mn}(\mathbb{R})$, the blocks H_{st} commute, but $\psi_\alpha^{[m]}[H] \notin \mathbb{P}_{mn}(\mathbb{R})$.*

Proof. The proof is similar to the proof of Theorem B. Let U be a unitary matrix and P be a permutation matrix such that defining $H := (H_{st})$ and $T := U^{\oplus n}$, we have

$$H = (TP)(A_1 \oplus \cdots \oplus A_m)(TP)^{-1}, \tag{5.3}$$

where $A_1, \dots, A_m$ are $n \times n$ matrices containing the eigenvalues of the blocks H_{st}. If $f = \phi_\alpha$ or ψ_α, we have $f^{[m]}[H] = (TP)(f[A_1] \oplus \cdots \oplus f[A_m])(TP)^{-1}$. It follows from Theorem 2.3 that $\phi_\alpha^{[m]}[H] \in \mathbb{P}_{mn}(\mathbb{C})$ if $\alpha \in 2\mathbb{N} \cup [n-2,\infty)$ and $\psi_\alpha^{[m]}[H] \in \mathbb{P}_{mn}(\mathbb{C})$ if $\alpha \in (-1+2\mathbb{N}) \cup [n-2,\infty)$. Conversely, if $f = \phi_\alpha$ and $\alpha \notin 2\mathbb{N} \cup [n-2,\infty)$ or $f = \psi_\alpha$ and $\alpha \notin (-1+2\mathbb{N}) \cup [n-2,\infty)$, then by [10, Theorem 2.5, Proposition 6.2] there exists a matrix $A \in \mathbb{P}_n$ such that $f[A] \notin \mathbb{P}_n$. Using the same construction as in Equation (5.2), we conclude that $f^{[m]}[-]$ does not preserve positivity. □

Remark 5.2. We now address the case $\alpha = 0$, which was omitted from Theorem 5.1 for ease of exposition. We first claim that if $n = 2$ and $H := (H_{st}) \in \mathbb{P}_{2m}(\mathbb{C})$ with Hermitian commuting blocks H_{st}, then $\phi_0^{[m]}[H], \psi_0^{[m]}[H] \in \mathbb{P}_{2m}(\mathbb{C})$. Indeed, as in Equation (5.3), the block matrix H can be factored as $H = (TP)(A_1 \oplus \cdots \oplus A_m)(TP)^{-1}$, where $A_1, \dots, A_m \in \mathbb{P}_2$. Moreover, ϕ_0, ψ_0 preserve positivity when applied entrywise to $\mathbb{P}_2$, since the only possible resulting matrices are $\mathbf{0}_{2\times 2}, \mathbf{1}_{2\times 2}$, $I_{2\times 2}$, and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, which are all positive semidefinite. However, when $n \geq 3$, we claim that $\phi_\alpha^{[m]}[H], \psi_\alpha^{[m]}[H]$ are not always positive semidefinite. Indeed, as in [10, Equation 6.2], define

$$A := \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{pmatrix} \oplus \mathbf{0}_{(n-3)\times(n-3)} \in \mathbb{P}_n. \tag{5.4}$$

One easily verifies that $\phi_0[A] = \psi_0[A] \notin \mathbb{P}_n$. Using the same construction as in Equation (5.2), we conclude that there exist commuting blocks $H_{st} := \Lambda_{st} \in \mathbb{P}_m(\mathbb{C})$ such that $H = (H_{st}) \in \mathbb{P}_{mn}(\mathbb{C})$, but $\phi_\alpha^{[m]}[H], \psi_\alpha^{[m]}[H] \notin \mathbb{P}_{mn}(\mathbb{C})$ when $\alpha = 0$.

Remark 5.3. An interesting consequence of Theorem 5.1 is that when the blocks commute, preserving positivity is in fact independent of the block size m (see part (2) of Theorem 5.4). This is in contrast to Theorem A, in which increasing the block size to $m \geq 2$ drastically reduces the set of powers preserving positivity, when the commutativity assumption is omitted.

