AXIOMATIC FRAMEWORK FOR THE BGG CATEGORY $\mathcal{O}$

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ABSTRACT. In this paper we introduce a general axiomatic framework for algebras with triangular decomposition, which allows for a systematic study of the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$. Our axiomatic framework can be stated via three relatively simple axioms, and it encompasses a very large class of algebras studied in the literature. We term the algebras satisfying our axioms as *regular triangular algebras (RTAs)*; these include (a) generalized Weyl algebras, (b) symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$, (c) quantum groups $U_q(\mathfrak{g})$ over "lattices with possible torsion", (d) infinitesimal Hecke algebras, (e) higher rank Virasoro algebras, and others.

In order to incorporate these special cases under a common setting, our theory distinguishes between roots and weights, and does not require the Cartan subalgebra to be a Hopf algebra. We also allow RTAs to have roots in arbitrary monoids rather than root lattices, and the roots of the Borel subalgebras to lie in cones with respect to a strict subalgebra of the Cartan subalgebra. These relaxations of the triangular structure have not been explored in the literature.

We then define and study the BGG Category $\mathcal{O}$ over an arbitrary RTA. In order to work with general RTAs – and also bypass the use of central characters – we introduce certain conditions (termed the *Conditions (S)*), under which distinguished subcategories of Category $\mathcal{O}$, termed "blocks", possess desirable homological properties including: (a) being a finite length, abelian, self-dual category; (b) having enough projectives and injectives; or (c) being a highest weight category satisfying BGG Reciprocity. We discuss the above examples and whether they satisfy the various Conditions (S). We also discuss two new examples of RTAs that cannot be studied by using previous theories of Category $\mathcal{O}$, but require the full scope of our framework. These include the first construction of a family of algebras for which the "root lattice" is non-abelian.

CONTENTS

1. Introduction

This paper is motivated by the study of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG] associated with a complex semisimple Lie algebra $\mathfrak{g}$. The definition of $\mathcal{O}$ depends on the fact that $U\mathfrak{g}$ has a triangular decomposition. This category has been studied quite intensively for both classical and modern reasons, and has connections to geometry, combinatorics, crystals, categorification, primitive ideals, abelian ideals, Kac-Moody theory, quantum algebras, and mathematical physics. To name but a few references, see [AnSt, H2, Ja, Jos, Kac2, Maz, MP, Soe] (and the references therein). One important property of the category $\mathcal{O}$ is that its blocks are highest weight categories in the sense of [CPS], and hence satisfy BGG Reciprocity.

Subsequently, Category $\mathcal{O}$ has been studied over a large number of algebras with triangular decomposition, and similar results on BGG Reciprocity and other homological properties of blocks have been shown in these settings. Thus the main goal of this paper is to simultaneously generalize both (a) the structure of the algebra over which to define and study $\mathcal{O}$, and (b) the setup of Category $\mathcal{O}$ over semisimple $\mathfrak{g}$, in several different ways. We do so in order to (a) systematize and unify the treatment of a large number of examples studied in the literature, and at the same time, (b) preserve the homological and representation-theoretic properties that are desirable in the case of semisimple Lie algebras.

Thus, the present paper studies algebras with triangular decomposition $A \cong B^- \otimes H_1 \otimes B^+$, with the "middle" subalgebra H_1 called the *Cartan subalgebra*. We begin by discussing the ways in which the structure of the underlying algebras is simultaneously generalized in the present paper, in order to incorporate a very large class of examples in the literature:

1. First, Lie algebras with triangular decomposition as well as their quantum analogues are combined under a common framework. Recall that several well-known Lie algebras in representation theory possess a triangular decomposition similar to $U\mathfrak{g}$ – for example, symmetrizable Kac-Moody Lie algebras [Kac2], contragredient Lie algebras [KK], the (centerless) Virasoro algebra [FeFr], and extended (centerless) Heisenberg algebras. An analogue of Category $\mathcal{O}$ has been explored for such Lie algebras in [MP] (see also [RCW]).

At the same time, a closely related setting involves quantum analogs of the aforementioned algebras. These algebras have also been studied in detail in the literature (see e.g. [Ja, Jos]). Our common framework incorporates both of these settings as special cases of algebras with triangular decomposition $A \cong B^- \otimes H_1 \otimes B^+$, where the Cartan subalgebra H_1 is a commutative, cocommutative Hopf algebra.

There are similarities between our framework and that of [AnSch], in that Hopf algebras, weight spaces, and quantum groups are involved. However, our construction is significantly different as well: the algebras here are neither finite-dimensional, nor do they need to be Hopf algebras (and *a priori*, we also do not impose restrictions on the ground field).

2. While the case of the Cartan subalgebra being a Hopf algebra is incorporated into our framework, we do not require it to necessarily be thus. In particular, the framework proposed in this paper also encompasses algebras arising from topology as well as *low rank continuous Hecke algebras*, for which the Cartan subalgebras are not Hopf algebras. See Section 9.

3. In our framework, there is another strict weakening of the axioms for $\mathcal{O}$ used in the literature to date. In all of the examples mentioned above, if we denote the triangular decomposition as $A = B^- \otimes H_1 \otimes B^+$, then one requires the roots of $B^\pm$ to lie in positive and negative cones with respect to the entire Cartan subalgebra H_1. However, the present paper only requires this condition to hold with respect to a (possibly proper) subalgebra $H_0 \subset H_1$. This allows us to consider certain *higher (Lie) rank infinitesimal Hecke algebras*, for which Category $\mathcal{O}$ could not have been studied using traditional approaches in the literature.

4. Recall that in the theory of semisimple (or Kac-Moody) Lie algebras, the root lattice embeds in the weight space. Such a phenomenon also occurs in their quantum analogues. We provide an

explanation by showing that in all such cases in the literature, there are natural identifications between the two spaces, which we call the *weight-to-root map* and the *root-to-weight map*. See Definition 2.13 and Proposition 2.15.

However, such maps need not exist in general, because the Cartan subalgebra is not always a Hopf algebra. Thus we will differentiate between the group generated by the roots, and the space of all weights, for the Cartan subalgebra H_1. This dichotomy between roots and weights allows us to incorporate *generalized Weyl algebras* into our framework.

5. Usually, the group generated by the roots is a "lattice", generated by a finite base of simple roots. This is the case for both Lie algebras with triangular decomposition [MP] as well as quantum groups $U_q(\mathfrak{g})$ over Kac-Moody Lie algebras $\mathfrak{g}$. Our framework weakens this restriction by removing the lattice assumption. In fact, we remove the commutativity assumption altogether, and work with arbitrary "torsion-free" monoids. This allows us to incorporate many algebras from mathematical physics such as *generalized Virasoro and Schrödinger-Virasoro algebras* (as well as more traditional examples such as Witten's family of algebras and conformal $\mathfrak{sl}_2$-algebras).

Furthermore, in Section 4 we prove an Existence Theorem that allows us to construct algebras with triangular decomposition, in which the span of positive roots can be any monoid that satisfies certain "cocycle conditions" (4.2),(4.3). These conditions are novel and incorporate *all* abelian, torsion-free monoids as well as some non-abelian ones. This enables us to construct algebras with *non-abelian* "root lattices"; to our knowledge, no such algebras have been studied in the literature. Our results show that the proposed axiomatic framework is at once not unnecessarily "too broad", as well as broad enough to incorporate a very large class of settings in the literature – traditional as well as modern, classical as well as quantum.

6. Recall [BGG, H2] that in studying Category $\mathcal{O}$ over a semisimple Lie algebra $\mathfrak{g}$, and its decomposition into finite length blocks with enough projectives, central characters have played a crucial role, via a finiteness condition that we call (S4) in this paper (see Definition 3.14). The condition is useful in proving results in representation theory because the center of $U\mathfrak{g}$ is "large enough".

In general, however, this is false: there are algebras with triangular decomposition, whose center is trivial. In fact one of our motivating examples was the infinitesimal Hecke algebra of $\mathfrak{sl}_2$ (and $\mathbb{C}^2$) studied with Tikaradze in [Kh1, KT] – as well as its quantized analogue, which was studied in joint work [GK] with Gan. It was shown that the latter, quantum version has trivial center; yet a theory of $\mathcal{O}$ and its block decomposition (with BGG Reciprocity) was developed in [GK].

Thus, we do not use the center in this paper. Instead, we propose a strictly weaker condition which we call (S3), and which holds for semisimple $\mathfrak{g}$ because of condition (S4) involving central characters. We show that Condition (S3) already implies a block decomposition into highest weight categories. Thus our framework allows us to incorporate relatively modern constructions such as *rank one (quantum) infinitesimal Hecke algebras*, even though they may have trivial center.

Additionally, we now describe two ways in which we extend in this paper, the treatment of Category $\mathcal{O}$ found in the literature.

7. In studying representations of Lie algebras $\mathfrak{g}$ with triangular decomposition, one often focuses on representations on which the center $Z(\mathfrak{g}) \subset \mathfrak{g}$ acts by a fixed linear functional, or *level*. This is indeed the case for Kac-Moody Lie algebras and for other algebras such as higher rank Virasoro algebras; see e.g. [FeFr, HWZ]. Similarly, in the present paper we define distinguished subcategories of $\mathcal{O}$ that satisfy conditions such as (S3) (discussed above and defined in Section 3.3). In other words, we identify "good parts" of $\mathcal{O}$ that possess desirable homological properties.

8. Finally, the framework we propose is "functorial", in that the structure of Category $\mathcal{O}$ (or its "good parts" as in the previous point) over a tensor product of commuting factors can be deduced from similar structural facts for $\mathcal{O}$ over each individual factor. For example, the connection between

modules over a semisimple Lie algebra and those over its simple ideals, is a specific manifestation of a broader phenomenon that holds in the general setting studied in this paper.

Given the phenomena discussed above, we develop in this paper a general framework of a *regular triangular algebra (RTA)* for which the notion of Category $\mathcal{O}$ makes sense, and which encompasses all of the aforementioned examples. We conclude this paper with Example 10.11, which describes an RTA A for which one has to use the full level of generality of our framework to study Category $\mathcal{O}$ (and one can show $\mathcal{O}$ has very desirable properties), but the previously developed treatments of $\mathcal{O}$ are not adequate to describe its representation theory. See Theorem 10.13.

Organization of the paper. This paper is organized as follows. In Section 2, we present our axiomatic framework, which encompasses a wide variety of algebras. Section 2.1 discusses the special case when the Cartan subalgebra H_1 is a Hopf algebra, and ends by characterizing such algebras inside our framework. Next, we introduce Verma modules and other key concepts in Section 3. We then state in Section 3.3 – and show in Section 3.4 – the main theoretical results about Category $\mathcal{O}$, including block decompositions and homological properties.

The second half is devoted to examples. In Section 4 we provide the first example of a regular triangular algebra for which the analogue of the root lattice is not abelian. We also prove an Existence Theorem for all (abelian) variants of the root lattice. Sections 5 and 6 discuss familiar examples, including Lie algebras with "regular triangular decomposition", and a family of "extended quantum groups" for every symmetrizable Kac-Moody Lie algebra. Section 7 discusses further examples of our broad framework, including one in which the center is trivial and yet $\mathcal{O}$ has a block decomposition. Section 8 studies generalized Weyl algebras in detail – as an additional result, we prove that generalized down-up algebras admit quantizations, which are themselves deformations of quantum $\mathfrak{sl}_2$. In Section 9 we provide two examples of such algebras where the Cartan subalgebra is not a Hopf algebra. Finally in Section 10, we study infinitesimal Hecke algebras of higher (Lie) rank, for which the root lattice and weight space are not contained in the same vector space. We end with Example 10.11, which uses the full generality of our framework to study Category $\mathcal{O}$.

2. The main definition: Regular Triangular Algebras

We work throughout over a ground field $\mathbb{F}$. Unless otherwise specified, char $\mathbb{F}$ is arbitrary, and all tensor products below are over $\mathbb{F}$. We will often abuse notation and claim that two modules or functors are equal, when they are isomorphic (e.g. double duals). Now define $\mathbb{Z}^\pm := \pm(\mathbb{N} \cup \{0\})$. Given $S \subset \mathbb{Z}$ and a subset Δ of an abelian group Θ_0, define $S\Delta$ to be the set of all finite S-linear combinations $\sum_{\alpha \in \Delta} n_\alpha \alpha$, where $n_\alpha \in S \ \forall \alpha$. Finally, given any group Θ and a subset $\mathcal{Q}^+ \subset \Theta$, define $-\mathcal{Q}^+ := \{\theta^{-1} : \theta \in \mathcal{Q}^+\} \subset \Theta$, $\langle \mathcal{Q}^+ \rangle$ to be the subgroup of Θ generated by $\mathcal{Q}^+$, and $\mathbb{F}\Theta$ to be the group algebra of Θ.

Definition 2.1. Fix a ground field $\mathbb{F}$, and $\mathbb{F}$-algebras $H \subset A$.

(1) Define the spaces of *roots* and *weights* of H to be $\mathrm{Aut}_{\mathbb{F}-alg}(H)$ and $\widehat{H} := \mathrm{Hom}_{\mathbb{F}-alg}(H, \mathbb{F})$ respectively.

(2) Given a weight $\lambda \in \widehat{H}$ and an H-module M, the *λ-weight space* of M is $M_\lambda := \{m \in M : hm = \lambda(h)m \ \forall h \in H\}$. The *set of H-weights* of M is $\mathrm{wt}_H(M) := \{\lambda \in \widehat{H} : M_\lambda \neq 0\}$, and M is an *H-weight module* if $M = \bigoplus_{\lambda \in \widehat{H}} M_\lambda$ is H-semisimple.

(3) Define the *θ-root space* of A corresponding to a root $\theta \in \mathrm{Aut}_{\mathbb{F}-alg}(H)$, as well as the *set of H-roots of A*, to respectively equal

$$A_\theta := \{a \in A : ah = \theta(h)a \ \forall h \in H\}, \qquad \mathrm{root}_H(A) := \{\theta \in \mathrm{Aut}_{\mathbb{F}-alg}(H) : A_\theta \neq 0\}. \qquad (2.2)$$

(4) If $H_0 \subset H$ is an $\mathbb{F}$-subalgebra, let $\pi'_{H_0} : \mathrm{End}_{\mathbb{F}-alg}(H) \to \mathrm{Hom}_{\mathbb{F}-alg}(H_0, H)$ denote the restriction map. Similarly, denote by $\pi_{H_0} : \widehat{H_1} \to \widehat{H_0}$ the restriction map to H_0.

The axiomatic framework introduced in this paper will display the aforementioned dichotomy between *roots*, which pertain to algebras and belong to $\mathrm{Aut}_{\mathbb{F}-alg}(H_0)$; and *weights*, which pertain to representations and live in $\widehat{H_1}$. In this paper, we use θ to refer to roots.

Equipped with the above terminology, it is now possible to propose a broad framework in which to study the BGG Category $\mathcal{O}$, and which incorporates many examples in the literature.

Definition 2.3. An associative $\mathbb{F}$-algebra A, together with data $(B^\pm, H_1, H_0, \mathcal{Q}_0^+, i)$ satisfying the following conditions, is called a *regular triangular algebra* (denoted also by *RTA*).

(RTA1) There exist associative unital $\mathbb{F}$-subalgebras $B^\pm, H_1$ of A, such that the multiplication map
: $B^- \otimes_\mathbb{F} H_1 \otimes_\mathbb{F} B^+ \to A$ is a vector space isomorphism (the *triangular decomposition*).

(RTA2) There exist a unital subalgebra $H_0 \subset H_1$ and a monoid $\mathcal{Q}_0^+ \subset \mathrm{Aut}_{\mathbb{F}-alg}(H_0)$, such that $\mathcal{Q}_0^+ \setminus \{\mathrm{id}_{H_0}\}$ is a semigroup, and moreover,

$$B^+ = \bigoplus_{\theta_1 \in \mathcal{Q}_1^+} B_{\theta_1}^+, \qquad \text{where } \mathcal{Q}_1^+ := (\pi'_{H_0})^{-1}(\mathcal{Q}_0^+) \cap \mathrm{Aut}_{\mathbb{F}-alg}(H_1). \qquad (2.4)$$

Moreover, $B_{\mathrm{id}_{H_0}}^+ = \mathbb{F} \cdot 1$, and $\dim_\mathbb{F} B_{\theta_0}^+ < \infty$ for all $\theta_0 \in \mathcal{Q}_0^+$ (the *regularity* assumption).

(RTA3) There exists an anti-involution i of A (i.e., $i^2|_A = \mathrm{id}|_A$) that fixes H_1, and sends $B^\pm$ into the image under the multiplication map of $H_1 \otimes B^\mp$.

As explained in Proposition 3.8(1) below, the assumption that $\mathcal{Q}_0^+ \setminus \{\mathrm{id}_{H_0}\}$ is a semigroup helps construct a partial order on the set of weights. It also implies that $\mathcal{Q}_0^+$ is either trivial or infinite.

As we will discuss through many examples, most of the traditionally well-studied RTAs in the literature satisfy two additional restrictions: (a) $H_0 = H_1$; and (b) $\mathcal{Q}_0^+$ is generated by a finite $\mathbb{Z}$-basis Δ of "simple roots". These restrictions are encoded as follows for a general RTA.

Definition 2.5. An RTA A (together with $(B^\pm, H_1, H_0, \mathcal{Q}_0^+, i)$) is *strict* if $H_1 = H_0$. An RTA is *based* if there exists a pairwise commuting $\mathbb{Z}$-linearly independent set $\Delta \subset \mathcal{Q}_0^+$, called the *(base of) simple roots*, such that $\mathcal{Q}_0^+ = \mathbb{Z}^+\Delta$. In this case we may also denote the RTA by $(B^\pm, H_1, H_0, \Delta, i)$. The *rank* of a *strict*, based RTA is defined to be $|\Delta|$ for the smallest such Δ (or $\mathcal{Q}_0^+$).

Remark 2.6. In Section 10 we will see examples of non-strict RTAs (called infinitesimal Hecke algebras) which involve simple Lie algebras of arbitrary Lie rank, but for which it is possible to choose precisely one simple root to generate $\mathcal{Q}_0^+$. In order to avoid this discrepancy, we do not talk about the rank of a non-strict, based RTA in this paper.

Example 2.7. Definition 2.3 is quite technical; here is the prototypical and "initial" example - a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then $A = U\mathfrak{g} = U\mathfrak{n}^- \otimes \mathrm{Sym}(\mathfrak{h}) \otimes U\mathfrak{n}^+$ is a strict, based RTA with:

- $H_1 = H_0 = U\mathfrak{h}$ the Cartan subalgebra – this is a commutative, cocommutative Hopf algebra, so $A = U\mathfrak{g}$ is in fact a strict *Hopf RTA* (see Section 2.1);
- $B^\pm = U\mathfrak{n}^\pm$; and
- i the anti-involution obtained by composing the Chevalley involution and the Hopf algebra antipode – so i sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ for all roots α (and hence $B^\pm$ to $B^\mp$), and fixes $\mathfrak{h}$.

Now identify $\mathrm{Aut}_{\mathbb{F}-alg}(H_0)$ with $\widehat{H_0} = \mathfrak{h}^*$ as follows (also see Proposition 2.15): for every weight $\mu \in \mathfrak{h}^*$, define the root $\rho_{H_0}(\mu) : H_0 \to H_0$ via: $\rho_{H_0}(\mu)(h_1 \cdots h_n) := \prod_{j=1}^n (h_j - \mu(h_j))$ (and extend by linearity). It follows easily that $\mu \mapsto \rho_{H_0}(\mu)$ is an isomorphism of additive groups $\rho_{H_0} : \mathfrak{h}^* \to \mathrm{Aut}_{\mathbb{F}-alg}(H_0)$. Define $\mathcal{Q}_0^+$ to be the monoid generated by the simple roots Δ (or more precisely, $\{\rho_{H_0}(\alpha) : \alpha \in \Delta\}$) – this is usually denoted in the literature as Q^+, the "positive" part of the root lattice. Now note that the above parameters equip $U\mathfrak{g}$ with the structure of a strict, based RTA of finite rank.