Powers of the trace function. Problems similar to the ones above have been considered in the literature, with the power function $H_{st} \mapsto H_{st}^\alpha$ replaced by other functions mapping $m \times m$ blocks to $p \times p$ matrices (see e.g. [35, 30, 6, 31, 39]). In particular, de Pillis [6] studies the map $(H_{st})_{s,t=1}^n \mapsto (\mathrm{tr}(H_{st}))_{s,t=1}^n$ and demonstrates that it preserves positivity. See also [39] for a nice short proof of the same result. To conclude this section, we extend de Pillis's result by characterizing the values $\alpha \geq 0, \beta \in \mathbb{Z}$ such that $(H_{st}) \mapsto (\Psi_{\alpha,\beta}(\mathrm{tr}(H_{st})))$ preserves positivity.

Theorem 5.4. *Fix $\alpha \geq 0$, $\beta \in \mathbb{Z}$, and $m, n \in \mathbb{N}$. Then the following are equivalent:*

(1) *$\Psi_{\alpha,\beta}[(\mathrm{tr}(H_{st}))_{s,t=1}^n] \in \mathbb{P}_n(\mathbb{C})$ for all $(H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}(\mathbb{C})$.*

(2) $\Psi_{\alpha,\beta}[-]$ *preserves positivity on* $\mathbb{P}_n(\mathbb{C})$.
(3) $\Psi_{\alpha,\beta}[(\mathrm{tr}(H_s^* H_t))_{s,t=1}^n] \in \mathbb{P}_n(\mathbb{C})$ *for all* $m \times m$ *complex matrices* $H_1, \ldots, H_n$.
(4) $\Psi_{\alpha,\beta}^{[m]}[(H_{st})] \in \mathbb{P}_{mn}(\mathbb{C})$ *if* $(H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}^{[m]}(\mathbb{C})$ *and all blocks* H_{st} *commute.*

Proof. Suppose first (1) holds and let $A = (a_{st})_{s,t=1}^n \in \mathbb{P}_n(\mathbb{C})$. Define $H_{st} \in \mathbb{P}_m(\mathbb{C})$ by $(H_{st})_{qr} := a_{st}$ if $q = r = 1$ and 0 otherwise. Then $(H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}(\mathbb{C})$, so $\Psi_{\alpha,\beta}[A] \in \mathbb{P}_n(\mathbb{C})$ by (1). Thus (1) $\Rightarrow$ (2). Conversely, if $(H_{st})_{s,t=1}^n \in \mathbb{P}_{mn}(\mathbb{C})$, then $(\mathrm{tr}(H_{st}))_{s,t=1}^n \in \mathbb{P}_n(\mathbb{C})$ by [6, Proposition 2.3], and (2) $\Rightarrow$ (1) follows immediately. Next, (2) $\Leftrightarrow$ (3) because matrices of the form $(\mathrm{tr}(H_s^* H_t))$ are general Gram matrices in the inner product space $\mathbb{C}^{m\times m}$ with $\langle A, B\rangle := \mathrm{tr}(A^* B)$, so that the set of such matrices coincides with $\mathbb{P}_n(\mathbb{C})$. Finally, that (2) $\Leftrightarrow$ (4) follows by simultaneously diagonalizing the blocks H_{st} and proceeding as in the proof of Theorem B. $\square$

Note that when β is even or odd, the function $\Psi_{\alpha,\beta}$ reduces on $\mathbb{R}$ to ϕ_α and ψ_α respectively. Thus the powers α such that $\phi_\alpha[-]$ or $\psi_\alpha[-]$ preserves positivity on $\mathbb{P}_n(\mathbb{R})$ in Theorem 5.4 are known (see Theorem 2.3). In the next section, we explore the general problem of characterizing the values α, β for which $\Psi_{\alpha,\beta}[-]$ preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$.

6. Entrywise powers preserving positivity on Hermitian matrices

This section is devoted to proving Theorem C. As the proof is long and intricate, we show the $n = 3$ case in Section 6.1, and then the general case in Section 6.2.