Remark 2.8.

(1) Henceforth we denote an RTA by A alone, and do not explicitly write out all of the additional data $(B^\pm, H_1, H_0, \mathcal{Q}_0^+, i)$, even though it will also be assumed to be fixed.

(2) If A is a based RTA and $\mathrm{Aut}_{\mathbb{F}-alg}(H_0)$ is a subgroup of an $\mathbb{F}$-vector space under addition, then we also require $\mathbb{F}$ to have characteristic zero, since otherwise $\mathcal{Q}_0^+ \setminus \{\mathrm{id}_{H_0}\}$ has torsion and hence cannot be a semigroup. This explains why we will assume $\mathrm{char}\,\mathbb{F} = 0$ for Lie algebras (as in Example 2.7), but not necessarily for quantum groups.

(3) In the definition of an RTA, we do not insist that the anti-involution $i : A \to A$ sends B^+ to B^-, as is the case for Lie algebras with triangular decompositions. The reason is that for quantum algebras i may not send B^+ to B^-; see Section 6 or Example 7.1.

We now list some basic properties of regular triangular algebras (RTAs). These properties will be used henceforth without further reference.

Lemma 2.9. *(A is an RTA.) Suppose $\mathcal{Q}_r^+$ generates the subgroup $\langle \mathcal{Q}_r^+ \rangle \subset \mathrm{Aut}_{\mathbb{F}-alg}(H_r)$ for $r = 0, 1$.*

(1) *The groups $\langle \mathcal{Q}_r^+ \rangle$ act on the sets $\widehat{H_r}$ for $r = 0, 1$ via: $\theta_r * \lambda_r := \lambda_r \circ \theta_r^{-1}$ for $\lambda_r \in \widehat{H_r}, \theta_r \in \langle \mathcal{Q}_r^+ \rangle$. The actions are functorial, in that the following square commutes for all pairs of $\mathbb{F}$-algebras $H_0 \hookrightarrow H_1$:*

$$
\begin{array}{ccc}
(\pi'_{H_0})^{-1}(\mathrm{Aut}_{\mathbb{F}-alg}(H_0)) \times \widehat{H_1} & \xrightarrow{\quad * \quad} & \widehat{H_1} \\
{\scriptstyle \pi'_{H_0} \times \pi_{H_0}} \downarrow & & \downarrow {\scriptstyle \pi_{H_0}} \\
\mathrm{Aut}_{\mathbb{F}-alg}(H_0) \times \widehat{H_0} & \xrightarrow{\quad * \quad} & \widehat{H_0}
\end{array}
$$

(2) *If M is any A-module, then for $r = 0, 1$, we have:*

$$
A_{\theta_r} \cdot M_{\lambda_r} \subset M_{\theta_r * \lambda_r} = M_{\lambda_r \circ \theta_r^{-1}}, \qquad \forall \theta_r \in \langle \mathcal{Q}_r^+ \rangle, \ \lambda_r \in \widehat{H_r}. \tag{2.10}
$$

(3) *H_r is commutative for $r = 0, 1$, whence $H_r = (H_r)_{\mathrm{id}_{H_r}} = (H_r)_{\mathrm{id}_{H_{3-r}}}$.*

(4) *$i(A_{\theta_r}) = A_{\theta_r^{-1}}$ for $r = 0, 1$ and $\theta_r \in \langle \mathcal{Q}_r^+ \rangle$.*

The proofs are straightforward. For instance, part (4) holds because $i(a_{\theta_r})h_r = i(h_r a_{\theta_r}) = i(a_{\theta_r} \theta_r^{-1}(h_r)) = \theta_r^{-1}(h_r) i(a_{\theta_r})$ for all $r = 0, 1$, $\theta_r \in \langle \mathcal{Q}_r^+ \rangle$, $a_{\theta_r} \in A_{\theta_r}, h_r \in H_r$.

In turn, Lemma 2.9 helps prove that the subalgebras $B^\pm$ are "symmetric" in a precise sense:

Proposition 2.11. *A is an RTA as above.*

(1) *B^- has a decomposition similar to that of B^+ in (RTA2), i.e., there exists a monoid $\mathcal{Q}_0^- \subset \mathrm{Aut}_{\mathbb{F}-alg}(H_0)$, such that*

$$
B^- = \bigoplus_{\theta_1 \in \mathcal{Q}_1^-} B_{\theta_1}^-, \qquad where \ \mathcal{Q}_1^- := \{\theta_1 \in \mathrm{Aut}_{\mathbb{F}-alg}(H_1) : \pi'_{H_0}(\theta_1) \in \mathcal{Q}_0^-\}.
$$

Moreover, $\mathcal{Q}_r^- = -\mathcal{Q}_r^+$ for $r = 0, 1$, $B_{\mathrm{id}_{H_0}}^- = \mathbb{F}$, and $\dim_{\mathbb{F}} B_{\theta_r^{-1}}^- = \dim_{\mathbb{F}} B_{\theta_r}^+ < \infty \ \forall \theta_r \in \mathcal{Q}_r^+$.

(2) *$H_r \otimes B^\pm$ (more precisely, their images under multiplication) are unital $\mathbb{F}$-subalgebras of A.*

(3) *For $r = 0, 1$, $\mathcal{Q}_r^\pm$ are sub-monoids of $\mathrm{Aut}_{\mathbb{F}-alg}(H_r)$, such that $\mathcal{Q}_r^\pm \setminus \{\mathrm{id}_{H_r}\}$ are semigroups. Moreover, $\pi'_{H_0} : \langle \mathcal{Q}_1^+ \rangle \to \langle \mathcal{Q}_0^+ \rangle$ is a group homomorphism that restricts to the monoid maps $: \mathcal{Q}_1^\pm \twoheadrightarrow \mathcal{Q}_0^\pm$, and $A = \bigoplus_{\theta_r \in \langle \mathcal{Q}_r^+ \rangle} A_{\theta_r}$ is $\langle \mathcal{Q}_r^+ \rangle$-graded for $r = 0, 1$.*

(4) *The algebras $B^\pm$ have subalgebras (in fact, augmentation ideals) defined respectively as*

$$
N^\pm := \bigoplus_{\theta_r \in \pm \mathcal{Q}_r^+ \setminus \{\mathrm{id}_{H_r}\}} B_{\theta_r}^\pm, \qquad r = 0, 1.
$$

Proof. First observe that any sum of H_r-root subspaces of B^+ or B^- is direct. The statement is part of (RTA2) for B^+, and hence follows for B^- using Lemma 2.9(4). We now proceed with the

proof. The meat of the result lies in proving part (1). Compute using the H_1-root-semisimplicity of B^+ and the multiplication map m_A on A:

$$B^- = i(i(B^-)) \subset i(m_A(H_1 \otimes B^+)) = m_A(i(B^+) \otimes H_1) = m_A(i(\bigoplus_{\theta_1 \in \mathcal{Q}_1^+} B_{\theta_1}^+) \otimes H_1)$$

$$= \bigoplus_{\theta_1 \in \mathcal{Q}_1^+} m_A(H_1 \otimes B_{\theta_1^{-1}}^- \otimes H_1) = \bigoplus_{\theta_1 \in \mathcal{Q}_1^+} m_A(H_1 \otimes B_{\theta_1^{-1}}^-).$$

where all decompositions are direct from above, and the last equality follows by definition of root spaces. It follows by (RTA1) that B^- decomposes as a direct sum of H_1-root spaces with roots in $\mathcal{Q}_1^- := -\mathcal{Q}_1^+$. Restricting $\mathcal{Q}_1^-$ to H_0 proves the same assertion for $\mathcal{Q}_0^- := -\mathcal{Q}_0^+$. Moreover, $i(1_{B^+}) = 1_{H_1 \otimes B^-} = 1_{B^-}$. Thus the remaining assertions in (1) follow if we show that $\dim_{\mathbb{F}} B_{\theta_r^{-1}}^- = \dim_{\mathbb{F}} B_{\theta_r}^+$ for $\theta_r \in \mathcal{Q}_r^+$.

Before doing so, we first prove (2) using only the aforementioned $\mathcal{Q}_r^{\pm}$-root-space decomposition of $B^{\pm}$. Indeed, observe for $r = 0, 1$ that $b_{\theta_r} h_r = \theta_r(h_r) b_{\theta_r} \in m_A(H_r \otimes B^{\pm})$ whenever $h_r \in H_r, \theta_r \in \mathcal{Q}_r^{\pm} \subset \langle \mathcal{Q}_r^+ \rangle, b_{\theta_r} \in B_{\theta_r}^{\pm}$. Thus (2) follows from (RTA1).

We now complete the proof of (1), by showing that $\dim_{\mathbb{F}} B_{\theta_r^{-1}}^- = \dim_{\mathbb{F}} B_{\theta_r}^+$ for $\theta_r \in \mathcal{Q}_r^+$. First suppose $r = 0$, and fix an H_0-root-basis $b_1, \ldots, b_n$ of $B_{\theta_0}^+$ for fixed $\theta_0 \in \mathcal{Q}_0^+$. Also fix any finite-dimensional subspace $V \subset B_{\theta_0^{-1}}^-$ such that $i(b_j) \in m_A(H_1 \otimes V)$ for all j. Then using (2),

$$B_{\theta_0^{-1}}^- = i(i(B_{\theta_0^{-1}}^-)) \subset i(m_A(H_1 \otimes B_{\theta_0}^+)) \subset m_A(i(B_{\theta_0}^+) \otimes H_1) \subset m_A(H_1 \otimes V \otimes H_1) = m_A(H_1 \otimes V),$$

which shows (by (RTA1)) that $B_{\theta_0^{-1}}^- = V$ must be finite-dimensional for all $\theta_0 \in \mathcal{Q}_0^+$. Now fix r and $\theta_r \in \mathcal{Q}_r^+$, as well as bases $b_1, \ldots, b_n$ of $B_{\theta_r}^+$ and $v_1, \ldots, v_m$ of $B_{\theta_r^{-1}}^-$. Suppose $i(v_j) = \sum_k m_A(h_{jk} \otimes b_k)$ and $i(b_k) = \sum_k m_A(v_j \otimes t_{kj})$ for some choices of elements $h_{jk}, t_{kj} \in H_1$. Then the (possibly rectangular) matrices $H := (h_{jk}), T := (t_{kj})$ satisfy: HT, TH are identity matrices. Equating their traces yields $m = n$, i.e., $\dim_{\mathbb{F}} B_{\theta_r^{-1}}^- = \dim_{\mathbb{F}} B_{\theta_r}^+$ as claimed.

The remaining parts are easily shown: (3) is straightforward given (1) and Lemma 2.9, and (4) follows from (3). $\square$

2.1. **Hopf regular triangular algebras.** We now analyze regular triangular algebras in the special case when H_1 is a Hopf algebra, H_0 a Hopf subalgebra, and the Hopf structure is used to define an adjoint action with respect to which A is semisimple. This is in itself a very general setup that encompasses many well-studied examples in the literature, including Kac-Moody Lie algebras and their quantum groups. To proceed further, it is convenient to fix some notation.

Notation. Let H be a Hopf algebra (not necessarily commutative) over a field $\mathbb{F}$, and denote by m_H (or $\Delta_H, \eta_H, \varepsilon_H, S_H$) the multiplication in H (or comultiplication, unit, counit, antipode respectively) – see e.g. [Kas]. We will use Sweedler notation: $\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}$ for $h \in H$. Now note that $\widehat{H} \subset H^*$ is precisely the set of grouplike elements in H^*. Also define *convolution* on $\widehat{H}$, via $\langle \mu * \lambda, h \rangle := \langle \mu \otimes \lambda, \Delta_H(h) \rangle = \sum \langle \mu, h_{(1)} \rangle \langle \lambda, h_{(2)} \rangle$. Then ([Kas, Exercise III.8.11]) $(\widehat{H}, *)$ is a group, with unit ε_H, and inverse given by $\lambda \mapsto \lambda \circ S_H$ in $\widehat{H}$.

We now introduce a Hopf-theoretic framework that encompasses many well-known algebras in the literature, as we illustrate through examples later in the paper.

Definition 2.12. Suppose H is a Hopf algebra and A is an $\mathbb{F}$-algebra containing H. Define the *adjoint action* $\operatorname{ad} : H \to \operatorname{End}_{\mathbb{F}}(A)$ via: $(\operatorname{ad} h)(a) := \sum h_{(1)} a S(h_{(2)})$ for all $h \in H$ and $a \in A$.

Next, a *Hopf regular triangular algebra* (denoted also by *Hopf RTA*, or *HRTA* in short), is an $\mathbb{F}$-algebra A, together with the data $(B^{\pm}, H_1, H_0, \mathcal{Q}_0'^+, i)$ that satisfies (RTA1), (RTA3), and the following condition:

(HRTA2) H_1 is a Hopf algebra that contains a sub-Hopf algebra H_0. Moreover, there exists a monoid $\mathcal{Q}_0'^+ \subset \widehat{H_0}$ such that $\mathcal{Q}_0'^+ \setminus \{\varepsilon_{H_0}\}$ is a semigroup, which satisfies:

$$B^+ = \bigoplus_{\mu_1 \in \mathcal{Q}_1'^+} B_{\mu_1}^+, \qquad \mathcal{Q}_1'^+ := \pi_{H_0}^{-1}(\mathcal{Q}_0'^+) \subset \widehat{H_1},$$

where $\pi_{H_0} : \widehat{H_1} \to \widehat{H_0}$ is the restriction map, and $B_{\mu_1}^+$ is the μ_1-weight space for the adjoint action of H_1 on A. Furthermore, $B_{\varepsilon_{H_0}}^+ = \mathbb{F}$, and $\dim_{\mathbb{F}} B_{\mu_0}^+ < \infty$ for all $\mu_0 \in \mathcal{Q}_0'^+$.

Note that the definition of an HRTA is in some sense parallel to that of an RTA. However, the conditions (RTA2) and (HRTA2) are significantly different, in that the monoid $\mathcal{Q}_0'^+$ is contained in "weight space" $\widehat{H_0}$ and involves the adjoint action of H_0 on A, instead of being contained in "root space" $\text{Aut}_{\mathbb{F}-alg}(H_0)$ as in the RTA case. In fact, Definition 2.12 was primarily designed to incorporate Lie algebras as well as their quantum analogues into a common framework, and the properties of HRTAs were extensively studied in previous work [Kh2].[1] Thus the definition of an HRTA is *a priori* similar, but not related to the notion of an RTA. However, it turns out that the two are indeed closely related. To explain their precise connection, additional notation is required.

Definition 2.13. We say that an HRTA is *strict* if $H_1 = H_0$. An HRTA is *based* if there exists a $\mathbb{Z}$-linearly independent set of weights $\Delta' \subset \mathcal{Q}_0'^+$, such that $\mathcal{Q}_0'^+ = \mathbb{Z}^+ \Delta'$. In this case we may also denote the HRTA by $(B^{\pm}, H_1, H_0, \Delta', i)$. Finally, given a Hopf algebra H, define two maps:

- The *weight-to-root map* $\rho_H : \widehat{H} \to \text{End}_{\mathbb{F}}(H)$ is defined via: $\rho_H(\mu)(h) := \sum \mu^{-1}(h_{(1)})h_{(2)} = \sum \mu(S(h_{(1)}))h_{(2)}$.
- The *root-to-weight map* $\Psi_{\varepsilon} : \text{Aut}_{\mathbb{F}-alg}(H) \to H^*$ is defined via: $\Psi_{\varepsilon}(\theta) := \varepsilon \circ \theta^{-1}$.

It is now possible to relate HRTAs to RTAs (and to justify why we call such algebras Hopf *RTA*s).

Theorem 2.14. *Suppose A is an RTA over a ground field $\mathbb{F}$. Then A is an HRTA if and only if $H_1 \supset H_0$ are Hopf algebras and there exists a choice of parameters such that $\mathcal{Q}_r^+ \subset \text{im}(\rho_{H_r})$ for $r = 0, 1$. In this case, A is a (strict) (based) Hopf RTA if and only if A is a (strict) (based) RTA.*

The proof of Theorem 2.14 uses the following preliminary results.

Proposition 2.15. *Suppose H is a Hopf algebra and A is an $\mathbb{F}$-algebra containing H.*

(1) The root-to-weight map is a surjective group homomorphism $\Psi_{\varepsilon} : \text{Aut}_{\mathbb{F}-alg}(H) \to \widehat{H}$. It has right inverse equal to the weight-to-root map, which is an injective group homomorphism $\rho_H : \widehat{H} \to \text{Aut}_{\mathbb{F}-alg}(H)$.

(2) The assignments $H_0 \mapsto \widehat{H_0}$ and $H_0 \to \text{Hom}_{\mathbb{F}-alg}(H_0, H)$ are contravariant functors from the category of sub-Hopf algebras H_0 of H and injective Hopf maps, to the categories of groups and sets respectively. Moreover, the family of weight-to-root maps $\{\rho_{H_0} : H_0 \subset H\}$ constitute a natural transformation $: \widehat{\quad} \to \text{Hom}_{\mathbb{F}-alg}(-, H)$.

(3) $\text{im}(\rho_H)$ acts freely on $\widehat{H}$ via: $\rho_H(\mu)(\nu) = \mu * \nu$.

(4) $\text{ad} : H \to \text{End}_{\mathbb{F}}(A)$ is an $\mathbb{F}$-algebra homomorphism.

---

[1]The notion of an RTA was also defined, albeit "incorrectly", in [Kh2]. The reason it is not "correct" is that it is overly restrictive, requiring six technical axioms (besides the triangular decomposition and anti-involution) and yet not able to incorporate several of the settings considered in the present paper - including non-based settings as in Sections 4 and 5.2, as well as generalized Weyl algebras as in Sections 8, 9. However, the notion of a (based) Hopf RTA in [Kh2] essentially agrees with Definition 2.12 in the present paper.

(5) For all $\mu \in \widehat{H}$, the weight space A_μ (for the adjoint action of H on A) and the root space $A_{\rho_H(\mu)}$ (see Definition 2.1) coincide. In particular, $A_{\varepsilon_H} = A_{\mathrm{id}_H} = Z_A(H)$.

Part (1) says in particular that every Hopf algebra is a module over its weights. To our knowledge (and that of some experts) it seems, somewhat surprisingly, to be a new formulation (at least). Parts (2) and (3) follow easily by using the definitions. Parts (4),(5) are nice exercises in Sweedler notation; the last assertion in (5) can be found in e.g. [Jos, Lemma 1.3.3].

Proof. The proofs are not hard to show, and consist of computations involving Hopf algebras. Thus we omit the proofs for brevity. Note that the categorical statement in the second part follows by observing that the following square commutes, given Hopf algebras $H_0 \hookrightarrow H_1 \hookrightarrow H$:

$$
\begin{array}{ccc}
\widehat{H_1} & \xrightarrow{\ \rho_{H_1}\ } & \mathrm{Aut}_{\mathbb{F}-alg}(H_1) \cap (\pi'_{H_0})^{-1}(\mathrm{Aut}_{\mathbb{F}-alg}(H_0)) \\
{\scriptstyle \pi_{H_0}} \downarrow & & {\scriptstyle \pi'_{H_0}} \downarrow \\
\widehat{H_0} & \xrightarrow{\ \rho_{H_0}\ } & \mathrm{Aut}_{\mathbb{F}-alg}(H_0)
\end{array}
\tag{2.16}
$$

$\square$

It is now possible to show how HRTAs relate to RTAs.