6.1. Preserving positivity on Hermitian matrices of order 3.

Note that for $n = 1, 2$, all maps $\Psi_{\alpha,\beta}$ preserve positivity when applied entrywise to every matrix in $\mathbb{P}_n(\mathbb{C})$. In this subsection we focus on the $n = 3$ case. We begin by identifying a smaller sub-family of matrices which it suffices to consider when verifying whether or not $\Psi_{\alpha,\beta}$ preserves Loewner positivity.

Lemma 6.1. *For* $j = 1, 2, 3$*, suppose* $r_j > 0, s_j \geq 0, t_j \in \mathbb{R}, \theta_j, \theta \in (-\pi, \pi]$*, and define* $\mathbf{t} := (t_1, t_2, t_3)$*. Now define:*

$$A := \begin{pmatrix} r_1 & s_3 e^{i\theta_3} & s_2 e^{i\theta_2} \\ s_3 e^{-i\theta_3} & r_2 & s_1 e^{i\theta 1} \\ s_2 e^{-i\theta_2} & s_1 e^{-i\theta_1} & r_3 \end{pmatrix}, \qquad T(\mathbf{t}, \theta) := \begin{pmatrix} 1 & t_3 & t_2 e^{i\theta} \\ t_3 & 1 & t_1 \\ t_2 e^{-i\theta} & t_1 & 1 \end{pmatrix}. \tag{6.1}$$

Then the following are equivalent:

(1) $A \in \mathbb{P}_3(\mathbb{C})$;
(2) $T(\mathbf{t}, \theta) \in \mathbb{P}_3(\mathbb{C})$*, where* $t_j := \frac{s_j \sqrt{r_j}}{\sqrt{r_1 r_2 r_3}}$ *for* $j = 1, 2, 3$*, and* $\theta = \theta_1 + \theta_3 - \theta_2$.
(3) *Given* $t_j := \frac{s_j \sqrt{r_j}}{\sqrt{r_1 r_2 r_3}}$*, we have* $t_j \in [0, 1]$ *for* $j = 1, 2, 3$*, and* $\det T(\mathbf{t}, \theta) = 1 - \sum_{j=1}^3 t_j^2 + 2t_1 t_2 t_3 \cos\theta \geq 0$.

Proof. Define $D := \mathrm{diag}(r_1^{-1/2}, r_2^{-1/2}, r_3^{-1/2})$. That (1) $\Leftrightarrow$ (2) follows from the fact that the principal minors of $T(\mathbf{t}, \theta)$ are equal to the corresponding principal minors of DAD, and hence are obtained from the principal minors of A by rescaling by positive factors. That (2) $\Leftrightarrow$ (3) is obvious. $\square$

The following corollary to Lemma 6.1 helps simplify the task of ascertaining if an entrywise power function $\Psi_{\alpha,\beta}$ preserves Loewner positivity.

Corollary 6.2. *Let* $n \geq 3$*,* $\alpha \in \mathbb{R}$*, and* $\beta \in \mathbb{Z}$*. Then* $\Psi_{\alpha,\beta}[-]$ *preserves positivity on* $\mathbb{P}_3(\mathbb{C})$ *if and only if* $T(\mathbf{t}^{\circ\alpha}, \beta\theta) \in \mathbb{P}_3(\mathbb{C})$ *for every* $\mathbf{t} \in [0, 1]^3$ *and* $\theta \in (-3\pi, 3\pi)$ *such that* $\det T(\mathbf{t}, \theta) \geq 0$.

Proof. Clearly $\Psi_{\alpha,\beta}$ preserves positivity on $\mathbb{P}_2(\mathbb{C})$, hence on matrices $A \in \mathbb{P}_3(\mathbb{C})$ with at least one zero diagonal entry. For all other matrices $A \in \mathbb{P}_3(\mathbb{C})$, we are now done by Lemma 6.1. $\square$

In order to prove our next result, we recall the notion of a generalized Dirichlet polynomial.

Definition 6.3. A *generalized Dirichlet polynomial* is a function $F : \mathbb{R} \to \mathbb{R}$ of the form $F(x) = \sum_{j=1}^{n} a_j t_j^x$, where $a_j, t_j, x \in \mathbb{R}$ and $t_1 > t_2 > \cdots > t_n > 0$.