Proof of Theorem 2.14. In proving the first assertion, we focus only on the conditions (RTA2) and (HRTA2). Suppose first that $H_1 \supset H_0$ are Hopf algebras and $\mathcal{Q}_r^+ \subset \mathrm{im}(\rho_{H_r})$ for $r = 0, 1$. Define $\mathcal{Q}_r'^+ := \rho_{H_r}^{-1}(\mathcal{Q}_r^+)$. Then $\mathcal{Q}_1'^+ = \pi_{H_0}^{-1}(\mathcal{Q}_0'^+)$ by (2.16). Moreover, if the root space $B_{\theta_1}^+ \neq 0$ for some $\theta_1 = \rho_{H_1}(\mu_1) \in \mathcal{Q}_1^+$, then $\pi'_{H_0}(\theta_1) = \rho_{H_0}(\pi_{H_0}(\mu_1)) \in \mathcal{Q}_0^+$ by (2.16). But then $\pi_{H_0}(\mu_1) \in \mathcal{Q}_0'^+$. This shows the decomposition in condition (HRTA2). That condition (HRTA2) holds now follows by using Proposition 2.15. Hence A is an HRTA.

Conversely, suppose A is an HRTA. Then $H_1 \supset H_0$ are clearly Hopf algebras. Now choose $\mathcal{Q}_r^+ \subset \mathrm{Aut}_{\mathbb{F}-alg}(H_r)$ to be $\rho_{H_r}(\mathcal{Q}_r'^+) \subset \mathrm{im}(\rho_{H_r})$ for $r = 0, 1$ (via Proposition 2.15). Moreover, Proposition 2.15 and the decomposition in condition (HRTA2) imply that the decomposition in (RTA2) holds as well.

Finally, the last assertion is easily verified, if we set $\Delta := \rho_{H_0}(\Delta')$ when A is a based HRTA. Note that since ρ_{H_0} is injective, the two possible notions of the rank of a strict, based HRTA coincide. $\square$

Remark 2.17. When $H_1 \supset H_0$ are Hopf algebras, Proposition 2.15(1) shows how the weight-to-root and root-to-weight maps help identify roots with weights. For general RTAs, the maps $\varepsilon, \Psi_\varepsilon, \rho_{H_r}$ need not exist, and so roots and weights necessarily lie in different spaces that need not be identifiable with one another.

3. The BGG Category $\mathcal{O}$

Having introduced the general framework of interest, the next step is to define and study the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$ for an RTA A. In this section we develop the theory of Category $\mathcal{O}$ for regular triangular algebras. The main results in this section are described in Section 3.3. Following the theory, in subsequent sections we discuss how the results in this section apply to a large number of examples, traditional as well as modern, classical as well as quantum.

Definition 3.1. Given an RTA A, the *BGG Category $\mathcal{O}$* is the full subcategory of all finitely generated H_1-semisimple A-modules with finite-dimensional H_1-weight spaces, on which B^+ acts locally finitely. (Henceforth by a weight space we mean an H_1-weight space, unless specified otherwise.)

3.1. **Verma modules; weights fixed by roots.** Category $\mathcal{O}$ was introduced by Bernstein, Gelfand, and Gelfand in their seminal paper [BGG] in the setting of complex semisimple Lie algebras. Since then, similar categories of modules have been studied in the literature in a wide variety of other settings, including Kac-Moody Lie algebras, quantum groups, and several other algebras with triangular decomposition. In studying $\mathcal{O}$ for these algebras, a common theme is to carefully examine the structure of a distinguished family of objects called Verma modules. We now introduce this and other notions in the general setting of regular triangular algebras.

Definition 3.2. A is an RTA.
  (1) Given $\lambda \in \widehat{H_1}$, the corresponding *Verma module* is $M(\lambda) := A/(A \cdot N^+ + A \cdot \ker \lambda)$.
  (2) The *Harish-Chandra projection* is $\xi : A = H_1 \oplus (N^- \cdot A + A \cdot N^+) \twoheadrightarrow H_1$.

We now begin to develop the theory of Category $\mathcal{O}$ via a careful study of Verma modules and related objects in $\mathcal{O}$. An attractive feature of our framework of RTAs is that it is robust enough that much of the "traditional" development of $\mathcal{O}$ in more classical settings goes through for RTAs as well. More precisely, several of the results in this section can be proved by adapting the arguments in [MP, Kh2] to RTAs. Thus, the proofs in this section will occasionally be omitted for brevity. This applies in particular to the following result.

Proposition 3.3. *Fix an RTA A and a weight $\lambda \in \widehat{H_1}$.*
  (1) Every submodule and quotient of an H_1-semisimple module M is also H_1-semisimple.
  (2) $M(\lambda)$ is an H_1-weight module generated by a one-dimensional subspace of its λ-weight space. It is a free rank one B^--module.
  (3) The center $Z(A)$ acts by a central character χ_λ on the Verma module $M(\lambda)$.
  (4) On $Z(A)$, the Harish-Chandra projection ξ is an algebra map that commutes with the anti-involution i (i.e., $\xi \circ i = \xi$), and $\chi_\lambda = \lambda \circ \xi$.

However, in the general setting of RTAs, one encounters certain technical issues involving Verma modules. More specifically, it is not always true that all Verma modules lie in Category $\mathcal{O}$. We now present such an example, which falls outside the traditional Hopf setting but is an RTA (and hence can be studied using the methods developed in this paper).

Example 3.4. Motivated by quantum algebras associated to Hecke R-matrices, Jing and Zhang [JZ] introduced and studied a family of noncommutative and non-cocommutative bialgebras that q-deform $U(\mathfrak{gl}_2)$. (These algebras were also studied later by Tang [Ta1] from the viewpoint of hyperbolic algebras.) More precisely, given $q \in \mathbb{F}^\times$ and char $\mathbb{F} \neq 2$, the algebra $U_q'(\mathfrak{gl}_2)$ is defined to be generated by u, d, h, a, with relations:

$$qhu - uh = 2u, \qquad hd - qdh = -2d, \qquad ud - qdu = a + h + \frac{1-q}{4}h^2,$$

where a is central. The algebra $U_q'(\mathfrak{sl}_2)$ is defined to be the quotient of $U_q'(\mathfrak{gl}_2)$ by the central ideal (a). Note that setting $q = 1$ yields the usual enveloping algebras of $\mathfrak{sl}_2$ and $\mathfrak{gl}_2$ respectively. Now it is not hard to show the following result.

Proposition 3.5. *Suppose $q \in \mathbb{F}^\times$ is not a root of unity, and char $\mathbb{F} \neq 2$. Then $U_q'(\mathfrak{gl}_2), U_q'(\mathfrak{sl}_2)$ are strict, based RTAs of rank one – but not Hopf RTAs – with $H_1 = H_0$ equal to $\mathbb{F}[a, h]$ and $\mathbb{F}[h]$ respectively, and*

$$B^- = \mathbb{F}[d], \quad B^+ = \mathbb{F}[u], \quad \Delta = \{\theta\}, \quad \theta(h) = qh - 2, \quad \theta(a) = a, \quad i(u) = d.$$

Moreover, $M(\lambda) \in \mathcal{O}$ if and only if $\lambda \neq -2/(1-q)$, and $M(-2/(1-q))$ is an H_1-weight module, with exactly one weight space of infinite dimension.

Proof. The only nontrivial property to check is (RTA1) – this is shown in far greater generality in Lemma 8.3. The remaining properties are easy to verify – e.g., that θ generates an infinite cyclic subgroup of $\mathrm{Aut}_{\mathbb{F}-alg}(H_0)$ follows from the fact that q is not a root of unity. $\square$

With this motivating example in mind, we introduce the following notation.

Definition 3.6. Suppose A is an RTA.

(1) Define $\widehat{H_0}^{free} := \{\lambda_0 \in \widehat{H_0} : \langle \mathcal{Q}_0^+ \rangle$ acts freely on $\lambda_0\}$ and $\widehat{H_1}^{free} := \pi_{H_0}^{-1}(\widehat{H_0}^{free})$. If A is a strict RTA, we will denote this common set by $\widehat{H}^{free}$.

(2) Define *partial orders* on the following four spaces:
   - Define $\geq_{\mathcal{Q}_0}$ on $\mathrm{Aut}_{\mathbb{F}-alg}(H_0)$ via: $\theta_0 \geq \theta_0'$ if there exists $\theta_0'' \in \mathcal{Q}_0^+$ such that $\theta_0 = \theta_0'' * \theta_0'$.
   - Define $\geq_{\mathcal{Q}_1}$ on $\mathrm{Aut}_{\mathbb{F}-alg}(H_1)$ via: $\theta_1 \geq \theta_1'$ if either $\pi_{H_0}(\theta_1) > \pi_{H_0}(\theta_1')$, or $\theta_1 = \theta_1'$.
   - Define $\geq_0$ on $\widehat{H_0}^{free}$ by: $\mu_0 \geq \mu_0'$ if there exists $\theta_0'' \in \mathcal{Q}_0^+$ such that $\mu_0 = \theta_0'' * \mu_0'$.
   - Define $\geq_1$ on $\widehat{H_1}^{free}$, via: $\mu_1 \geq \mu_1'$ if $\pi_{H_0}(\mu_1) > \pi_{H_0}(\mu_1')$ in $\widehat{H_0}$, or $\mu_1 = \mu_1'$ in $\widehat{H_1}$.

(3) A *maximal vector* of weight λ in an A-module M, is $m \in M_\lambda \cap \ker N^+$.

In the remainder of the paper, we will often use $\geq$ without specifying which of the four aforementioned partial orders is being used, when this is clear from context.

Remark 3.7. If A is an HRTA with $\mathcal{Q}_r^+ = \rho_{H_r}(\mathcal{Q}_r'^+)$ for $r = 0, 1$, then $\widehat{H_r}^{free} = \widehat{H_r}$ by Proposition 2.15(3). For this reason, in many examples in the literature (and below) one works with all of $\mathcal{O}$, since all Verma modules lie in $\mathcal{O}$.

In the rest of the paper, we work with Verma modules (and their quotients) with highest weights in the set $\widehat{H_1}^{free}$ – these modules are objects in $\mathcal{O}$. The following result summarizes the basic properties of Verma modules and their unique simple quotients.

Proposition 3.8. *Fix an RTA A and a weight $\lambda \in \widehat{H_1}^{free}$.*

(1) *The relations $\geq$ in $\widehat{H_r}$ are partial orders when restricted to $\widehat{H_r}^{free}$ for $r = 0, 1$. The map π_{H_0} is an order-preserving map when restricted to $\widehat{H_1}^{free}$.*

(2) *$M(\lambda)$ is an indecomposable object of $\mathcal{O}$, generated by its one-dimensional λ-weight space. All other weight spaces have weights $\mu < \lambda$ with $\mu \in \widehat{H_1}^{free}$.*

(3) *Every proper submodule of $M(\lambda)$ is H_1-semisimple and has zero λ-weight space.*

(4) *$M(\lambda)$ has a unique maximal submodule $\mathrm{Rad}\, M(\lambda)$, and a unique simple quotient $L(\lambda)$.*

(5) *$M(\lambda)$ is the "universal" cyclic module of highest weight λ.*

(6) *If $v \in M(\lambda)_\mu$ is maximal, then $\mu \leq \lambda$ in $\widehat{H}$, and $[M(\lambda) : L(\mu)] > 0$.*

(7) *The simple objects in $\mathcal{O}$ with at least one weight in $\widehat{H_1}^{free}$ are precisely $L(\lambda)$ for some $\lambda \in \widehat{H_1}^{free}$. All such modules are pairwise non-isomorphic.*

Proof. Most of the proofs are similar to those in [MP, Kh2], and are hence not included for brevity, except for the last part. In that part, fix a simple module V in $\mathcal{O}$ with nonzero weight space V_μ for some weight $\mu \in \widehat{H_1}^{free}$. Then the vector space $B^+ V_\mu$ is finite-dimensional and H_1-semisimple by the assumptions on $\mathcal{O}$. Thus it contains a weight vector v_λ of maximal H_1-weight λ in the partial order $\geq_{H_1}$. Since V is simple, it is generated by v_λ, whence $V \cong L(\lambda)$ by part (4). $\square$

Remark 3.9. It is also possible to introduce the *Shapovalov form* $Sh : A \times A \to H_1$ for a general RTA A, by defining: $Sh(x, y) := \xi(i(x)y)$ for $x, y \in A$. One verifies that the Shapovalov form satisfies the following properties for RTAs, which it satisfies for $A = U\mathfrak{g}$ for semisimple $\mathfrak{g}$: (a) The Shapovalov form is bilinear and symmetric. (b) $Sh(A_{\theta_r}, A_{\theta_r'}) = 0$ unless $\theta_r = \theta_r'$. (c) The Shapovalov form induces a symmetric bilinear form Sh_λ on every Verma module $M(\lambda)$ via: $Sh_\lambda(b_1 m_\lambda, b_2 m_\lambda) := \lambda(Sh(b_1, b_2))$ for $b_1, b_2 \in B^-$ and m_λ a nonzero highest weight vector in $M(\lambda)_\lambda$. (d) If $\lambda \in \widehat{H_1}^{free}$ then $\ker(Sh_\lambda) = \mathrm{Rad}(M(\lambda))$. (e) Given $\lambda \in \widehat{H_1}^{free}$ and $\theta_1 \in \mathrm{root}_{H_1}(B^-)$, consider the restriction of the form $Sh(-, -)$ to the root space $B_{\theta_1}^-$. Then if one applies λ to each entry of

the matrix of this bilinear form (with respect to any fixed basis of $B_{\theta_1}^-$), the resulting matrix has rank equal to $\dim L(\lambda)_{\theta_1 * \lambda}$.

Remark 3.10. Various other notions from the theory of semisimple Lie algebras also have analogues for general RTAs. For instance, the *Kostant partition function* has analogues $\mathcal{P}_r : \mathcal{Q}_r^+ \to \mathbb{Z}^+$ defined via $\mathcal{P}_r(\theta_r) := \dim_{\mathbb{F}} B_{\theta_r}^+$, for $r = 0, 1$. Next, *highest weight modules* $\mathbb{V}^\lambda$ are quotients of Verma modules $M(\lambda)$; if $\lambda \in \widehat{H_1}^{free}$ then $\mathbb{V}^\lambda \in \mathcal{O}$ since $\mathcal{O}$ is closed under quotienting.

Now suppose A is a strict, based RTA with a base of simple roots Δ of smallest possible size. One can then define the *height* of a "restricted root" $\theta_0 = \sum_{\theta \in \Delta} n_\theta \theta \in \mathbb{Z}\Delta = \langle \mathcal{Q}_0^+ \rangle \subset \mathrm{Aut}_{\mathbb{F}-alg}(H_0)$, to be $\mathrm{ht}(\theta_0) := \sum_{\theta \in \Delta} n_\theta$. Similarly, define *parabolic/Levi regular triangular subalgebras* as follows: for any subset $\Delta_0 \subset \Delta$, define $B_{\Delta_0}^\pm := \bigoplus_{\theta_1 \in (\pi'_{H_0})^{-1}(\mathbb{Z}\Delta_0)} B_{\theta_1}^\pm$, and

$$\mathfrak{P}_{\Delta_0}^\pm := B^\pm \otimes H_1 \otimes B_{\Delta_0}^\mp \quad \supset \quad \mathfrak{L}_{\Delta_0}^\pm := B_{\Delta_0}^\pm \otimes H_1 \otimes B_{\Delta_0}^\mp,$$

or more precisely, (the subalgebras of A generated by) their images under the multiplication map. Thus one can study notions such as *parabolic/generalized Verma modules*, as well as analogues of "parabolic" induction over based RTAs.

3.2. **Duality and extensions.** We next construct a duality functor on finite length objects in $\mathcal{O}$. In light of Proposition 3.8 and Example 3.4, henceforth we only work with objects in $\mathcal{O}$ whose weights lie in $\widehat{H_1}^{free}$. The following notation is required for this purpose.

Definition 3.11. Define $\mathcal{O}[\widehat{H_1}^{free}]$ and $\mathcal{O}_{\mathbb{N}}$ to respectively be the full subcategories of objects in $\mathcal{O}$ whose weights lie in $\widehat{H_1}^{free}$ and which are of finite length. Also define $\mathcal{O}_{\mathbb{N}}[\widehat{H_1}^{free}] := \mathcal{O}[\widehat{H_1}^{free}] \cap \mathcal{O}_{\mathbb{N}}$. Next, define the *formal character* of $M \in \mathcal{O}$ to be: $\mathrm{char}\, M := \sum_{\lambda \in \widehat{H_1}} \dim M_\lambda \cdot e^\lambda$, where e^λ is a formal variable for each $\lambda \in \widehat{H_1}$. Finally, given an object M in $\mathcal{O}$, use the anti-involution $i : A \to A$ to define its *restricted dual* $F(M) := \bigoplus_{\lambda \in \mathrm{wt}\, M} M_\lambda^*$, with A-module structure given by: $(am^*)(m) := m^*(i(a)m)$.

In particular, it follows from Proposition 3.8 that the simple objects in $\mathcal{O}[\widehat{H_1}^{free}]$ are parametrized by $\widehat{H_1}^{free}$. As discussed above, we work henceforth only in $\mathcal{O}[\widehat{H_1}^{free}]$; however, the next result holds in all of $\mathcal{O}$. The proof is as in the special case when $A = U\mathfrak{g}$ for semisimple $\mathfrak{g}$; see [MP, Kh2].

Proposition 3.12. $\mathcal{O}_{\mathbb{N}}$ *and* $\mathcal{O}_{\mathbb{N}}[\widehat{H_1}^{free}]$ *are abelian categories, and* $F : \mathcal{O}_{\mathbb{N}} \to \mathcal{O}_{\mathbb{N}}$ *is an exact, contravariant duality functor that sends each simple object* $L(\lambda)$ *for* $\lambda \in \widehat{H_1}^{free}$ *to itself. More generally,* F *preserves the length and formal character of all objects in* $\mathcal{O}_{\mathbb{N}}$ *and* $\mathcal{O}_{\mathbb{N}}[\widehat{H_1}^{free}]$ *respectively.*

The above results allow us to now consider *extensions*. A key result involves classifying all non-split objects in $\mathcal{O}[\widehat{H_1}^{free}]$ of length two.

Theorem 3.13. *Fix* $\lambda, \lambda' \in \widehat{H_1}^{free}$. *Then* $E(\lambda, \lambda') := \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\lambda'))$ *is nonzero if and only if* $\mathrm{Rad}\, M(\lambda) \twoheadrightarrow L(\lambda')$, *or* $\mathrm{Rad}\, M(\lambda') \twoheadrightarrow L(\lambda)$. *Moreover,* F *induces an isomorphism* $: E(\lambda, \lambda') \leftrightarrow E(\lambda', \lambda)$. *Finally,* $\mathrm{Ext}^1_{\mathcal{O}}(M, N)$ *is finite-dimensional for* $M, N \in \mathcal{O}_{\mathbb{N}}[\widehat{H_1}^{free}]$.

Proof. That F is an isomorphism $: E(\lambda, \lambda') \to E(\lambda', \lambda)$ follows from Proposition 3.12 (and standard arguments), since F is contravariant and exact. Now if $\mathrm{Rad}\, M(\lambda) \twoheadrightarrow L(\lambda')$ with kernel V, then

$$0 \to L(\lambda') = (\mathrm{Rad}\, M(\lambda)/V) \to M(\lambda)/V \to L(\lambda) \to 0,$$

and this is non-split, else $M(\lambda) \twoheadrightarrow M(\lambda)/V \twoheadrightarrow L(\lambda')$, whence $\lambda = \lambda'$ and $\dim M(\lambda)_\lambda \geq 2$, which is false. Conversely, suppose $0 \to L(\lambda') \to M \to L(\lambda) \to 0$ is nonsplit in $\mathcal{O}$, and let $v_\lambda \in L(\lambda)_\lambda, v_{\lambda'} \in L(\lambda')_{\lambda'}$ be nonzero highest weight vectors in the first and third terms of the short exact sequence

respectively. Also fix any lift $m_\lambda \in M_\lambda$ of v_λ, so that $N^+ m_\lambda \subset L(\lambda')$ (where N^+ is the augmentation ideal in B^+). There are three cases:

First if $\pi_{H_0}(\lambda) = \pi_{H_0}(\lambda') =: \lambda_0$, say, then by Proposition 3.8 M_{λ_0} is a two-dimensional H_0-weight space, spanned by $v_{\lambda'}$ and m_λ. Now $B^- m_\lambda$ is a nonzero submodule of M, and it has trivial intersection with the simple A-module $L(\lambda')$ since otherwise $v_{\lambda'} \in B^- m_\lambda$. Therefore the short exact sequence splits, which is impossible.