Given a sequence $(a_j)_{j=1}^n$, denote by $S[(a_j)]$ the number of sign changes in the sequence after discarding all zero terms a_j. Also define $A_j := a_1 + \cdots + a_j$ for all $1 \leq j \leq n$. Then $S[(A_j)] \leq S[(a_j)]$. We now recall the following classical result which extends Descartes' Rule of Signs to generalized Dirichlet polynomials.

Theorem 6.4 (Descartes' Rule of Signs, [23, 26])**.** *Suppose $F(x) = \sum_{j=1}^n a_j t_j^x : \mathbb{R} \to \mathbb{R}$ is a generalized Dirichlet polynomial (with $t_1 > \cdots > t_n > 0$ as above), and $A_j = a_1 + \cdots + a_j$ for all j. Then F has at most $S[(A_j)]$ positive zeros, and at most $S[(a_j)]$ real zeros.*

Before we fully classify the entrywise powers which preserve Loewner positivity on $\mathbb{P}_3(\mathbb{C})$, we first show that $\Psi_{\alpha,\beta}$ preserves positivity on $\mathbb{P}_3(\mathbb{C})$ if $\alpha \geq \max(1, |\beta|)$. We also prove that $\Psi_{\alpha,\beta}$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ if $\beta \notin \mathbb{Z}$. In Section 6.2, we will prove that $\Psi_{\alpha,\beta}$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ if $\alpha < \max(n-2, |\beta| + 2\lfloor(\sqrt{8n+1} - 5)/2\rfloor)$, thus completing the classification when $n = 3$.

Theorem 6.5. *For $n = 3$, the entrywise power function $\Psi_{\alpha,\beta}$ preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ if $\beta \in \mathbb{Z}$ and $\alpha \geq \max(1, |\beta|)$. Moreover, if $\beta \notin \mathbb{Z}$, then $\Psi_{\alpha,\beta}$ does not preserve Loewner positivity on $\mathbb{P}_n(\mathbb{C})$.*

Proof. Suppose $\beta \in \mathbb{Z}$ and $\alpha \geq \max(|\beta|, 1)$. By Corollary 6.2, it suffices to show that $\Psi_{\alpha,\beta}$ preserves positivity on all matrices $T(\mathbf{t}, \theta) \in \mathbb{P}_3(\mathbb{C})$ of the form (6.1). Using Lemma 6.1, this reduces to showing:

$$1 - \sum_{j=1}^{3} t_j^2 + 2t_1t_2t_3 \cos\theta \geq 0 \quad \Longrightarrow \quad g_\beta(\alpha) := 1 - \sum_{j=1}^{3} t_j^{2\alpha} + 2(t_1t_2t_3)^\alpha \cos(\beta\theta) \geq 0. \tag{6.2}$$

In (6.2) we may assume without loss of generality that $\beta > 0$. There are now three cases: first, if $t_j = 0$ for some j, then Equation (6.2) is easy to show. Next, suppose t_j are all nonzero and $\max_j t_j = 1$, say $t_1 = 1$. Then $g_1(1) = -t_2^2 - t_3^2 + 2t_2t_3 \cos\theta \geq 0$ if and only if $t_2 = t_3$ and $\cos\theta = 1$. But then $\theta = 0$ or $\pm 2\pi$ and (6.2) again follows. The third case is if $t_j \in (0, 1)\ \forall j$. In this case we use Theorem 6.4: the partial sums of the coefficients are $1, 0, -1, -2, -2 + 2\cos(\beta\theta)$, and hence the generalized Dirichlet polynomial has at most one positive root. First suppose θ is not an integer multiple of $2\pi/\beta$. Note that $g_\beta(0) = 1 - 3 + 2\cos(\beta\theta) < 0$. Also, by the Schur product theorem, $g_\beta(\beta) \geq 0$ since $\beta \in \mathbb{N}$. Thus, the generalized Dirichlet polynomial g_β has a unique root between 0 and β. It follows that $g_\beta(\alpha) \geq 0$ for all $\alpha \geq \beta$, since $g_\beta(\alpha) \to 1$ as $\alpha \to \infty$. Finally, suppose $\theta = 2\pi k/\beta$ for some $k \in \mathbb{Z}$. To show (6.2), note that

$$1 - \sum_{j=1}^{3} t_j^2 + 2t_1t_2t_3 \cos\theta \geq 0 \quad \Longrightarrow \quad 1 - \sum_{j=1}^{3} t_j^2 + 2t_1t_2t_3 \geq 0. \tag{6.3}$$