The second case is if $N^+ m_\lambda = 0$ and $\pi_{H_0}(\lambda) \neq \pi_{H_0}(\lambda')$. Then $M(\lambda) \twoheadrightarrow B^- m_\lambda \twoheadrightarrow L(\lambda)$, so if the extension is nonsplit then $B^- m_\lambda$ is not simple and hence $L(\lambda') \subset B^- m_\lambda$. But then $B^- m_\lambda$ has length 2, hence $M = B^- m_\lambda = M(\lambda)/V$, say. It follows that $L(\lambda') = (\operatorname{Rad} M(\lambda))/V$, proving the assertion. Furthermore, the nonsplit extension class is completely determined by θ and $b_- \in B_\theta^-$ such that $\theta * \lambda = \lambda'$ and $b_- m_\lambda = v_{\lambda'}$. Thus using condition (RTA2),

$$\dim \operatorname{Ext}^1_{\mathcal{O}}(L(\lambda), L(\lambda')) \leq \dim B^-_{\pi'_{H_0}(\theta)} < \infty.$$

Finally, suppose $0 \neq N^+ m_\lambda \subset L(\lambda')$, so that $\lambda < \lambda'$. In this case we use the duality functor F to reduce to the previous case. This proves the first two assertions of the theorem. The final assertion now follows by using Proposition 3.12 and standard homological arguments in $\mathcal{O}_{\mathbb{N}}[\widehat{H_1}^{free}]$. $\qquad \square$

3.3. **Blocks in $\mathcal{O}$, Conditions (S), and main results.** We now describe the two main results in this paper, on Category $\mathcal{O}$ over an arbitrary RTA. The results provide sufficient conditions under which a large subcategory of $\mathcal{O}$ – in fact of $\mathcal{O}[\widehat{H_1}^{free}]$ – acquires an increasing number of desirable homological properties. To state and prove these results requires the following notation.

Definition 3.14. Suppose A is an RTA.

(1) For each weight $\lambda \in \widehat{H_1}^{free}$, define the following four sets:
  - $S^4(\lambda) := \{\mu \in \widehat{H_1} : \chi_\mu \equiv \chi_\lambda \text{ on } Z(A)\}$ (where χ_λ denotes the central character defined in Proposition 3.3).
  - $S^3(\lambda)$ is the equivalence closure of $\{\lambda\}$ in $\widehat{H_1}$, under the relation:

    $$\mu \to \lambda \text{ if and only if } L(\mu) \text{ is a subquotient of } M(\lambda).$$

  - $S^2(\lambda) := \{\pi_{H_0}(\mu) : \mu \in S^3(\lambda)\}$.
  - $S^1(\lambda) := \{\pi_{H_0}(\mu) : \mu \in S^3(\lambda), \mu \leq \lambda\}$.

(2) For $1 \leq m \leq 4$, define

$$S^m(A) := \{\lambda \in \widehat{H_1}^{free} : S^m(\lambda) \text{ is finite}\} \subset \widehat{H_1}^{free}. \tag{3.15}$$

  (Note that $S^1(A), S^2(A) \subset \widehat{H_1}^{free}$ although $S^1(\lambda), S^2(\lambda) \subset \widehat{H_0}$.) We say that the algebra A satisfies *Condition (S1), (S2), (S3), or (S4)* if the corresponding set $S^m(A)$ equals $\widehat{H_1}^{free}$.

(3) Given $T \subset \widehat{H_1}^{free}$, define $\mathcal{O}[T]$ to be the full subcategory of $\mathcal{O}$, such that every simple subquotient of every object is of the form $L(\lambda)$ for some $\lambda \in T$. (This is consistent with the definition of $\mathcal{O}[\widehat{H_1}^{free}]$.) Now given $\lambda \in \widehat{H_1}^{free}$, define the corresponding *block* of $\mathcal{O}$ to be $\mathcal{O}[S^3(\lambda)]$.

The idea behind the conditions (S) is that each of them implies increasingly desirable homological and representation-theoretic properties for $\mathcal{O}$. (For instance, the sets $S^4(\lambda)$ are related to central characters, while $S^3(\lambda)$ are concerned with linkage.) Thus, in some sense $\mathcal{O}[S^m(A)]$ is *the part of Category $\mathcal{O}$ that satisfies these (desirable) properties*. This is made clearer in our "first main result", Theorem A below. In later sections, we show that a large number of well-explored settings in representation theory are all examples of RTAs, and explore whether or not these algebras satisfy the various Conditions (S). We are also motivated by settings such as [FeFr], in which it is often the

case that distinguished subcategories/sums of blocks in $\mathcal{O}$ are shown to have desirable properties or a tractable analysis.

Remark 3.16. The S-sets should not be confused with the antipode map on H in the event that H is a Hopf algebra. In fact we do not use the antipode in the remainder of the paper, except in Proposition 6.3.

In order to state our main results, we need a further piece of notation.

Definition 3.17. Suppose A is an RTA. Define

$$\overline{S^2}(A) := \{\lambda \in S^2(A) : \mu_0' \leq \pi_{H_0}(\mu) \leq \mu_0'' \text{ and } \mu_0', \mu_0'' \in S^2(\lambda) \implies \mu \in S^1(A)\}. \qquad (3.18)$$

Also say that an RTA is *discretely graded* if for all $\theta_0 \in \mathcal{Q}_0^+$, the interval $[0, \theta_0]$ (in the partial order $\leq$ on $\mathcal{Q}_0^+$) is finite.

Note that $\overline{S^2}(A)$ is precisely the set of weights $\lambda \in S^2(A)$ such that $\pi_{H_0}^{-1}([S^2(\lambda)]_{\leq}) \subset S^1(A)$, where $[T]_{\leq}$ denotes the closure of $T \subset \widehat{H_0}^{free}$ in the partial order induced by $\mathcal{Q}_0^+$.

Remark 3.19. The assumption of being discretely graded is satisfied by most algebras studied in the literature, which are moreover based with a finite set of simple roots. In fact based RTAs with an *infinite* base of simple roots are also discretely graded. There are other examples of non-based but discretely graded RTAs that arise from mathematical physics, such as generalized Heisenberg algebras for discrete, totally ordered groups. See Section 5.2 for more details.

We now discuss some results on the S-sets and the Conditions (S). First, the nomenclature is inspired by the "T"-properties of separation/Hausdorffness in topology, in the following sense.

Lemma 3.20. $S^3(\lambda) \subset S^4(\lambda) \cap (\langle \mathcal{Q}_1^+ \rangle * \lambda)$ *for all* $\lambda \in \widehat{H}$, *so* $S^4(A) \subset S^3(A) \subset S^2(A) \subset S^1(A)$. *Therefore the following implications hold among the Conditions (S):* $(S4) \Rightarrow (S3) \Rightarrow (S2) \Rightarrow (S1)$. *Moreover, if* $S^2(A) = \widehat{H_1}^{free}$ *then* $\overline{S^2}(A) = \widehat{H_1}^{free}$ *as well.*

Additionally (like the separation properties), the S^m-sets/conditions yield increasingly (in m) useful homological information about Category $\mathcal{O}$. The following is one of the two main results involving the S-sets for a general regular triangular algebra.

Theorem A. *Suppose A is a discretely graded (e.g. based) RTA.*
  (1) *$\mathcal{O}[S^1(A)]$ is finite length, and hence splits into a direct sum of blocks $\mathcal{O}[S^1(A) \cap S^3(\lambda)]$, each of which is abelian and self-dual.*
  (2) *Suppose $\lambda \in \overline{S^2}(A)$. Then the block $\mathcal{O}[S^3(\lambda)]$ is abelian and self-dual with enough projectives, each with a filtration whose subquotients are Verma modules.*
  (3) *Suppose $\lambda \in S^3(A) \cap \overline{S^2}(A)$. Then the block $\mathcal{O}[S^3(\lambda)]$ is equivalent to the category (Mod-$B_\lambda)^{fg}$ of finitely generated right modules over a finite-dimensional $\mathbb{F}$-algebra B_λ. Moreover, $\mathcal{O}[S^3(\lambda)]$ is a highest weight category; equivalently, the algebra B_λ is quasi-hereditary.*

In particular, if A satisfies condition (Sm) for some m, then the corresponding assertion above (numbered $\min(m, 3)$) holds for all of $\mathcal{O}[\widehat{H_1}^{free}]$. Thus if $(S3)$ holds, we obtain a block decomposition

$$\mathcal{O} = \mathcal{O}[\widehat{H_1} \setminus \widehat{H_1}^{free}] \oplus \bigoplus_{\lambda \in \widehat{H_1}^{free}/S^3} \mathcal{O}[S^3(\lambda)].$$

Remark 3.21. (For the definition of a highest weight category, see [CPS].) Thus, if A satisfies (S3), then Theorem A implies that each block $\mathcal{O}[S^3(\lambda)]$ has enough projectives (each filtered with Verma subquotients), finite cohomological dimension, *tilting modules* (i.e., modules simultaneously filtered in $\mathcal{O}$ by standard as well as costandard subquotients – see [Rin, Don]), and the property

of *BGG reciprocity.* These properties then transfer to all of $\mathcal{O}[\widehat{H_1}^{free}]$. Thus, Theorem A implies that the algebras B_λ are *BGG algebras* (see [Irv]). We do not discuss these results in great detail as they are homological properties valid in all highest weight categories; however, some of these results are stated in Theorem 3.27 to give the reader a flavor of highest weight categories. We also refer the interested reader to the comprehensive program developed by Cline, Parshall, and Scott for more on such categories.

We state our second main result about Category $\mathcal{O}$ and the Conditions (S) over regular triangular algebras: these constructions are all *functorial.*

Theorem B. *Suppose* $A_j = B_j^- \otimes H_{1j} \otimes B_j^+$ *(with $H_{1j} \supset H_{0j}$) is a (Hopf) RTA for $1 \le j \le n$.*

(1) *Then so is $A := \otimes_{j=1}^n A_j$. Moreover, A is strict and/or based (and discretely graded), if and only if so is A_j for all j.*

(2) *A module $V \in \mathcal{O}[\widehat{H_1}^{free}]$ is simple if and only if $V = \otimes_{j=1}^n V_j$, with V_j simple (and unique up to isomorphism) in $\mathcal{O}[\widehat{H_{1j}}^{free}]$ for all j.*

(3) *Each of the Conditions (S) holds for A if and only if it holds for all A_j. More generally, for $1 \le m \le 4$,*

$$S_A^m((\lambda_1, \ldots, \lambda_n)) = \times_{j=1}^n S_{A_j}^m(\lambda_j), \qquad S^m(A) = \times_{j=1}^n S^m(A_j), \qquad \forall 1 \le j \le n, \ \lambda_j \in \widehat{H_{1j}}^{free} \tag{3.22}$$

as subsets of $\widehat{H_r} = \times_{j=1}^n \widehat{H_{rj}}$ for suitable $r = 0, 1$. Furthermore, $\overline{S^2}(A) = \times_{j=1}^n \overline{S^2}(A_j)$.

(4) *Complete reducibility for finite-dimensional modules holds in $\mathcal{O}[\widehat{H_1}^{free}]$ if and only if it holds in $\mathcal{O}[\widehat{H_{1j}}^{free}]$ for all $1 \le j \le n$.*

In other words, it is possible to relate all of these notions for a tensor product $A = \otimes_{j=1}^n A_j$ of commuting RTAs A_i, with their counterparts for the individual tensor factors A_i. This is akin to (and more general than) relating representations of semisimple Lie algebras with those of the individual simple ideals. In fact, Theorem B provides a useful approach to take in studying Category $\mathcal{O}$ over newly introduced and studied classes of RTAs. For example, this was the approach adopted in [Zhi], where Zhixiang showed the algebra of interest to be a strict, based Hopf RTA of rank one that satisfies Condition (S4). See Example 7.4 for more details.

3.4. **Proofs of main results.** The remainder of this section is devoted to proving Theorems A and B. We will sketch those arguments which are along the lines of similar results in [MP, Kh2]; but we will spell out the details when illustrating how the more general structure of a (discretely graded) regular triangular algebra is used.

We begin with results in the spirit of the original paper [BGG], which help explicitly construct projective modules in Category $\mathcal{O}$ over discretely graded RTAs. To do so, we introduce the following notation.

Definition 3.23. Suppose A is an RTA. Given a subset $\Theta_0 \subset \mathcal{Q}_0^+$ and $\lambda \in \widehat{H_1}$, define

$$B_{\Theta_0+} := \sum_{\theta \in \mathcal{Q}_0^+, \ \theta \not\le \theta_0 \ \forall \theta_0 \in \Theta_0} B_\theta^+, \qquad P(\lambda, \Theta_0) := A/(A \cdot B_{\Theta_0+} + A \cdot \ker \lambda). \tag{3.24}$$

Also define $\mathcal{O}(\lambda, \Theta_0+)$ to be the full subcategory of $\mathcal{O}$ consisting of the objects M for which $B_{\Theta_0+} M_\lambda = 0$. Finally, an A-module M is said to have a *standard filtration* (respectively, a *highest weight filtration*) if M has a finite descending chain of A-submodules such that the successive quotients are Verma modules (respectively, quotients of Verma modules).

Note that if Θ_0 is finite, then $B_{\Theta_0+} = B_{\max(\Theta_0)+}$, where $\max(\Theta_0)$ denotes the $\le$-maximal elements of Θ_0. We now list some properties of the modules $P(\lambda, \Theta_0+)$ that are used to prove Theorem A.

Proposition 3.25. *Suppose A is a discretely graded RTA, $\Theta_0 \subset \mathcal{Q}_0^+$ is finite, and $\lambda \in \widehat{H_1}^{free}$.*

(1) *The subspace B_{Θ_0+} is a left ideal in B^+ of finite codimension. Moreover, $B_{\{id_{H_0}\},+} = N^+$ and $P(\lambda, \{id_{H_0}\}+) = M(\lambda)$.*

(2) *$P(\lambda, \Theta_0+)$ is an object of $\mathcal{O}(\lambda, \Theta_0+) \subset \mathcal{O}$. Moreover, $\operatorname{Hom}_{\mathcal{O}}(P(\lambda, \Theta_0+), M) = \dim M_\lambda$ for all objects M in $\mathcal{O}(\lambda, \Theta_0+)$.*

(3) *$P(\lambda, \Theta_0+)$ has a standard filtration in $\mathcal{O}$, and surjects onto $M(\lambda)$. If $\operatorname{root}_{H_1}(B^+/B_{\Theta_0+}) = S_{\Theta_0}$ as multisets, then the multiset of Verma subquotients of $P(\lambda, \Theta_0+)$ equals $\{M(\theta * \lambda) : \theta \in S_{\Theta_0}\}$.*

(4) *An H_1-semisimple module M is in $\mathcal{O}[\widehat{H_1}^{free}]$ if and only if M is a quotient of a finite direct sum of modules of the form $P(\lambda, \Theta_0+)$ for $\lambda \in \widehat{H_1}^{free}$, if and only if M has a highest weight filtration with highest weights in $\widehat{H_1}^{free}$.*

(5) *Given objects $M_1, M_2 \in \mathcal{O}[\widehat{H_1}^{free}]$, $M_1 \oplus M_2$ has a standard filtration if and only if each of M_1 and M_2 has a standard filtration.*

We omit the proof as the arguments in [BGG], [Don, Appendix A], and [Kh2] can be suitably modified to work for all discretely graded RTAs. Note that when the discretely graded RTA is $U\mathfrak{g}$ for semisimple $\mathfrak{g}$, we set $H_1 = H_0 := \mathfrak{h}^*$ and $\mathcal{Q}_0^+ := \mathbb{Z}^+\Delta = \mathbb{Z}^+\rho_{H_0}(\Delta')$ to lie in the simple root lattice, and work with the modules $P(\lambda, l) := P(\lambda, \Theta_l+)$ for $l \in \mathbb{Z}^+$, where $\Theta_l := \{\theta_0 \in \mathbb{Z}^+\Delta : \operatorname{ht}(\theta_0) = l\}$, with $\operatorname{ht}(\theta_0)$ defined in Remark 3.10. Indeed, this was the approach adopted in the seminal work [BGG] to explicitly construct projective objects in blocks of $\mathcal{O}$.

It is now possible to prove our first main theorem.

Proof of Theorem A. Along the way to showing the assertions, we prove some intermediate steps that are useful facts in their own right. The first claim is that part (1) already holds for $\mathcal{O}_\mathbb{N}$, i.e., for all $T \subset \widehat{H_1}^{free}$, $\mathcal{O}_\mathbb{N}[T]$ has a block decomposition:

$$\mathcal{O}_\mathbb{N}[T] = \bigoplus_{\lambda \in T/S^3} (\mathcal{O}_\mathbb{N}[T] \cap \mathcal{O}[S^3(\lambda)]) = \bigoplus_{\lambda \in T/S^3} \mathcal{O}_\mathbb{N}[T \cap S^3(\lambda)], \tag{3.26}$$

where we sum over distinct blocks, and where each summand is an abelian, finite-length, and self-dual Serre subcategory of $\mathcal{O}_\mathbb{N}[T] \subset \mathcal{O}_\mathbb{N}[\widehat{H_1}^{free}]$. Indeed, most of the claim follows by Proposition 3.12 and standard arguments, once we show the direct sum decomposition for all finite length objects in $\mathcal{O}_\mathbb{N}[T]$. That there are no morphisms or extensions between objects of distinct blocks follows from the same statement for *simple* objects of distinct blocks, by using Theorem 3.13 and the long exact sequence of $\operatorname{Ext}_{\mathcal{O}_\mathbb{N}}$s.

It remains to prove the direct sum decomposition of $\mathcal{O}_\mathbb{N}[T]$ into blocks. This is done by induction on the length l of the object in $\mathcal{O}_\mathbb{N}[T]$. For $l = 0, 1, 2$, the result is immediate or follows from Theorem 3.13. Now suppose the result holds for some object $N = \bigoplus_{\lambda \in T/S^3} N[S^3(\lambda)]$, and we have $0 \to N \to M \to L(\mu) \to 0$ for some $\mu \in T$. Now use the following general fact that holds in any abelian category $\mathscr{C}$: if $0 \to A \oplus B' \to C \to B'' \to 0$ and $\operatorname{Ext}^1_{\mathscr{C}}(B'', A) = 0$, then the sequence $0 \to A \to C \to C/A \to 0$ splits, and we have $0 \to B' \to C/A \to B'' \to 0$. Write $N = N' \oplus N[S^3(\mu)]$, and set $A := N', B' := N[S^3(\mu)], C := M, B'' := L(\mu)$. Applying the above general fact yields $M = N' \oplus M[S^3(\mu)]$, where $0 \to N[S^3(\mu)] \to M[S^3(\mu)] \to L(\mu) \to 0$; thus Equation (3.26) follows. We now prove the various parts of the theorem.

(1) Given $M \in \mathcal{O}[S^1(A)]$, observe by Proposition 3.25 that M has a highest weight filtration. Moreover, the corresponding highest weights $\lambda_1, \ldots, \lambda_k$ can be shown to lie in $S^1(A)$. Now for each $\mu_0 \in S^1(\lambda_j)$, by Proposition 3.8 there exists a unique $\theta_{j,\mu_0} \in -\mathcal{Q}_0^+$ such that

$\theta_{j,\mu_0} * \pi_{H_0}(\lambda_j) = \mu_0$. Thus,

$$l(M) \leq \sum_{j=1}^{k} l(M(\lambda_j)) \leq \sum_{j=1}^{k} \sum_{\mu_0 \in S^1(\lambda_j)} \dim_{\mathbb{F}} B^-_{\theta_{j,\mu_0}} < \infty,$$

since every simple subquotient of a Verma module $M(\lambda)$ is generated by (a lift of) its highest weight vector, whose H_0-weight lies in $S^1(\lambda)$. It follows that $\mathcal{O}[S^1(A)]$ is finite length. Now use the above analysis (before this first part) to complete the proof.

(2) We first introduce some notation. Fix $\lambda \in S^2(A)$, with $S^2(\lambda) = \{\lambda_1, \ldots, \lambda_k\}$. Given $\mu \in \widehat{H_1}^{free}$, define $\theta_j(\mu)$ to be the unique element of $\mathcal{Q}_0^+$ such that $\theta_j(\mu) * \pi_{H_0}(\mu) = \lambda_j$ if there exists such a $\theta_j(\mu) \in \langle \mathcal{Q}_0^+ \rangle$, else set $\theta_j(\mu) := 0 = \mathrm{id}_{H_0}$. Now define $\Theta_\mu := \bigcup_{j=1}^{k} [\mathrm{id}_{H_0}, \theta_j(\mu)]_{\leq}$, where $\leq$ is the partial order induced on $\widehat{H_0}^{free}$ by $\mathcal{Q}_0^+$. (Alternatively, we may define Θ_μ to be the set $\{\theta_j(\mu)\}$, discounting repetitions.) Note that Θ_μ is a finite subset for all $\mu \in \widehat{H_0}^{free}$ since $\mathcal{Q}_0^+$ is discretely graded.

We now prove the result. Suppose $\lambda \in \overline{S^2}(A)$ and $M \in \mathcal{O}[S^3(\lambda)]$. By Proposition 3.25(4), M is generated by the lifts to M of the highest weight vectors in each of its highest weight module subquotients. Each of these highest weights μ_l lies in $\pi_{H_0}^{-1}(S^2(\lambda))$; thus, M is generated by its H_0-weight spaces of weights λ_j for $1 \leq j \leq k$. It follows by Proposition 3.25 and the previous paragraph that $P_M := \bigoplus_{l=1}^{N} P(\mu_l, \Theta_{\mu_l}+) \twoheadrightarrow M$, where $\pi_{H_0}(\mu_l) \in \{\lambda_1, \ldots, \lambda_k\}$ $\forall l$. Now use Proposition 3.25(4) as well as the definition of $\overline{S^2}(A)$ to show that $\mathcal{O}[S^3(\lambda)] \subset \mathcal{O}(\mu_l, \Theta_{\mu_l}+) \cap \mathcal{O}[S^1(A)]$ for all $1 \leq l \leq N$. Moreover, $P(\mu_l, \Theta_{\mu_l}+)$ is an object of $\mathcal{O}[S^1(A)]$ by Proposition 3.25(3) and the definition of $\overline{S^2}(A)$. Denote its summand in the block $\mathcal{O}[S^1(A) \cap S^3(\lambda)] = \mathcal{O}[S^3(\lambda)]$ by P_l, say. Then $\mathrm{Hom}_{\mathcal{O}}(P_l, -)$ is exact in $\mathcal{O}[S^3(\lambda)]$ by Proposition 3.25(2), whence $\oplus_l P_l$ is projective in $\mathcal{O}[S^3(\lambda)]$ and surjects onto M. This shows that the block $\mathcal{O}[S^3(\lambda)]$ has enough projectives.

It remains to show that each indecomposable projective P in $\mathcal{O}[S^3(\lambda)]$ has a standard filtration. Since P is finite length, it has a simple quotient $L(\mu)$ for some $\mu \in S^3(\lambda)$. Now $P(\mu, \Theta_\mu+) \twoheadrightarrow L(\mu)$ from above, so its $\mathcal{O}[S^3(\lambda)]$-summand P_λ surjects onto $L(\mu)$. By universality, this surjection factors through a nonzero map $: P_\lambda \to P \twoheadrightarrow L(\mu)$. Now replace P_λ by some indecomposable (projective) summand $P' \in \mathcal{O}[S^3(\lambda)]$ to obtain nonzero maps $: (P' \leftrightarrow P) \twoheadrightarrow L(\mu)$. Then standard arguments involving Fitting's Lemma show that P, P' are both isomorphic to the projective cover in $\mathcal{O}[S^3(\lambda)]$ of $L(\mu)$. Since P' is a summand of $P(\mu, \Theta_\mu+)$ and $P \cong P'$, it follows by Proposition 3.25(5) that P has a standard filtration.

(3) Suppose $\lambda \in S^3(A) \cap \overline{S^2}(A)$. First note by Proposition 3.25(3) and the RTA axioms that in the set of Verma subquotients in any standard filtration of $P(\mu, \Theta_0+)$ (for any $\mu \in \widehat{H_1}^{free}$ and finite subset $\Theta_0 \subset \mathcal{Q}_0^+$), the multiplicity of $M(\mu)$ is always 1, and any Verma module with nonzero multiplicity is of the form $M(\theta * \mu)$ for some $\theta \in \mathrm{root}_{H_1}(B^+/B_{\Theta_0+})$. Hence the same applies to the projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{O}[S^3(\lambda)]$, for all $\lambda \in S^3(A) \cap \overline{S^2}(A)$.

Now continue the analysis in the previous part and recall that $\mathcal{O}[S^3(\lambda)]$ has only finitely many simple objects up to isomorphism. Thus, standard category-theoretic and homological arguments using Fitting's Lemma show that the set of indecomposable projectives in $\mathcal{O}[S^3(\lambda)]$ is precisely the set of projective covers $P(\mu)$ for $\mu \in S^3(\lambda)$ (and a dual statement holds for injective hulls as well). Now define $P := \bigoplus_{\mu \in S^3(\lambda)} P(\mu)^{\oplus n_\mu}$ for any choice of integers $n_\mu > 0$. Since $S^3(\lambda)$ is finite, $P \in \mathcal{O}[S^3(\lambda)]$ is a projective generator of the block $\mathcal{O}[S^3(\lambda)]$. Moreover, if $B_\lambda := \mathrm{End}_{\mathcal{O}}(P)$, then by standard computations in [Don, Appendix A] or [CPS], the functor $\mathrm{Hom}_{\mathcal{O}}(P, -) = \mathrm{Hom}_{\mathcal{O}[S^3(\lambda)]}(P, -)$ is an equivalence from $\mathcal{O}[S^3(\lambda)]$

into the category of finitely generated right B_λ-modules. That B_λ is finite-dimensional follows from a more general result:

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)], \qquad \forall M \in \mathcal{O}[S^1(A)], \ \lambda \in \overline{S^2}(A).$$

Now the first paragraph in this part implies that each block is a highest weight category. Finally, suppose A satisfies Condition (Sm) for some $1 \le m \le 4$. Then the corresponding assertion (numbered $\min(m,3)$) holds on $\mathcal{O}[\widehat{H_1}^{free}]$ because $S^m(A) = \widehat{H_1}^{free}$. $\qquad \square$

The proof of our second main result (Theorem B) is of a very different flavor. Before proceeding to this proof, we first write down some additional facts in order to give the reader a flavor of highest weight categories. More precisely, we list various desirable properties for the blocks of Category $\mathcal{O}$ over regular triangular algebras satisfying Condition (S3). See [Rin, Don, Kh2] for proofs.

Theorem 3.27. *Suppose A is an RTA, and $\lambda, \mu \in S^3(A) \cap \overline{S^2}(A)$. Then $\mathcal{A} := \mathcal{O}[S^3(A) \cap \overline{S^2}(A)]$ has the following properties:*

(1) *(BGG Reciprocity.) The multiplicity of $M(\mu)$ in any standard filtration of $P(\lambda)$ in $\mathcal{O}[S^3(\lambda)]$ (or in $\mathcal{A}$) equals the multiplicity of $L(\lambda)$ in any Jordan-Holder series for $M(\mu)$.*

(2) *(Neidhardt's theorem.) If B^- is an integral domain, then every nonzero map of Verma modules with highest weights in $\widehat{H_1}^{free}$ is an embedding. Moreover, $\mathrm{Hom}_{\mathcal{A}}(M(\mu), M(\lambda)) \ne 0$ if and only if $M(\lambda)$ has a subquotient $L(\mu)$.*

(3) $\mathrm{Ext}_{\mathcal{A}}^n(L(\lambda), L(\mu)) = 0$ *for all $n > 2|S^3(\lambda)|$. In particular, $\mathcal{O}[S^3(\lambda)]$ has finite cohomological (or global) dimension, bounded above by $2|S^3(\lambda)|$.*

(4) $\mathrm{Ext}_{\mathcal{A}}^n(M, N)$ *is finite-dimensional for all $n \ge 0$ and $M, N \in \mathcal{A}$.*

(5) *If $X, Y \in \mathcal{A}$ have standard filtrations, then*

$$\dim_{\mathbb{F}} \mathrm{Ext}_{\mathcal{A}}^n(X, F(Y)) = \begin{cases} \sum_{\lambda \in S^3(A) \cap \overline{S^2}(A)} [X : M(\lambda)][Y : M(\lambda)], & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.28}$$

In particular, if $\mathrm{Ext}_{\mathcal{A}}^n(M(\lambda), F(M(\mu)))$ is nonzero, then $n = 0$ and $\lambda = \mu$.

Note that some of the assertions hold more generally; moreover, if Condition (S3) holds, then $S^3(A) \cap \overline{S^2}(A) = \widehat{H_1}^{free}$ by Lemma 3.20.

We end this section with the proof of our second main result. The proof repeatedly uses the following standard result.

Lemma 3.29. *Given a ring R, every simple (sub)quotient of a direct sum of R-modules is automatically a simple (sub)quotient of some summand.*

Proof of Theorem B.

(1) This part involves some (relatively straightforward) bookkeeping. In particular, define

$$\mathcal{Q}_r^+ = \times_{j=1}^n \mathcal{Q}_{rj}^+, \qquad \Delta := \coprod_{j=1}^n \Delta_j, \qquad \Delta' := \coprod_{j=1}^n \Delta'_j, \qquad B^\pm := \otimes_{j=1}^n B_j^\pm, \qquad H_r = \otimes_{j=1}^n H_{rj}.$$

Then $\widehat{H_r} = \times_{j=1}^n \widehat{H_{rj}}$ for $r = 0, 1$; moreover, $(B_j^\pm)_{\theta_j} \subset B^\pm_{\mathrm{id}_{H_1}, \dots, \mathrm{id}_{H_{j-1}}, \theta_j, \mathrm{id}_{H_{j+1}}, \dots, \mathrm{id}_{H_n}}$ for all $1 \le j \le n$ and $\theta_j \in \mathrm{Aut}_{\mathbb{F}-alg}(H_j)$. Conversely if A is based, then defining $\Delta_j := \Delta \cap \mathcal{Q}_{0j}^+$ for all j shows that A_j is also based. The assertion about the equivalence of discrete gradings follows from the fact that $[\mathrm{id}_{H_0}, (\theta_j)_{j=1}^n] = \times_{j=1}^n [\mathrm{id}_{H_{0j}}, \theta_j]$.

(2) By Proposition 3.8(7), simple modules in $\mathcal{O}[\widehat{H_1}^{free}]$ are characterized by $\widehat{H_1}^{free}$. Now verify using the previous part that $\widehat{H_1}^{free} = \times_{j=1}^n \widehat{H_{1j}}^{free}$. Moreover, given $\lambda_j \in \widehat{H_{1j}}^{free}$ for all j, set $\lambda := (\lambda_1, \dots, \lambda_n) \in \widehat{H_1}^{free}$. Then $\otimes_j L_j(\lambda_j)$ is generated by its one-dimensional λ-weight

space, which is spanned by a maximal vector. Moreover, it is easily verified that $\otimes_j L_j(\lambda_j)$ is a simple highest weight A-module in $\mathcal{O}_A$, whence it is isomorphic to $L(\lambda)$. Finally, the uniqueness of the λ_j (given some $\lambda \in \widehat{H_1}^{free}$) follows because $L(\lambda)$ is isomorphic to a direct sum of copies of $L_j(\lambda_j)$ for any fixed j, so by Lemma 3.29, λ_j is uniquely determined from $L(\lambda)$ as well.

(3) We first claim that $S_A^3(\lambda) = \times_{j=1}^n S_{A_j}^3(\lambda_j)$, where $\lambda_j \in \widehat{H_{1j}}^{free}$ for all j and $\lambda = (\lambda_1, \dots, \lambda_n)$ as above. To do so, first note that $M(\lambda) = \otimes_{j=1}^n M_j(\lambda_j)$. Now if $M(\lambda_j)$ has a simple subquotient $L_j(\mu_j)$, then there exist submodules $N_j \subset M_j \subset M(\lambda_j)$ such that $M_j/N_j \cong L_j(\mu_j)$. But then by the previous part,

$$(\otimes_{j=1}^n M_j)/N \cong L(\mu) = \otimes_{j=1}^n L_j(\mu_j), \qquad N := \sum_{j=1}^n (N_j \otimes \otimes_{k \neq j} M_k).$$

Moreover, suppose (exactly) one of $[M(\lambda_j) : L(\mu_j)]$, $[M(\mu_j), L(\lambda_j)]$ is nonzero for each j. Then by the previous paragraph, the simple A-module $\otimes_{j=1}^n L_j(\min(\lambda_j, \mu_j))$ occurs as a subquotient of both $M(\lambda) = \otimes_{j=1}^n M_j(\lambda_j)$ and $M(\mu) = \otimes_{j=1}^n M_j(\mu_j)$. From this analysis it follows that $\times_{j=1}^n S_{A_j}^3(\lambda_j) \subset S_A^3(\lambda)$.

To prove the reverse inclusion, suppose $[M(\lambda) : L(\mu)] > 0$ in $\mathcal{O}[\widehat{H_1}^{free}]$. Fix $1 \leq j \leq n$, and consider both modules over their restriction to A_j. Thus (a direct sum of copies of) $L_j(\mu_j)$ occurs as a subquotient of a direct sum of copies of $M_j(\lambda_j)$. It follows using Lemma 3.29 that $[M_j(\lambda_j) : L_j(\mu_j)] > 0$ for all j. Now it is not hard to show that $S_A^3(\lambda) \subset \times_{j=1}^n S_{A_j}^3(\lambda_j)$.

In turn, applying $\pi_{H_0} = \times_{j=1}^n \pi_{H_{0,j}}$ shows that $S_A^2(\lambda) = \times_{j=1}^n S_{A_j}^2(\lambda_j)$. Moreover, the partial order on $\widehat{H_r}^{free}$ holds precisely when it holds in each component (i.e., $\widehat{H_{rj}}^{free}$). This implies that $S_A^1(\lambda) = \times_{j=1}^n S_{A_j}^1(\lambda_j)$. Finally, one verifies that $Z(A) = \otimes_{j=1}^n Z(A_j)$, which implies the assertion for the S^4-sets. The assertion involving $S^m(A)$ now follows easily. Finally, verify that

$$\begin{aligned} \overline{S^2}(A) &= \{\lambda = (\lambda_j) \in S^2(A) = \times_j S^2(A_j) : \pi_{H_0}^{-1}([S^2(\lambda)]_{\leq}) \subset S^1(A) = \times_j S^1(A_j)\} \\ &= \{\lambda = (\lambda_j) \in \times_j S^2(A_j) : \times_j \pi_{H_{0j}}^{-1}([S^2(\lambda_j)]_{\leq}) \subset \times_j S^1(A_j)\} \\ &= \times_{j=1}^n \{\lambda_j \in S^2(A_j) : \pi_{H_{0j}}^{-1}([S^2(\lambda_j)]_{\leq}) \subset S^1(A_j)\} = \times_{j=1}^n \overline{S^2}(A_j). \end{aligned}$$

(4) The proof is similar to that of [Kh2, Theorem 15.2] and is therefore omitted.

$\square$

4. Existence results for RTAs: semidirect product constructions, non-abelian root lattice

Having studied the structure of Category $\mathcal{O}$ over a general RTA, we turn to the construction and study of examples of RTAs. We begin by presenting examples of RTAs that are "completely different" from all examples considered to date in the literature, in the following sense: all previously studied RTAs have the property that the "root lattice" $\mathcal{Q}_0^+$ is abelian. In fact, with the exception of stratified Virasoro algebras (see Section 5.2), all previously studied RTAs are moreover based with a finite set of simple roots. Thus a natural question to ask (and whose answer is *a priori* not yet known) is: do there exist examples of RTAs for which $\mathcal{Q}_0^+$ is not abelian? In this section we provide a positive answer to this question, which further reinforces the generality of our framework.

Before constructing a concrete example of an RTA with non-abelian monoid $\mathcal{Q}_0^+$ of positive roots, we first introduce the following notation.

Definition 4.1. A monoid Q_0^+ is said to be a *regular triangular monoid (RTM)* if it satisfies the following properties:

(RTM1) $Q_0^+ \setminus \{1_{Q_0^+}\}$ is a semigroup, which generates a group $\langle Q_0^+ \rangle$.

(RTM2) There exists a left-action $\ltimes$ of Q_0^+ on $-Q_0^+$, which fixes $1_{Q_0^+}$ and satisfies the following "cocycle conditions":

$$\theta_1 \cdot \theta_2^{-1} = (\theta_1 \ltimes \theta_2^{-1}) \cdot (\theta_2 \ltimes \theta_1^{-1})^{-1}, \tag{4.2}$$

$$\theta_1 \ltimes (\theta_2^{-1} \cdot \theta_3^{-1}) = (\theta_1 \ltimes \theta_2^{-1}) \cdot ((\theta_2 \ltimes \theta_1^{-1})^{-1} \ltimes \theta_3^{-1}). \tag{4.3}$$

Now say that a monoid M acts *admissibly* on a regular triangular monoid Q_0^+ if there exists a monoid map $: M \to \mathrm{End}_{monoid}(\langle Q_0^+ \rangle) \bigcap \mathrm{End}_{monoid}(Q_0^+)$, such that

$$m(\theta_1 \ltimes \theta_2^{-1}) = m(\theta_1) \ltimes m(\theta_2)^{-1}, \qquad \forall m \in M, \ \theta_1, \theta_2 \in Q_0^+. \tag{4.4}$$

Remark 4.5. Note that the condition (RTM2) is equivalent to defining a *matched pairing* of the monoids Q_0^+ and $-Q_0^+$, as in [GM, Section 3]. This is because (RTM2) can be reformulated in terms of a right action $\rtimes$ of $-Q_0^+$ on Q_0^+, via similar looking "cocycle conditions" as (4.2),(4.3). The relationship between these two actions is: $\theta_1 \rtimes \theta_2^{-1} = (\theta_2 \ltimes \theta_1^{-1})^{-1}$. Further note that the first cocycle condition (4.2) is unchanged under taking inverses. Moreover as in [GM, Section 3], the matched pairing/RTM structure above shows that $\langle Q_0^+ \rangle = (-Q_0^+) \bowtie Q_0^+$, the bicrossed product of the two monoids. However, the matched pairing is not strong, because the multiplication map $: -Q_0^+ \times Q_0^+ \to \langle Q_0^+ \rangle$ is not a bijection unless Q_0^+ is a singleton.