This implies that the real matrix $T(\mathbf{t}, 0)$ as in Equation (6.1) is positive semidefinite. Now (6.2) follows by applying Theorem 2.1 to $T(\mathbf{t}, 0)$, since $\alpha \geq 1$.

To conclude the proof, we now provide a "universal" example of a matrix $A \in \mathbb{P}_3(\mathbb{C})$ such that $\Psi_{\alpha,\beta}[A \oplus \mathbf{0}_{(n-3)\times(n-3)}] \notin \mathbb{P}_n(\mathbb{C})$ whenever $\beta \in \mathbb{R} \setminus \mathbb{Z}$. Define

$$A = \begin{pmatrix} 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ e^{-2\pi i/3} & 1 & e^{2\pi i/3} \\ e^{2\pi i/3} & e^{-2\pi i/3} & 1 \end{pmatrix}. \tag{6.4}$$

Clearly $A \in \mathbb{P}_3(\mathbb{C})$, but $\det \Psi_{\alpha,\beta}[A] = -2 + 2\cos(2\pi\beta)$, which is negative precisely when $\beta \notin \mathbb{Z}$. Thus $\Psi_{\alpha,\beta}$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ when $\beta \notin \mathbb{Z}$. $\square$

6.2. **Bounds for arbitrary dimension** n**.** We now prove Theorem C, which addresses the case of general $n \geq 3$. The proof will use the following preliminary result, which generalizes an idea from FitzGerald and Horn [8, Theorem 2.2].

Proposition 6.6. *Let $\alpha > 1$ and fix an integer $n \geq 3$. Suppose $\Psi_{\alpha-1,1}[A] \in \mathbb{P}_{n-1}(\mathbb{C})$ for all $A \in \mathbb{P}_{n-1}(\mathbb{C})$. Then $\Psi_{\alpha,0}[A] \in \mathbb{P}_n(\mathbb{C})$ for all $A \in \mathbb{P}_n(\mathbb{C})$.*

Proof. Suppose $\Psi_{\alpha-1,1}[A] \in \mathbb{P}_{n-1}(\mathbb{C})$ for all $A \in \mathbb{P}_{n-1}(\mathbb{C})$. Fix $z = z_1 + z_2 i, w = w_1 + w_2 i \in \mathbb{C}$, where $z_1, z_2, w_1, w_2 \in \mathbb{R}$, and denote by $z_\lambda := \lambda z + (1-\lambda)w$. Then

$$\begin{aligned}\frac{d}{d\lambda}\Psi_{\alpha,0}(z_\lambda) &= \frac{\alpha}{2}\Psi_{\alpha-2,0}(z_\lambda)\left[2(\lambda z_1 + (1-\lambda)w_1)(z_1 - w_1) + 2(\lambda z_2 + (1-\lambda)w_2)(z_2 - w_2)\right] \\ &= \alpha\Psi_{\alpha-2,0}(z_\lambda)\operatorname{Re}(z_\lambda\overline{z-w}) = \alpha\operatorname{Re}(\Psi_{\alpha-2,0}(z_\lambda)z_\lambda\overline{z-w}) \\ &= \alpha\operatorname{Re}(\Psi_{\alpha-1,1}(z_\lambda)\overline{z-w}).\end{aligned}$$

We now proceed as in the proof of [8, Theorem 2.2]. Note that

$$\Psi_{\alpha,0}(z) = \Psi_{\alpha,0}(w) + \int_0^1 \frac{d}{d\lambda}\Psi_{\alpha,0}(z_\lambda)\,d\lambda = \Psi_{\alpha,0}(w) + \alpha\int_0^1 \operatorname{Re}(\Psi_{\alpha-1,1}(z_\lambda)\overline{z-w})\,d\lambda. \tag{6.5}$$