For completeness, we also note that matched pairs of monoids were defined and studied by Gateva-Ivanova and Majid in connection with solutions of the Yang-Baxter Equation. More generally, an example of the bicrossed product of two Hopf algebras is the Drinfeld quantum double, which is a braided Hopf algebra and hence provides solutions of the Yang-Baxter Equation. There are further connections to Hopf algebras and Lie theory (see [GM, Kas] and their references), which we do not pursue further in this paper.

In this section we prove the following existence theorem for RTAs over RTMs. The theorem shows that RTAs with certain additional properties exist over monoids Q_0^+, if and only if these are regular triangular monoids:

Theorem 4.6 (RTM-RTA Correspondence). *Given a regular triangular monoid Q_0^+, there exists a strict RTA $A := \mathcal{A}(Q_0^+)$ such that (a) $\mathcal{Q}_0^+ = Q_0^+$; (b) $B_{\theta_0}^+ \neq 0 \ \forall \theta_0 \in \mathcal{Q}_0^+$; (c) $B^\pm$ do not contain zerodivisors; (d) the multiplication map $m_A : B^+ \otimes H_0 \otimes B^- \to A$ is also a vector space isomorphism; and (e) for each $\theta_1, \theta_2 \in \mathcal{Q}_0^+$, the image of $m_A(B_{\theta_1}^+ \otimes \mathbb{F}1_{H_0} \otimes B_{\theta_2^{-1}}^-)$ is contained in $B_{\theta_-}^- \otimes H_0 \otimes B_{\theta_+}^+$ for unique $\theta^\pm \in \pm\mathcal{Q}_0^+$.*

Conversely, if A is a strict RTA that satisfies (b)–(e), then $\mathcal{Q}_0^+$ is a regular triangular monoid.

Remark 4.7. Thus the notion of an RTM is intimately related to the notion of an RTA. In fact, RTMs provide a natural answer (via Theorem 4.6) to the question: "Given a (sufficiently nice) RTA, what structure can one expect from its underlying semigroup $\mathcal{Q}_0^+$?" Moreover, the existence result in Theorem 4.6 shows that the RTM/RTA frameworks are not "unnecessarily broad".

We prove Theorem 4.6 below; for now, we observe that Theorem 4.6 immediately shows the existence of RTAs over arbitrary abelian monoids Q_0^+:

Corollary 4.8. *Suppose Q_0^+ is a commutative monoid such that $Q_0^+ \setminus \{1_{Q_0^+}\}$ is a semigroup. Then there exists a strict RTA $A = \mathcal{A}(Q_0^+)$ such that $\mathcal{Q}_0^+ = Q_0^+$.*

Proof. Note that every commutative monoid Q_0^+ satisfying (RTM1) is a regular triangular monoid with $\theta_1 \ltimes \theta_2^{-1} := \theta_2^{-1}$ for all $\theta_1, \theta_2 \in Q_0^+$. Now apply Theorem 4.6. $\qquad\square$

This section is organized as follows. In Section 4.1 we prove Theorem 4.6. In Section 4.2, we then provide several recipes that yield RTMs (which in turn lead to RTA constructions). Finally, in Section 4.3 we carry out the aforementioned construction of an RTA A with non-abelian span of roots, and study its associated Category $\mathcal{O}$.

4.1. **Existence theorem for RTAs over regular triangular monoids.** This subsection is devoted to proving Theorem 4.6. We begin by showing a more general result.

Theorem 4.9 (RTA Existence Theorem). *Suppose Q_0^+ is a regular triangular monoid, and $\mathbb{Z}^k$ acts admissibly on $\langle Q_0^+ \rangle$ for some $k \in \mathbb{Z}^+$. Then for all $\mathbf{c} \in \mathbb{F}^k$, there exists a strict RTA $A = \mathcal{A}(Q_0^+, \mathbf{c})$ such that: (i) $\mathcal{Q}_0^+$ equals the regular triangular monoid $(\mathbb{Z}^+)^k \ltimes Q_0^+$; (ii) $\mathcal{A}(Q_0^+, \mathbf{c})$ satisfies properties (b)–(d) in Theorem 4.6, as well as property (e) for all $\theta_1, \theta_2 \in Q_0^+$; and (iii) there exist nonzero elements $\{x_r^\pm : 1 \leq r \leq k\}$ in $\mathcal{A}(Q_0^+, \mathbf{c})$ such that $[x_q^+, x_r^+] = [x_q^-, x_r^-] = 0$ and $[x_q^+, x_r^-] = \delta_{q,r} c_r$ for all $1 \leq q, r \leq k$.*

Proof. Suppose $\{\varepsilon_j : 1 \leq j \leq k\}$ denotes the standard $\mathbb{Z}$-basis of $\mathbb{Z}^k$. Define the $\mathbb{F}$-algebra $\mathcal{A}(Q_0^+, \mathbf{c})$ to be generated by the algebra $H_0 := \mathbb{F}[\mathbb{Z}^k \ltimes \langle Q_0^+ \rangle] = (\mathbb{F}(\mathbb{Z}^k \ltimes \langle Q_0^+ \rangle))^*$ of $\mathbb{F}$-valued functions on the group $\mathbb{Z}^k \ltimes \langle Q_0^+ \rangle$ (with coordinatewise addition and multiplication), together with elements $\{t^{\theta_0^{\pm 1}} : \theta_0 \in Q_0^+\}$ and $\{x_j^\pm : 1 \leq j \leq k\}$, modulo the following relations:

$$t^{\theta_1^{\pm 1}} t^{\theta_2^{\pm 1}} = t^{\theta_1^{\pm 1} \cdot \theta_2^{\pm 1}}, \qquad t^{1_{Q_0^+}} = 1, \qquad t^{\theta_1} t^{\theta_2^{-1}} = t^{\theta_1 \ltimes \theta_2^{-1}} t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}},$$

$$t^{\theta_1^{\pm 1}} f(-) = f((0, \theta_1^{\mp 1}) \cdot -) t^{\theta_1^{\pm 1}}, \qquad x_j^\pm f(-) = f((\mp \varepsilon_j, 0) \cdot -) x_j^\pm, \tag{4.10}$$

$$[x_j^\pm, x_{j'}^+] = [x_j^\pm, x_{j'}^-] = 0, \qquad [x_j^+, x_j^-] = c_j, \qquad x_j^\pm t^{\theta_0} = t^{\varepsilon_j^{\pm 1}(\theta_0)} x_j^\pm,$$

for all $1 \leq j \neq j' \leq k$, $\theta_1, \theta_2 \in Q_0^+$, $\theta_0 \in \langle Q_0^+ \rangle$, and $f : \mathbb{Z}^k \ltimes \langle Q_0^+ \rangle \to \mathbb{F}$ in H_0.

We now claim that $\mathcal{A}(Q_0^+, \mathbf{c})$ is a strict RTA satisfying the aforementioned properties. The meat of the proof lies in showing that the algebra $\mathcal{A}(Q_0^+, \mathbf{c})$ satisfies (RTA1). More precisely, we **claim** that the following holds:

The $\mathbb{F}$-algebra $\mathcal{A}(Q_0^+, \mathbf{c})$ satisfies (RTA1), with B^- having an $\mathbb{F}$-basis of the form

$$X_{irr}^- := \{ t^{\theta_0^{-1}} \prod_{j=1}^k (x_{k+1-j}^-)^{n_j} : n_j \in \mathbb{Z}^+, \theta_0 \in Q_0^+ \} \quad (\text{with } t^{1_{Q_0^+}} = 1), \tag{4.11}$$

and B^+ having an $\mathbb{F}$-basis $X_{irr}^+ := \{ \prod_{j=1}^k (x_j^+)^{n_j} \cdot t^{\theta_0} : n_j \in \mathbb{Z}^+, \theta_0 \in Q_0^+ \}$.

We show the claim after proving the remaining assertions. Given the above claim, the decomposition in (RTA2) also holds, with $\mathcal{Q}_0^+ = (\mathbb{Z}^+)^k \ltimes Q_0^+$. Moreover, every weight space of B^+ is one-dimensional, of the form $B_{(\mathbf{n}, \theta_0)}^+ = \mathbb{F} \prod_{j=1}^k (x_j^+)^{n_j} t^{\theta_0}$. This includes the space $B_{\mathrm{id}_{H_0}}^+ = B_{(\mathbf{0}, 1_{Q_0^+})}^+$.

Next, the symmetric form of the algebra relations (4.10) implies that (RTA3) also holds with the anti-involution sending t^{θ_0}, x_j^+ to $t^{\theta_0^{-1}}, x_j^-$ respectively, for all $\theta_0 \in Q_0^+$ and $1 \leq j \leq k$. (In fact, (RTA3) can be verified at the very outset, even before/without verifying (4.11).) Now property (i) from the statement is clear from the algebra relations and the above claim, except for the fact that $(\mathbb{Z}^+)^k \ltimes Q_0^+$ is an RTM. This last fact is proved more generally in Theorem 4.20(1),(4) below. Property (iii) is immediate from Equations (4.10),(4.11). To show property (ii), the algebra relations (4.10) yield:

$$m_A(B_{\theta_1}^+ \otimes H_0 \otimes B_{\theta_2^{-1}}^-) \subset B_{\theta_1 \ltimes \theta_2^{-1}}^- \otimes H_0 \otimes B_{(\theta_2 \ltimes \theta_1^{-1})^{-1}}^+, \qquad \forall \theta_1, \theta_2 \in Q_0^+.$$

Next, that $B^\pm$ do not contain zerodivisors follows since $B^\pm$ are H_0-root-semisimple, each root space is one-dimensional, and by Equation (4.10) and (RTA1) there exist no root vectors that are

zerodivisors. Finally, that the multiplication map : $B^+ \otimes H_0 \otimes B^- \to \mathcal{A}(Q_0^+, \mathbf{c})$ is a vector space isomorphism is proved similarly as the proof (below) of the claim (4.11).

It remains to show that (RTA1) holds, or more precisely, the claim (4.11) is true. For this we use the Diamond Lemma from [Be]. More precisely, Bergman has shown a variant for rings of the following result from graph theory: if a directed graph satisfies (a) the *descending chain condition* (every directed path has finite length) and (b) the *diamond condition* (any two distinct directed edges with common source extend to directed paths with common target), then every connected graph component has a unique "maximal" vertex. We now apply Bergman's result to $\mathcal{A}(Q_0^+, \mathbf{c})$, to prove the above claim (4.11) as follows:

- Let $\{h_i : i \in I\}$ be a fixed $\mathbb{F}$-basis of H_0 with $h_0 = 1_{H_0}$ for a fixed element $0 \in I$. Then a set of generators of $\mathcal{A}(Q_0^+, \mathbf{c})$ is given by:

$$X := \{x_j^{\pm} : 1 \leq j \leq k\} \coprod \{t^{\theta_0} : \theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}\} \coprod \{h_i : i \in I\}. \tag{4.12}$$

Then one has relations $h_q h_r = \sum_{s \in I} c_{q,r}^s h_s$ that encode the multiplication in H_0, as well as other relations that we rewrite for reasons explained below:

$$t^{\theta_1^{\pm 1}} t^{\theta_2^{\pm 1}} = t^{\theta_1^{\pm 1} \cdot \theta_2^{\pm 1}}, \qquad t^{\theta_1} t^{\theta_2^{-1}} = t^{\theta_1 \ltimes \theta_2^{-1}} t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}},$$

$$t^{\theta_1} h_i(-) = ((0, \theta_1)(h_i))(-) \cdot t^{\theta_1}, \qquad h_i(-) t^{\theta_1^{-1}} = t^{\theta_1^{-1}} \cdot ((0, \theta_1)(h_i))(-),$$

$$x_j^+ h_i(-) = ((\varepsilon_j, 1_{Q_0^+})(h_i))(-) \cdot x_j^+, \qquad h_i(-) x_j^- = x_j^- \cdot ((\varepsilon_j, 1_{Q_0^+})(h_i))(-), \tag{4.13}$$

$$x_{j_1}^+ x_{j_2}^- = x_{j_2}^- x_{j_1}^+, \qquad x_j^+ x_j^- = x_j^- x_j^+ + c_j, \qquad x_j^- x_{j'}^- = x_{j'}^- x_j^-, \qquad x_{j'}^+ x_j^+ = x_j^+ x_{j'}^+,$$

$$t^{\theta_1} x_j^{\pm} = x_j^{\pm} t^{\mp \varepsilon_j(\theta_1)}, \qquad x_j^{\pm} t^{\theta_1^{-1}} = t^{\pm \varepsilon_j(\theta_1^{-1})} x_j^{\pm}, \qquad h_0 x = x h_0 = x,$$

for all $i \in I$, $1 \leq j_1 \neq j_2 \leq k$, $1 \leq j < j' \leq k$, $\theta_1, \theta_2 \in Q_0^+ \setminus \{1_{Q_0^+}\}$, and $x \in X$. (Note that some of these relations are obtained from the presentation (4.10) of $\mathcal{A}(Q_0^+, \mathbf{c})$, via the fact that H_0 is a contragredient representation of the group $\mathbb{Z}^k \ltimes \langle Q_0^+ \rangle$.) We label the set of all these relations by the index set Σ_X.

- The next ingredient is to define a total order on $\langle X \rangle$. To do so, first define and fix a total order $\prec$ on the set of generators X as follows. Use the Axiom of Choice (more specifically, the Ultrafilter Theorem – or equivalently, the Compactness Theorem for first-order logic), to construct a total order $\prec$ on Q_0^+, which extends the partial order $\theta_0 \prec \theta_0' \cdot \theta_0 \ \forall \theta_0, \theta_0' \in Q_0^+$. Also fix a basis $\{h_i : i \in I\}$ of H_0 that includes $h_0 := 1_{H_0}$ (with $0 \in I$). Next, define a total ordering on I such that $0 < i \ \forall i \in I$. Now use the following total order on X:

$$h_0 \prec \quad t^{\theta_0^{-1}} \ (\downarrow \theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}) \quad \prec x_k^- \prec \cdots \prec x_1^- \prec \quad h_i \ (\uparrow i \in I \setminus \{0\}) \tag{4.14}$$

$$\prec x_1^+ \prec \cdots \prec x_k^+ \prec \quad t^{\theta_0} \ (\uparrow \theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}),$$

where t^{θ_0} ($\uparrow \theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}$) means that $\{t^{\theta_0} : \theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}\}$ inherits the total order from Q_0^+ via the map $\theta_0 \mapsto t^{\theta_0}$, and similarly in the other cases (where $\downarrow$ indicates order-reversing).

This order then extends to a semigroup partial order on the monoid $\langle X \rangle$ generated by X (which is a basis for the tensor algebra on the $\mathbb{F}$-span of X), which satisfies: if $x > x'$ in X then $AxB > Ax'B$ for all $A, B \in \langle X \rangle$. To do so, we in fact write down a *total order* on $\langle X \rangle$ as follows: compare two words by setting larger words to be greater, and via the lexicographic order induced by $\prec$ on two words of equal lengths.

- In place of directed edges in the graph-theoretic version of the Diamond Lemma, Bergman uses the algebra relations to work with *reductions*. Namely, we need to verify that every relation discussed above can be written as $w_\sigma \to f_\sigma \ \forall \sigma \in \Sigma_X$, with $w_\sigma \in \langle X \rangle$, $f_\sigma \in \mathbb{F}\langle X \rangle = T_{\mathbb{F}}(\text{span}_{\mathbb{F}} X)$, and such that every monomial in f_σ is strictly less than w_σ in the semigroup

partial (in fact total) order above. It is easy to verify that this procedure applies to every relation in (4.13) by replacing the equality by $\to$.

- The previous step verifies that the semigroup partial order on $\langle X \rangle$ is compatible with the reductions. This provides us with a directed path structure on the graph whose nodes are the monomials in $\langle X \rangle$. In this structure, "maximal" vertices are those from which no directed path starts, i.e., monomials that are left unchanged by every reduction. In other words, maximal vertices are precisely the "irreducible" monomials, given by $\{x^- \cdot h_i \cdot x^+ : x^\pm \in X_{irr}^\pm, i \in I\}$ (via Equation (4.11)).

- It remains to check that the descending chain condition (DCC) and the diamond condition are satisfied in our setting. A standard tool used to verify the DCC is a *misordering index* $\mathrm{mis}(w_\sigma)$, where $\mathrm{mis} : \langle X \rangle \to \mathbb{Z}^+$ is zero on all irreducible words, and we show that $w \succ w'$ in $\langle X \rangle$ implies $\mathrm{mis}(w) > \mathrm{mis}(w')$. Thus each reduction reduces the $\mathrm{mis}(\cdot)$-value, proving the DCC.

  In our setting, define $\mathrm{mis} : \langle X \rangle \to \mathbb{Z}^+$ via: $\mathrm{mis}(s_1 \cdots s_n) := T'_+ + T'_- + H' + N'$, where $T'_\pm, H'$ denote respectively the number of letters s_j of type $t^{\theta_0^{\pm 1}}, h_i$ for $\theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}$ and $i \in I$; and N denotes the number of pairs (j, j') such that: (a) $1 \le j < j' \le n$ and (b) $s_j \succ s_{j'}$ with not both $s_j, s_{j'}$ of the same type (T'_+, T'_-, or H'). We claim that every reduction of $w \in \langle X \rangle$ strictly decreases $\mathrm{mis}(w)$. This is not hard to verify using the presentation of the algebra $\mathcal{A}(Q_0^+, \mathbf{c})$ given in (4.13).