Now let $A \in \mathbb{P}_n(\mathbb{C})$ and let $\zeta := (a_{1n}, a_{2n}, \dots, a_{nn})^T/a_{nn}^{1/2}$ if $a_{nn} \neq 0$ and $\zeta := \mathbf{0}_{n\times 1}$ otherwise. By [8, Lemma 2.1], the matrix $A - \zeta\zeta^* \in \mathbb{P}_n(\mathbb{C})$. Also, note that the entries of the last row and column of $A - \zeta\zeta^*$ are zero. Applying (6.5) entrywise, we obtain that

$$\Psi_{\alpha,0}[A] = \Psi_{\alpha,0}[\zeta\zeta^*] + \alpha\int_0^1 \operatorname{Re}\left(\Psi_{\alpha-1,1}[\lambda A + (1-\lambda)\zeta\zeta^*] \circ \overline{A - \zeta\zeta^*}\right)\,d\lambda. \tag{6.6}$$

Note that the Schur product $\Psi_{\alpha-1,1}[\lambda A + (1-\lambda)\zeta\zeta^*] \circ \overline{A - \zeta\zeta^*}$ in the integrand in Equation (6.6) is positive semidefinite by hypothesis and the fact that the last row and column of $A - \zeta\zeta^*$ are zero. It follows immediately that $\Psi_{\alpha,0}[A] \in \mathbb{P}_n(\mathbb{C})$. This concludes the proof. □

We now have all the ingredients necessary to prove our last main result.

Proof of Theorem C.
Proof of (1). Suppose first that $\beta \in \mathbb{Z}$ and $\alpha \in |\beta| - 2 + 2\mathbb{N}$, say $\alpha = |\beta| + 2m$ with $m \geq 0$. Note that $A = (a_{st}) \in \mathbb{P}_n(\mathbb{C})$ if and only if $\overline{A} := (\overline{a_{st}}) \in \mathbb{P}_n(\mathbb{C})$. Then,

$$\Psi_{\alpha,\beta}[A] = \begin{cases}\Psi_{2m,0}[A] = (A \circ \overline{A})^{\circ m}, & \text{if } \beta = 0,\\ \Psi_{2m+\beta,\beta}[A] = A^{\circ\beta} \circ (A \circ \overline{A})^{\circ m}, & \text{if } \beta > 0,\\ \Psi_{2m+|\beta|,\beta}[A] = \overline{A}^{\circ|\beta|} \circ (A \circ \overline{A})^{\circ m}, & \text{if } \beta < 0.\end{cases} \tag{6.7}$$

In all three cases, we obtain that $\Psi_{\alpha,\beta}[A] \in \mathbb{P}_n(\mathbb{C})$ by the Schur product theorem.

Suppose instead $\beta \in \mathbb{Z}$ and $\alpha \geq \max(n-2, |\beta| + 2n - 6)$. We claim that in that case, $\Psi_{\alpha,\beta}[-]$ also preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$. The proof is by induction on $n \geq 3$. For $n = 3$ we are done by Theorem 6.5. Now suppose the assertion holds for $n - 1 \geq 3$. Then $\Psi_{\alpha,1}[-]$ preserves Loewner positivity on $\mathbb{P}_{n-1}(\mathbb{C})$ for $\alpha \geq 2(n-1-3) + 1 = 2n - 7$. Hence by Proposition 6.6, $\Psi_{\alpha,0}[-]$ preserves Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ for $\alpha \geq 2n - 7 + 1 = 2n - 6$. Thus if $\alpha \geq 2n - 6 + |\beta|$ and $A \in \mathbb{P}_n(\mathbb{C})$, then

$$\Psi_{\alpha,|\beta|}[A] = \Psi_{\alpha-|\beta|,0}[A] \circ A^{\circ|\beta|}, \qquad \Psi_{\alpha,-|\beta|}[A] = \Psi_{\alpha-|\beta|,0}[A] \circ \overline{A}^{\circ|\beta|},$$

and these are both in $\mathbb{P}_n(\mathbb{C})$ by the Schur product theorem. Therefore the claim is proved by induction.

Proof of (2). If $\beta \notin \mathbb{Z}$, then Theorem 6.5 shows that $\Psi_{\alpha,\beta}$ does not preserve Loewner positivity on $\mathbb{P}_n(\mathbb{C})$. Thus assume $\beta \in \mathbb{Z}$. If $\alpha < 1$, it is easy to see that $\Psi_{\alpha,\beta}[-]$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ (see Equation (5.4)). It thus remains to prove that $\Psi_{\alpha,\beta}[-]$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ if $1 \leq \alpha < |\beta| + 2\lfloor(\sqrt{8n+1}-5)/2\rfloor$, but $\alpha - |\beta|$ is not a nonnegative even integer. To show this statement, first note for each integer $k \geq 0$ that

$$\lfloor(\sqrt{8n+1}-5)/2\rfloor \geq k \qquad \Longleftrightarrow \qquad n \geq \binom{k+3}{2}.$$

Thus, we first show the assertion for $n = \binom{k+3}{2}$, from which it immediately follows for all $n > \binom{k+3}{2}$ by padding with zeros. Moreover, it suffices to show that $\Psi_{\alpha,\beta}[-]$ does not preserve Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ when $\alpha \in (|\beta| + 2k - 2, |\beta| + 2k)$, since the smaller values of $\alpha \in (|\beta|, |\beta| + 2k) \setminus (\alpha - 2\mathbb{Z})$ do not preserve positivity on $\mathbb{P}_n(\mathbb{C})$ by considering lower values of k (and then padding by zeros).

Thus, suppose $n = \binom{k+3}{2}$ and $\alpha \in (|\beta| + 2k - 2, |\beta| + 2k)$. It suffices to show that $\Psi_{\alpha,\beta}[-]$ does not preserve positivity on $\mathbb{P}_n(\mathbb{C})$. Since $\Psi_{\alpha,-\beta}[A] = \overline{\Psi_{\alpha,\beta}[A]}$, we may assume $\beta \geq 0$. Now fix $z \in \mathbb{C}^\times$ and consider the function $f : (-1/|z|, 1/|z|) \to \mathbb{C}$, given by:

$$f(\epsilon) := \Psi_{\alpha,\beta}(1 + \epsilon z) = (1 + \epsilon z)^{(\alpha+\beta)/2}(1 + \epsilon\overline{z})^{(\alpha-\beta)/2}.$$

Defining $Z(\epsilon) := 1 + \epsilon z$, one has:

$$\frac{df}{d\epsilon} = \frac{d\Psi_{\alpha,\beta}(Z(\epsilon))}{d\epsilon} = \frac{\partial\Psi_{\alpha,\beta}}{\partial Z}\frac{dZ}{d\epsilon} + \frac{\partial\Psi_{\alpha,\beta}}{\partial\overline{Z}}\frac{d\overline{Z}}{d\epsilon}.$$

Repeatedly using this formula and the general Leibniz rule, we obtain for any integer $l \geq 0$:

$$\begin{aligned}\frac{d^l f}{d\epsilon^l}(0) &= \sum_{j=0}^{l} \binom{l}{j} \prod_{t=0}^{j-1}\left(\frac{\alpha+\beta}{2} - t\right) \prod_{t=0}^{l-j-1}\left(\frac{\alpha-\beta}{2} - t\right) \cdot \left.\frac{z^j \overline{z}^{l-j} f(\epsilon)}{(1+\epsilon z)^j (1+\epsilon\overline{z})^{l-j}}\right|_{\epsilon=0} \\ &= \sum_{j=0}^{l} \binom{l}{j} \Psi_{l,l-2j}(z) \prod_{t=0}^{j-1}\left(\frac{\alpha+\beta}{2} - t\right) \prod_{t=0}^{l-j-1}\left(\frac{\alpha-\beta}{2} - t\right).\end{aligned}$$