- Finally, we verify the diamond condition. By [Be], one only needs to work with directed paths of reductions starting from monomials; and one only needs to resolve "minimal ambiguities". Moreover, there are no "inclusion ambiguities" in our setting (i.e., for no σ, σ' is it true that $w_\sigma \in \langle X \rangle w_{\sigma'} \langle X \rangle$). Thus it suffices to show that the diamond condition holds for all overlap ambiguities ABC, where $AB = w_\sigma$ and $BC = w_{\sigma'}$ for some $\sigma, \sigma' \in \Sigma_X$. In the present case this involves computations with words of length precisely 3, in the $t^{\theta_0^{\pm 1}}, x_j^\pm, h_i$. Some of these computations are straightforward using the algebra relations and so we do not write them all down; for instance, overlap ambiguities involving all three alphabets being of the same "type" – t, or x, or h – are trivially resolved using the algebra relations and the cocycle conditions (4.2),(4.3). The other overlap ambiguities are also not hard to work out; for illustrative purposes we carry out a few of the verifications in the equations (4.15), using the RTM-axioms and the hypotheses of the theorem. We will also use 0 instead of $\mathbf{0}$ for the zero element in $\mathbb{Z}^k$.

$$
\begin{aligned}
(\theta_1 > 1_{Q_0^+}) \quad & (t^{\theta_1} x_j^+) h_i(-) \to \quad x_j^+(t^{(-\varepsilon_j)(\theta_1)} h_i(-)) \to (x_j^+(0, (-\varepsilon_j)(\theta_1))(h_i)(-)) t^{(-\varepsilon_j)(\theta_1)} \\
& \to ((\varepsilon_j, 1_{Q_0^+})(0, (-\varepsilon_j)(\theta_1))h_i)(-) \cdot x_j^+ t^{(-\varepsilon_j)(\theta_1)}, \\
& t^{\theta_1}(x_j^+ h_i(-)) \to \quad (t^{\theta_1}((\varepsilon_j, 1_{Q_0^+})h_i)(-)) x_j^+ \to ((0, \theta_1)(\varepsilon_j, 1_{Q_0^+})h_i)(-) \cdot (t^{\theta_1} x_j^+) \\
& \to ((0, \theta_1)(\varepsilon_j, 1_{Q_0^+})h_i)(-) \cdot x_j^+ t^{(-\varepsilon_j)(\theta_1)}.
\end{aligned}
$$

$$
\begin{aligned}
(\theta_1 > 1_{Q_0^+}) \quad & (t^{\theta_1} h_i(-)) x_j^- \to \quad ((0, \theta_1)h_i)(-) \cdot (t^{\theta_1} x_j^-) \to (((0, \theta_1)h_i)(-) x_j^-) t^{\varepsilon_j(\theta_1)} \\
& \to x_j^- \cdot ((\varepsilon_j, 1_{Q_0^+})(0, \theta_1)h_i)(-) \cdot t^{\varepsilon_j(\theta_1)}, \\
& t^{\theta_1}(h_i(-) x_j^-) \to (t^{\theta_1} x_j^-)((\varepsilon_j, 1_{Q_0^+})h_i)(-) \to x_j^-(t^{\varepsilon_j(\theta_1)}((\varepsilon_j, 1_{Q_0^+})h_i)(-)) \\
& \to x_j^- \cdot ((0, \varepsilon_j(\theta_1))(\varepsilon_j, 1_{Q_0^+})h_i)(-) \cdot t^{\varepsilon_j(\theta_1)}.
\end{aligned}
$$

$$
\begin{aligned}
(\theta_l > 1_{Q_0^+}) \quad & (t^{\theta_1} h_i(-)) t^{\theta_2^{-1}} \to \quad ((0, \theta_1)h_i)(-)(t^{\theta_1} t^{\theta_2^{-1}}) \to (((0, \theta_1)h_i)(-) t^{\theta_1 \ltimes \theta_2^{-1}}) t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}} \\
& \to t^{\theta_1 \ltimes \theta_2^{-1}} \cdot ((0, \theta_1 \ltimes \theta_2^{-1})^{-1}(0, \theta_1)h_i)(-) \cdot t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}},
\end{aligned}
$$

(4.15)

$$t^{\theta_1}(h_i(-)t^{\theta_2^{-1}}) \to \quad (t^{\theta_1}t^{\theta_2^{-1}})((0,\theta_2)h_i)(-) \to t^{\theta_1 \ltimes \theta_2^{-1}}(t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}}((0,\theta_2)h_i)(-))$$

$$\to t^{\theta_1 \ltimes \theta_2^{-1}} \cdot ((0,(\theta_2 \ltimes \theta_1^{-1})^{-1})(0,\theta_2)h_i)(-) \cdot t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}}.$$

$(\theta_l > 1_{Q_0^+})$ $(t^{\theta_1}t^{\theta_2})x_j^{\pm} \to t^{\theta_1 \theta_2}x_j^{\pm} \to x_j^{\pm}t^{(\mp\varepsilon_j)(\theta_1\theta_2)},$

$$t^{\theta_1}(t^{\theta_2}x_j^{\pm}) \to (t^{\theta_1}x_j^{\pm})t^{(\mp\varepsilon_j)(\theta_2)} \to x_j^{\pm}(t^{(\mp\varepsilon_j)(\theta_1)}t^{(\mp\varepsilon_j)(\theta_2)}) \to x_j^{\pm}t^{(\mp\varepsilon_j)(\theta_1\theta_2)}.$$

$(\theta_l > 1_{Q_0^+})$ $(t^{\theta_1}x_j^{\pm})t^{\theta_2^{-1}} \to x_j^{\pm}(t^{(\mp\varepsilon_j)(\theta_1)}t^{\theta_2^{-1}}) \to (x_j^{\pm}t^{(\mp\varepsilon_j)(\theta_1)\ltimes\theta_2^{-1}})t^{(\theta_2\ltimes(\mp\varepsilon_j)(\theta_1^{-1}))^{-1}}$

$$\to t^{\theta_1\ltimes(\pm\varepsilon_j)(\theta_2^{-1})}x_j^{\pm}t^{(\theta_2\ltimes(\mp\varepsilon_j)(\theta_1^{-1}))^{-1}},$$

$$t^{\theta_1}(x_j^{\pm}t^{\theta_2^{-1}}) \to (t^{\theta_1}t^{(\pm\varepsilon_j)(\theta_2^{-1})})x_j^{\pm} \to t^{\theta_1\ltimes(\pm\varepsilon_j)(\theta_2^{-1})} \cdot (t^{(((\pm\varepsilon_j)(\theta_2^{-1})^{-1})\ltimes\theta_1^{-1})^{-1}}x_j^{\pm})$$

$$= t^{\theta_1\ltimes(\pm\varepsilon_j)(\theta_2^{-1})}(t^{((\pm\varepsilon_j)(\theta_2)\ltimes\theta_1^{-1})^{-1}}x_j^{\pm}) \to t^{\theta_1\ltimes(\pm\varepsilon_j)(\theta_2^{-1})}x_j^{\pm}t^{(\mp\varepsilon_j)((\pm\varepsilon_j)(\theta_2)\ltimes\theta_1^{-1})^{-1}}$$

$$= t^{\theta_1\ltimes(\pm\varepsilon_j)(\theta_2^{-1})}x_j^{\pm}t^{(\theta_2\ltimes(\mp\varepsilon_j)(\theta_1^{-1}))^{-1}}.$$

$(\theta_1 > 1_{Q_0^+})$ $(t^{\theta_1}x_j^+)x_l^- \to x_j^+(t^{(-\varepsilon_j)(\theta_1)}x_l^-) \to (x_j^+x_l^-)t^{(\varepsilon_l-\varepsilon_j)(\theta_1)} \to (x_l^-x_j^+ + \delta_{jl}c_j)t^{(\varepsilon_l-\varepsilon_j)(\theta_1)},$

$$t^{\theta_1}(x_j^+x_l^-) \to (t^{\theta_1}x_l^-)x_j^+ + \delta_{jl}c_jt^{\theta_1} \to x_l^-(t^{\varepsilon_l(\theta_1)}x_j^+) + \delta_{jl}c_jt^{\theta_1}$$

$$\to x_l^-x_j^+t^{(\varepsilon_l-\varepsilon_j)(\theta_1)} + \delta_{jl}c_jt^{\theta_1}.$$

$(i \in I)$ $(x_j^+h_i(-))x_l^- \to \quad ((\varepsilon_j,1_{Q_0^+})h_i)(-)(x_j^+x_l^-) \to ((\varepsilon_j,1_{Q_0^+})h_i)(-)(x_l^-x_j^+ + \delta_{jl}c_j)$

$$\to x_l^- \cdot ((\varepsilon_j+\varepsilon_l,1_{Q_0^+})h_i)(-) \cdot x_j^+ + \delta_{jl}c_j \cdot ((\varepsilon_j,1_{Q_0^+})h_i)(-),$$

$$x_j^+(h_i(-)x_l^-) \to \quad (x_j^+x_l^-)((\varepsilon_l,1_{Q_0^+})h_i)(-) \to (x_l^-x_j^+ + \delta_{jl}c_j)((\varepsilon_l,1_{Q_0^+})h_i)(-)$$

$$\to x_l^- \cdot ((\varepsilon_j+\varepsilon_l,1_{Q_0^+})h_i)(-) \cdot x_j^+ + \delta_{jl}c_j \cdot ((\varepsilon_l,1_{Q_0^+})h_i)(-).$$

$(j > l)$ $(x_j^+x_l^+)x_i^- \to x_l^+(x_j^+x_i^-) \to x_l^+(x_i^-x_j^+ + \delta_{ij}c_j) \to x_i^-x_l^+x_j^+ + \delta_{il}c_lx_j^+ + \delta_{ij}c_jx_l^+,$

$$x_j^+(x_l^+x_i^-) \to \quad x_j^+(x_i^-x_l^+ + \delta_{il}c_l) \to x_i^-(x_j^+x_l^+) + \delta_{ij}c_jx_l^+ + \delta_{il}c_lx_j^+.$$

$(\theta_1 > 1_{Q_0^+})$ $(t^{\theta_1}h_q(-))h_r(-) \to ((0,\theta_1)h_q)(-)(t^{\theta_1}h_r(-)) \to ((0,\theta_1)h_q)(-) \cdot ((0,\theta_1)h_r)(-)t^{\theta_1},$

$$t^{\theta_1}(h_q(-)h_r(-)) \to \quad \sum_s c_{q,r}^s t^{\theta_1}h_s(-) \to \sum_s c_{q,r}^s((0,\theta_1)h_s)(-) \cdot t^{\theta_1}.$$

Both computations in the last reduction yield the same quantity because H_0 is a module-algebra over the group $\mathbb{Z}^k \ltimes \langle Q_0^+ \rangle$ (via its contragredient representation), and this imposes a compatibility constraint on the structure constants for H_0. Further note that several overlap ambiguities that are not listed in (4.15) can be resolved without any further computation, by applying the anti-involution $i : \mathcal{A}(Q_0^+, \mathbf{c}) \to \mathcal{A}(Q_0^+, \mathbf{c})$ from (RTA3). For instance, the first overlap ambiguity resolved in (4.15) implies that the overlap ambiguity $h_i(-)x_j^- t^{\theta_1^{-1}}$ can also be resolved for $\theta_1 \in Q_0^+ \setminus \{1_{Q_0^+}\}$. $\qquad \square$

Remark 4.16. Our construction of the algebra $\mathcal{A}(Q_0^+, \mathbf{c})$ can be thought as a generalization of continuous Cherednik algebras (see [EGG]). Note that the algebras $B^{\pm}$ are "dual" to one another in some sense; however, they need not be polynomial algebras as in [EGG].

Remark 4.17. In light of the algebra relations (4.10) in $\mathcal{A}(Q_0^+, \mathbf{c})$, the first cocycle condition (4.2) can be thought of as a group/monoid version of a so-called "straightening identity" in the flavor of Garland [Gar], Beck-Chari-Pressley [BCP], and several other works in the literature. For more on the subject, see [BC] and the references therein.

Equipped with the above theorem, we now prove our initial result in this section.

Proof of Theorem 4.6. The meat of the proof lies in proving the existence of an RTA $\mathcal{A}(Q_0^+)$ over an arbitrary RTM Q_0^+; but this is the special case of Theorem 4.9 where $k = 0$. Conversely, suppose

there exists a strict RTA A satisfying properties (b)–(e). Define $\theta_1 \ltimes \theta_2^{-1}, \theta_1 \rtimes \theta_2^{-1}$ via:

$$m_A(B_{\theta_1}^+ \otimes \mathbb{F} \cdot 1_{H_0} \otimes B_{\theta_2^{-1}}^-) \subset B_{\theta_1 \ltimes \theta_2^{-1}}^- \otimes H_0 \otimes B_{\theta_1 \rtimes \theta_2^{-1}}^+.$$

Note here that both sides are nonzero subspaces of A. It follows by considering their H_0-roots that $\ltimes$ is an action of $\mathcal{Q}_0^+$ on $-\mathcal{Q}_0^+$, and that $\theta_1 \cdot \theta_2^{-1} = (\theta_1 \ltimes \theta_2^{-1}) \cdot (\theta_1 \rtimes \theta_2^{-1})$. Now applying the anti-involution i to the above subspace (via Lemma 2.9) and again considering the H_0-roots, we obtain (via a slight abuse of notation):

$$(\theta_1 \rtimes \theta_2^{-1})^{-1} = i(\theta_1 \rtimes \theta_2^{-1}) = i(\theta_2^{-1}) \ltimes i(\theta_1) = \theta_2 \ltimes \theta_1^{-1}, \qquad \forall \theta_1, \theta_2 \in \mathcal{Q}_0^+.$$

This shows the first cocycle condition (4.2) for $\mathcal{Q}_0^+$. Next, consider $\theta_1, \theta_2, \theta_3 \in \mathcal{Q}_0^+$, and compute $m_A(B_{\theta_1}^+ \otimes B_{\theta_2^{-1}}^- \otimes B_{\theta_3^{-1}}^-)$ in two ways by using the associativity of m_A and properties (b)–(e). This is an easy computation that yields the second cocycle condition (4.3) for $\mathcal{Q}_0^+$. $\qquad\square$

We end this subsection with two further results on the algebras $\mathcal{A}(Q_0^+)$. The first discusses Casimir operators.

Proposition 4.18. *Fix a regular triangular monoid Q_0^+ and a subset $\mathcal{Q}^- \subset -Q_0^+$ such that $\theta_1 \ltimes -$ is a bijection on $\mathcal{Q}^-$ for all $\theta_1 \in Q_0^+$.*

(1) Then a suitable completion of the algebra $\mathcal{A}(Q_0^+)$ contains a central "Casimir" operator
$$\Omega(\mathcal{Q}^-) := \sum_{\theta_0 \in \mathcal{Q}^-} t^{\theta_0} t^{\theta_0^{-1}}.$$

(2) The operators $\Omega(\mathcal{Q}^-)$ act on all objects in $\mathcal{O}[\widehat{H_0}^{free}]$. They kill every highest weight module in $\mathcal{O}$, hence act nilpotently on $\mathcal{O}[\widehat{H_0}^{free}]$.

Examples of subsets $\mathcal{Q}^-$ include any subset of $-Q_0^+$ when Q_0^+ is abelian; as well as $\mathcal{Q}^- = \{1_{Q_0^+}\}$, which corresponds to $\Omega(\mathcal{Q}^-) = 1$.

Proof. For the first part, observe using the first cocycle condition (4.2) that $(\theta_2 \ltimes \theta_1^{-1})^{-1} \cdot \theta_2 = (\theta_1 \ltimes \theta_2^{-1})^{-1} \cdot \theta_1$, for all $\theta_1, \theta_2 \in Q_0^+$. Now fix $\theta_2 \in -\mathcal{Q}^-$ and compute using the algebra relations:

$$t^{\theta_1} \cdot t^{\theta_2^{-1}} t^{\theta_2} = t^{\theta_1 \ltimes \theta_2^{-1}} t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}} t^{\theta_2} = t^{\theta_1 \ltimes \theta_2^{-1}} t^{(\theta_1 \ltimes \theta_2^{-1})^{-1}} t^{\theta_1}.$$

By the assumptions on $\mathcal{Q}^-$, it follows that t^{θ_1} commutes with $\Omega(\mathcal{Q}^-)$ for all $\theta_1 \in Q_0^+$. In turn, this implies (using the anti-involution $i : t^{\theta_0} \leftrightarrow t^{\theta_0^{-1}}$ on $\mathcal{A}(Q_0^+)$) that $\Omega(\mathcal{Q}^-)$ is central.

To prove the second part, first observe as in the Kac-Moody setting, that the "Casimir" operator $\Omega(\mathcal{Q}^-)$ acts on arbitrary objects of Category $\mathcal{O}[\widehat{H_0}^{free}]$. Moreover, $\Omega(\mathcal{Q}^-)$ kills the highest vector in any highest weight module in the respective Categories $\mathcal{O}$, since the Harish-Chandra projection to H_0 kills all such operators. It follows that these central elements annihilate the entire module. The final assertion now follows from Proposition 3.25(4). $\qquad\square$

We also discuss the Conditions (S) for the algebras $\mathcal{A}(Q_0^+)$.

Proposition 4.19. *Suppose Q_0^+ is a nontrivial regular triangular monoid, whose action $\ltimes$ on $-Q_0^+$ stabilizes $-Q_0^+ \setminus \{1_{Q_0^+}\}$. Then the algebra $\mathcal{A}(Q_0^+)$ satisfies none of the Conditions (S), because $S^3(\lambda) = \langle Q_0^+ \rangle * \lambda \ \forall \lambda \in \widehat{H_0}^{free}$, so that $\dim L(\lambda) = 1$.*

Note that $\widehat{H_0}^{free}$ is nonempty because $\langle Q_0^+ \rangle \subset \widehat{H_0}^{free}$.

Proof. Given $\lambda \in \widehat{H_0}^{free}$, we first claim that every nonzero weight vector $t^{\theta_2^{-1}} m_\lambda$ of the Verma module $M(\lambda)$ is maximal (for $\theta_2 \in Q_0^+$). To show the claim, compute using the assumptions and the algebra relations (4.10), for $\theta_1 \in Q_0^+ \setminus \{1_{Q_0^+}\}$:

$$t^{\theta_1} \cdot t^{\theta_2^{-1}} m_\lambda = t^{\theta_1 \ltimes \theta_2^{-1}} t^{(\theta_2 \ltimes \theta_1^{-1})^{-1}} m_\lambda \in t^{\theta_1 \ltimes \theta_2^{-1}} \cdot N^+ m_\lambda = 0.$$

This proves the claim. In particular, $N^- m_\lambda \subset M(\lambda)$ is a codimension 1 submodule, whence $\mathbb{F} \cong M(\lambda)/N^- m_\lambda \twoheadrightarrow L(\lambda)$. Therefore $\dim L(\lambda) = 1$. Finally, recall by the first cocycle condition (4.2) that an arbitrary element $\theta \in \langle Q_0^+ \rangle$ can be written as $\theta = \theta_+ \theta_-$, where $\theta_+, \theta_-^{-1} \in Q_0^+$. Now note using the above claim:

$$[M(\lambda) : L(\theta_- * \lambda)] > 0, \qquad [M(\theta * \lambda) : L(\theta_+^{-1} * (\theta * \lambda))] > 0 \quad \implies \quad \theta * \lambda \in S^3(\lambda) \ \forall \theta \in \langle Q_0^+ \rangle.$$

This proves the statement about $S^3(\lambda)$ since $S^3(\lambda) \subset \langle \mathcal{Q}_0^+ \rangle * \lambda$ for every RTA and all $\lambda \in \widehat{H_0}^{free}$. Moreover, $\lambda = \pi_{H_0}(\lambda) > \theta_0^{-n} * \lambda$ for all $n \in \mathbb{N}$ and $\theta_0 \in Q_0^+ \setminus \{1_{Q_0^+}\}$, so that $|S^1(\lambda)| = \infty \ \forall \lambda \in \widehat{H_0}^{free}$. This concludes the proof. $\qquad \square$

4.2. **Examples of regular triangular monoids.** Having proved the RTM-RTA Correspondence (Theorem 4.6) and the more general RTA Existence Theorem 4.9, we now describe several recipes to construct examples of RTMs, which in turn admit RTA constructions.

Theorem 4.20. *Suppose $k \in \mathbb{Z}^+$, and for each $j = 0, \ldots, k$, Q_j^+ is a monoid contained in a group $\langle Q_j^+ \rangle$ such that $Q_j^+ \setminus \{1_{Q_j^+}\}$ is a semigroup.*

> (1) *If Q_0^+ is abelian, then it is an RTM with $\theta_1 \ltimes \theta_2^{-1} = \theta_2^{-1}$ for $\theta_1, \theta_2 \in Q_0^+$.*
> (2) *If all Q_j^+ are RTMs, then so is $\times_{j=0}^k Q_j^+$.*
> (3) *Suppose $k = 0$ and Q_0^+ is an RTM. Suppose Q_0^+ contains a submonoid Q^+ whose action $\ltimes$ on $-Q_0^+$ stabilizes $-Q^+$. Then Q^+ is also an RTM.*
> (4) *Suppose $k = 1$, Q_0^+, Q_1^+ are RTMs, and $\langle Q_0^+ \rangle$ acts admissibly on Q_1^+. Then $Q_0^+ \cdot Q_1^+$ is an RTM (where $\langle Q_0^+ \rangle \cdot \langle Q_1^+ \rangle$ denotes the semidirect product group).*

Note that parts (1),(4) are used in the proof of Theorem 4.9.

Proof. The first two parts are easily verified; note for the second part that we define
$$(\theta_j^+)_{j=0}^k \ltimes (\theta_j^-)_{j=0}^k := (\theta_j^+ \ltimes \theta_j^-)_{j=0}^k, \qquad \forall \theta_j^\pm \in \pm Q_j^+.$$
To prove the third part, note that $\langle Q^+ \rangle \subset \langle Q_0^+ \rangle$ is a group; moreover, the cocycle conditions (4.2),(4.3) hold in $-Q^+$ because they hold in $-Q_0^+$.