Therefore by Taylor's theorem, as $\epsilon \to 0^+$ we have

$$\Psi_{\alpha,\beta}(1 + \epsilon z) = 1 + \sum_{l=1}^{k+1}\sum_{j=0}^{l} \frac{c_{l,j}\epsilon^l}{l!}\Psi_{l,l-2j}(z) + o(\epsilon^{k+2}), \tag{6.8}$$

$$\text{where} \quad c_{l,j} := \binom{l}{j} \prod_{t=0}^{j-1}\left(\frac{\alpha+\beta}{2} - t\right) \prod_{t=0}^{l-j-1}\left(\frac{\alpha-\beta}{2} - t\right) \ \forall 1 \leq l \leq k+1,\ 0 \leq j \leq l.$$

Now consider the family of power functions $S_k := \{\Psi_{l,l-2j} : 1 \leq l \leq k+1, 0 \leq j \leq l\} \cup \{K \equiv 1\}$. Note that S_k contains precisely $\binom{k+3}{2}$ functions, which are linearly independent on $\mathbb{C}^n$ by Lemma 3.1. Hence there exists a vector $u_{k,n} \in \mathbb{C}^n$ such that

$$\Psi_{k+1,k+1}[u_{k,n}] \notin \operatorname{span}_{\mathbb{C}}\{h[u_{k,n}] : h \in S_k \setminus \{\Psi_{k+1,k+1}\}\}. \tag{6.9}$$

Now define the matrix $A_\epsilon := \mathbf{1}_{n\times n} + \epsilon u_{k,n}u_{k,n}^* \in \mathbb{P}_n(\mathbb{C})$. Then,

$$\Psi_{\alpha,\beta}[A_\epsilon] = \mathbf{1}_{n\times n} + \sum_{l=1}^{k+1} \frac{c_{l,j}\epsilon^l}{l!}\Psi_{l,l-2j}[u_{k,n}]\Psi_{l,l-2j}[u_{k,n}]^* + o(\epsilon^{k+2})C,$$

where $C_{n\times n}$ is a fixed matrix independent of ϵ. Moreover, there exists $v_{k,n} \in \mathbb{C}^n$ orthogonal to $\{h[u_{k,n}] : h \in S_k \setminus \{\Psi_{k+1,k+1}\}\}$, but not to $\Psi_{k+1,k+1}[u_{k,n}]$. Now compute:

$$\begin{aligned} v_{k,n}^* \Psi_{\alpha,\beta}[A_\epsilon] v_{k,n} &= \frac{c_{k+1,0}\epsilon^{k+1}}{(k+1)!} |v_{k,n}^* \Psi_{k+1,k+1}[u_{k,n}]|^2 + o(\epsilon^{k+2}) v_{k,n}^* C v_{k,n} \\ &= \frac{|v_{k,n}^* \Psi_{k+1,k+1}[u_{k,n}]|^2}{2^{k+1}(k+1)!} \cdot \epsilon^{k+1} \prod_{t=0}^{k} (\alpha - \beta - 2t) + o(\epsilon^{k+2}) v_{k,n}^* C v_{k,n}. \end{aligned}$$

Since $\alpha \in (\beta + 2k - 2, \beta + 2k)$, the first term is negative, whence so is the entire expression for sufficiently small $\epsilon > 0$. This shows that $\Psi_{\alpha,\beta}[-]$ does not preserve Loewner positivity on $\mathbb{P}_n(\mathbb{C})$ if $\alpha \in (\beta + 2k - 2, \beta + 2k)$, which concludes the proof. $\square$

Remark 6.7. Since $n \geq \binom{k+3}{2}$, we observe that the vector $u_{k,n} \in \mathbb{C}^n$ satisfying (6.9) can in fact be chosen to have all its entries in the complex disc $D(0, R)$ for any fixed $0 < R \leq \infty$. Indeed, by Lemma 3.1, the characters in the set S_k are linearly independent on $D(0, R)$. Thus there exists $u = u_{k,n} \in D(0, R)^n$ such that the vectors $\{h[u] : h \in S_k\}$ are linearly independent.

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