It remains to prove the last part; for this we will write every element of $\langle Q_0^+ \rangle \cdot \langle Q_1^+ \rangle$ as $(\theta_1, \theta_0) = \theta_1 \cdot \theta_0$, with $\theta_j \in \langle Q_j^+ \rangle$ for $j = 0, 1$. That $Q_0^+ \cdot Q_1^+$ satisfies (RTM1) is not hard to verify, so we only verify here that (RTM2) holds. For this, define
$$(\theta_1^+, \theta_0^+) \ltimes (\theta_1^-, \theta_0^-) := (\theta_1^+ \ltimes \theta_0^+(\theta_1^-), \theta_0^+ \ltimes \theta_0^-), \qquad \forall \theta_j^\pm \in \pm Q_j^+.$$
Note that this is a natural definition to propose, given the actions $\ltimes$ of Q_j^+ on $-Q_j^+$ for $j = 0, 1$ and the semidirect product structure of $Q_0^+ \cdot Q_1^+$. Now compute:
$$(\nu_1^+, \nu_0^+) \ltimes ((\theta_1^+, \theta_0^+) \ltimes (\theta_1^-, \theta_0^-)) = (\nu_1^+, \nu_0^+) \ltimes (\theta_1^+ \ltimes \theta_0^+(\theta_1^-), \theta_0^+ \ltimes \theta_0^-)$$
$$= (\nu_1^+ \ltimes \nu_0^+(\theta_1^+ \ltimes \theta_0^+(\theta_1^-)), \nu_0^+ \ltimes (\theta_0^+ \ltimes \theta_0^-)),$$
$$((\nu_1^+, \nu_0^+) \cdot (\theta_1^+, \theta_0^+)) \ltimes (\theta_1^-, \theta_0^-) = ((\nu_1^+ \nu_0^+(\theta_1^+)) \ltimes (\nu_0^+ \theta_0^+)(\theta_1^-), (\nu_0^+ \theta_0^+) \ltimes \theta_0^-).$$
Using the admissibility of the Q_0^+-action on Q_1^+, as well as the actions $\ltimes$ of Q_j^+ on $-Q_j^+$ for $j = 0, 1$, it follows that both of the above quantities are equal. Therefore $\ltimes$ is indeed an action of $Q_0^+ \cdot Q_1^+$ on $(-Q_0^+) \cdot (-Q_1^+)$. The action fixes $(1_{Q_1^+}, 1_{Q_0^+})$ because Q_0^+, Q_1^+ are both RTMs.

It remains to verify the two cocycle conditions (4.2),(4.3). In what follows, denote $a \rtimes b := (b^{-1} \ltimes a^{-1})^{-1}$ for suitable a, b. Now to show the first cocycle condition for $Q_0^+ \cdot Q_1^+$, we compute:
$$(\theta_1^+, \theta_0^+) \cdot (\theta_1^-, \theta_0^-) = \theta_1^+ \theta_0^+ \theta_1^- \theta_0^- = \theta_1^+ \theta_0^+(\theta_1^-) \cdot (\theta_0^+ \ltimes \theta_0^-)(\theta_0^+ \rtimes \theta_0^-) \tag{4.21}$$
$$= (\theta_1^+ \ltimes \theta_0^+(\theta_1^-)) \cdot (\theta_0^+ \ltimes \theta_0^-) \cdot (\theta_0^+ \ltimes \theta_0^-)^{-1}(\theta_1^+ \rtimes \theta_0^+(\theta_1^-)) \cdot (\theta_0^+ \rtimes \theta_0^-).$$

Thus it suffices to prove that

$$\left((\theta_1^-,\theta_0^-)^{-1} \ltimes (\theta_1^+,\theta_0^+)^{-1}\right) \cdot (\theta_0^+ \ltimes \theta_0^-)^{-1}(\theta_1^+ \rtimes \theta_0^+(\theta_1^-)) \cdot (\theta_0^+ \rtimes \theta_0^-) = (1_{Q_1^+}, 1_{Q_0^+}) = 1,$$

i.e., that

$$\left((\theta_0^-)^{-1}(\theta_1^-)^{-1} \ltimes (\theta_0^+\theta_0^-)^{-1}(\theta_1^+)^{-1}, (\theta_0^-)^{-1} \ltimes (\theta_0^+)^{-1}\right) \cdot ((\theta_0^+ \ltimes \theta_0^-)^{-1}(\theta_1^+ \rtimes \theta_0^+(\theta_1^-)), (\theta_0^+ \rtimes \theta_0^-)) = 1,$$

i.e., that (using the first cocycle condition (4.2) for Q_0^+):

$$(\theta_0^-)^{-1}(\theta_1^-)^{-1} \ltimes (\theta_0^+\theta_0^-)^{-1}(\theta_1^+)^{-1} \cdot \left((\theta_0^-)^{-1} \ltimes (\theta_0^+)^{-1}\right)\left[(\theta_0^+ \ltimes \theta_0^-)^{-1}(\theta_1^+ \rtimes \theta_0^+(\theta_1^-))\right] = (1_{Q_1^+}, 1_{Q_0^+}) = 1.$$

But now the first cocycle condition (4.2) and action on Q_1^+ for Q_0^+ show that the second factor on the left-hand side equals $(\theta_0^+\theta_0^-)^{-1}\left(\theta_1^+ \rtimes \theta_0^+(\theta_1^-)\right)$. Similarly, the admissibility of the Q_0^+-action on Q_1^+ shows that the first factor on the left-hand side equals $(\theta_0^+\theta_0^-)^{-1}\left(\theta_0^+(\theta_1^-)^{-1} \ltimes (\theta_1^+)^{-1}\right)$. Multiplying these two factors, we are now done by using the first cocycle condition for Q_1^+.

Similarly, the second cocycle condition is verified as follows, using (4.21) and the cocycle conditions for Q_j^+:

$$\left((\theta_1^+,\theta_0^+) \ltimes (\theta_1^-,\theta_0^-)\right) \cdot \left(((\theta_1^+,\theta_0^+) \rtimes (\theta_1^-,\theta_0^-)) \ltimes (\nu_1^-,\nu_0^-)\right) = a \cdot b \cdot \left[(b^{-1}(c),d) \ltimes (\nu_1^-,\nu_0^-)\right],$$

where $a := \theta_1^+ \ltimes \theta_0^+(\theta_1^-)$, $b := \theta_0^+ \ltimes \theta_0^-$, $c := \theta_1^+ \rtimes \theta_0^+(\theta_1^-)$, and $d := \theta_0^+ \rtimes \theta_0^-$. In turn, this expression equals

$$= a \cdot b \cdot (b^{-1}(c) \ltimes d(\nu_1^-)) \cdot (d \ltimes \nu_0^-) = a \cdot (c \ltimes (bd)(\nu_1^-)) \cdot b \cdot (d \ltimes \nu_0^-).$$

Recall by the cocycle condition for Q_0^+ that $bd = \theta_0^+\theta_0^-$. Now we compute the other side of the second cocycle condition:

$$(\theta_1^+,\theta_0^+) \ltimes ((\theta_1^-,\theta_0^-) \cdot (\nu_1^-,\nu_0^-)) = (\theta_1^+ \ltimes \theta_0^+(\theta_1^- \cdot \theta_0^-(\nu_1^-)), \theta_0^+ \ltimes (\theta_0^-\nu_0^-))$$
$$= \left((\theta_1^+ \ltimes \theta_0^+(\theta_1^-)) \cdot \left[(\theta_1^+ \rtimes (\theta_0^+(\theta_1^-))) \ltimes (\theta_0^+(\theta_0^-(\nu_1^-)))\right], (\theta_0^+ \ltimes \theta_0^-) \cdot ((\theta_0^+ \rtimes \theta_0^-) \ltimes \nu_0^-)\right)$$
$$= a \cdot (c \ltimes (bd)(\nu_1^-)) \cdot b \cdot (d \ltimes \nu_0^-),$$

and this proves the second cocycle condition, as desired. $\qquad\square$

We now describe an application of Theorem 4.20(1),(4), which yields a natural class of solvable examples of RTMs:

Corollary 4.22. *Suppose G is a group with abelian subgroups $G_0, \ldots, G_n$ such that:*
- *G_j acts on G_k for $0 \le j \le k \le n$ by group homomorphisms, in a compatible manner such that $G = (\cdots((G_n \rtimes G_{n-1}) \rtimes G_{n-2}) \rtimes \cdots G_1) \rtimes G_0$; and*
- *For every $0 \le k \le n$, G_k contains a sub-monoid G_k^+ stable under the action of $(\cdots((G_k \rtimes G_{k-1}) \rtimes G_{k-2}) \rtimes \cdots G_1) \rtimes G_0$, such that $G_k^+ \setminus \{1_{G_k^+}\}$ is a semigroup that generates G_k.*

Then $G^+ := (\cdots((G_n^+ \rtimes G_{n-1}^+) \rtimes G_{n-2}^+) \rtimes \cdots G_1^+) \rtimes G_0^+$ is a regular triangular monoid.

Note that Corollary 4.8 is a particular special case with $n = 0$. Similarly, if $n = 1$ then this result implies Theorem 4.20(4) when G_1^+ is an abelian RTM with the usual (trivial) $\ltimes$-action.

Proof. The proof is by induction on n. For $n = 0$ the result follows from Theorem 4.20(1). Now suppose the result holds for $n - 1$, whence $M^+ := (\cdots((G_n^+ \rtimes G_{n-1}^+) \rtimes G_{n-2}^+) \rtimes \cdots G_1^+)$ is an RTM. Define the action map $\ltimes$ of G^+ on $-G^+$ as follows:

$$(g_n^+, \ldots, g_0^+) \ltimes (g_n^-, \ldots, g_0^-) := \left((g_{n-1}^+ \cdots g_0^+)(g_n^-), \ldots, (g_1^+ g_0^+)(g_2^-), g_0^+(g_1^-), g_0^-\right), \qquad (4.23)$$

where $g_k^\pm \in \pm G_k^+$ for $0 \le k \le n$. It is now a straightforward calculation to verify that G_0 acts admissibly on the regular triangular monoid M^+, whence we are done by induction via Theorem 4.20(4). $\qquad\square$

Remark 4.24. Corollary 4.22 holds for all groups that can be expressed as a tower of semidirect products. Clearly such groups G include all abelian groups; each such group G is solvable; and if all G_k are finitely generated, then G is polycyclic. A natural question to explore is if every solvable group with a given set of abelian Jordan-Holder factors generated by RTMs, is also generated by an RTM.

4.3. **Non-based example with non-abelian span of roots.** We conclude this section by studying an RTA $\mathcal{A}(Q_0^+, \mathbf{c})$ (constructed in the RTA Existence Theorem 4.9) for a specific non-abelian monoid Q_0^+, as well as its Category $\mathcal{O}$.

Fix $k \in \mathbb{N}$, $\boldsymbol{\zeta} \in (0, \infty)^k$, $\mathbf{c} \in \mathbb{F}^k$, and a nontrivial additive subgroup $\mathbb{E} \subset (\mathbb{R}, +)$ such that $\zeta_j^{\pm 1} \mathbb{E} \subset \mathbb{E}$ for $1 \leq j \leq k$. Then $\mathbb{E} \cap [0, \infty)$ is an abelian RTM with the trivial $\ltimes$-action on $\mathbb{E} \cap (-\infty, 0]$. Set $\boldsymbol{\zeta}^{\mathbf{n}} := \prod_{j=1}^k \zeta_j^{n_j}$ and $\mathbf{n}(e) := \boldsymbol{\zeta}^{\mathbf{n}} e$ for $\mathbf{n} \in \mathbb{Z}^k$ and $e \in \mathbb{E}$. Then $\mathbb{Z}^k$ acts admissibly on $\mathbb{E}$ (by Corollary 4.22), which allows us to define the $\mathbb{F}$-algebra $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c}) := \mathcal{A}((\mathbb{Z}^+)^k \ltimes_{\boldsymbol{\zeta}} (\mathbb{E} \cap [0, \infty)), \mathbf{c})$ as in the proof of Theorem 4.9. Here we use $\ltimes_{\boldsymbol{\zeta}}$ to denote the semidirect product of the groups $\mathbb{Z}^k$ and $\mathbb{E}$, in order to differentiate it from the RTM action $\ltimes$.

Theorem 4.25. *Fix $k \in \mathbb{N}$, $\boldsymbol{\zeta} \in (0, \infty)^n$, $\mathbb{E} \subset \mathbb{R}$, and $\mathbf{c} \in \mathbb{F}^k$ as above.*
  (1) The algebra $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ is a strict RTA with $\mathcal{Q}_0^+ = (\mathbb{Z}^+)^k \ltimes_{\boldsymbol{\zeta}} (\mathbb{E} \cap [0, \infty))$.
  (2) The algebra $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ is based if and only if it is discretely graded, if and only if $\mathbb{E} = \eta \mathbb{Z}$ for some $\eta \in \mathbb{R}^\times$ and $\zeta_j = 1$ for all j.

Thus, to our knowledge the algebras $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ with $\boldsymbol{\zeta} \neq (1, \ldots, 1)$ provide the first explicitly constructed examples of regular triangular algebras with non-abelian group of roots $\langle \mathcal{Q}_0^+ \rangle$. These algebras cannot be studied by using existing theories of Category $\mathcal{O}$ in the literature (e.g. as in [H2, Kh2, MP]), because the "root lattice" is not abelian. In fact the monoid $\mathcal{Q}_0^+$ is abelian if and only if $\zeta_j = 1$ for all j.

Proof. The first part follows directly from Theorem 4.9. To show the second part, note that if $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ is based then it is discretely graded. In turn, this implies that the interval $[(0,0), (0,e)]$ is finite for every $0 < e \in \mathbb{E} \subset \mathbb{R}$. Thus $\mathbb{E}$ is a lattice $\eta \mathbb{Z}$ for $\eta \neq 0$, which contains $\zeta_j^{\mathbb{Z}} \eta \mathbb{Z}$ for all j. It follows that $\zeta_j = 1 \; \forall j$. Finally, if $\mathbb{E} = \eta \mathbb{Z}$ and $\zeta_j = 1 \; \forall j$, then A is indeed based with $\Delta := \{\varepsilon_1, \ldots, \varepsilon_k, (0, \eta)\}$. $\qquad \square$

We now study Category $\mathcal{O}$ over the algebra $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$, including computing the center and its action on Verma modules.

Proposition 4.26. *Fix $\boldsymbol{\zeta} \in (0, \infty)^n$, $\mathbb{E} \subset \mathbb{R}$, and $\mathbf{c} \in \mathbb{F}^k$ as above. Define $J := \{j \in [1, k] : c_j = 0\}$.*
  (1) $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ contains a central subalgebra $Z_0 := \mathbb{F}[\{x_j^- x_j^+ : j \in J\}]$ that is isomorphic to a polynomial algebra in $|J|$ variables. Now suppose $\mathrm{char}\, \mathbb{F} = 0$ if $J \subsetneq \{1, \ldots, k\}$. Then the center of $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ equals:

$$Z(\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})) = \begin{cases} Z_0 \otimes_{\mathbb{F}} \mathrm{span}_{\mathbb{F}}\{t^{-e} t^e : e \in \mathbb{E} \cap [0, \infty)\}, & \text{if } \boldsymbol{\zeta} = (1, \ldots, 1); \\ Z_0, & \text{otherwise.} \end{cases}$$

  (2) Suppose $\boldsymbol{\zeta} \neq (1, \ldots, 1)$. Then there are exactly $k+1$ isomorphism classes of algebras among the family $\{\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c}) : \mathbf{c} \in \mathbb{F}^k\}$ (where we assume $\mathrm{char}\, \mathbb{F} = 0$ if $J \subsetneq \{1, \ldots, k\}$).
  (3) The algebras $B^{\pm}$ do not contain zerodivisors. Thus every nonzero map of Verma modules is an embedding.
  (4) Define $K := \{j \in [1, k] : \zeta_j \neq 1\}$. Then a suitable completion of $\mathcal{A}_{\boldsymbol{\zeta}}(\mathbb{E}, \mathbf{c})$ contains central "Casimir" operators of the form

$$T(e) := \sum_{\mathbf{n} \in \mathbb{Z}^K} t^{-e \prod_{j \in K} \zeta_j^{n_j}} t^{e \prod_{j \in K} \zeta_j^{n_j}}, \qquad \forall e \in \mathbb{E} \cap (0, \infty),$$

where $T(e) = t^{-e}t^e \in \mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})$ if K is empty. Then the operators $T(e)$ act on all objects in $\mathcal{O}[\widehat{H_0}^{free}]$. Moreover, $T(e)$ and $Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c}))$ kill every highest weight module in $\mathcal{O}$, hence act nilpotently on $\mathcal{O}[\widehat{H_0}^{free}]$.

(5) *$\langle \mathcal{Q}_0^+ \rangle = \mathbb{Z}^k \ltimes_\zeta \mathbb{E} \subset \widehat{H_0}^{free}$, and the algebra $\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})$ satisfies none of the Conditions (S) because $\mathbb{Z}^J \ltimes_\zeta \mathbb{E}$ is in each block. More precisely, $S^3(\lambda) \supset (\mathbb{Z}^J \ltimes_\zeta \mathbb{E}) * \lambda \; \forall \lambda \in \widehat{H_0}^{free}$.*

Proof.

(1) The first assertion in this part is easily verified using the algebra relations. Next, the center is contained in the centralizer of H_0: $Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})) \subset Z_{\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})}(H_0)$, and Theorem 4.25 can be used to show that

$$Z_{\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})}(H_0) = H_0[x_1^- x_1^+, \dots, x_k^- x_k^+] \otimes_\mathbb{F} \mathrm{span}_\mathbb{F}\{t^{-e}t^e : e \in \mathbb{E} \cap [0, \infty)\}. \tag{4.27}$$

Now given $\mathbf{m} \in (\mathbb{Z}^+)^k$, define $\mathbf{x}^\mathbf{m} := \prod_{j=1}^k (x_j^- x_j^+)^{m_j}$. Then write an arbitrary element $z \in Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})) \subset Z_{\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})}(H_0)$ using (4.27):

$$z = \sum_{\mathbf{m}' \in (\mathbb{Z}^+)^J} \mathbf{x}^{\mathbf{m}'} \sum_{\mathbf{m} \in (\mathbb{Z}^+)^{J^c}} \mathbf{x}^\mathbf{m} \sum_{i=1}^{N(\mathbf{m}', \mathbf{m})} h_i t^{-e_i} t^{e_i},$$

for a suitable choice of elements $e_i \in \mathbb{E} \cap [0, \infty), 0 \neq h_i \in H_0$, and where $J^c := \{1, \dots, k\} \setminus J$. Now note by (RTA1) that z is central if and only if the inner double summation in the previous equation is central for each fixed $\mathbf{m}'$. Thus, assume without loss of generality that

$$z = \sum_\mathbf{m} \mathbf{x}^\mathbf{m} \sum_{i=1}^n h_{i,\mathbf{m}} t^{-e_{i,\mathbf{m}}} t^{e_{i,\mathbf{m}}}$$

for suitable $h_{i,\mathbf{m}}, e_{i,\mathbf{m}}$. Note by the algebra relations that the variables $\{x_j^\pm : j \notin J\}$ each generate a copy of the Weyl algebra in $\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})$, and this has trivial center since $\mathrm{char}\,\mathbb{F} = 0$. It follows via (RTA1) that the central element z in the above form has only one nonzero term in the outer sum, namely, the term corresponding to $\mathbf{m} = \mathbf{0}$. Thus we may assume that $z = \sum_{i=1}^n h_i t^{-e_i} t^{e_i}$. Now define $\mathbf{x}_+^\mathbf{m} := \prod_{j=1}^k (x_j^+)^{m_j}$ for $\mathbf{m} \in (\mathbb{Z}^+)^k$, and compute:

$$0 = \mathbf{x}_+^\mathbf{m} z - z \mathbf{x}_+^\mathbf{m} = \sum_{i=1}^n h_i((\mathbf{m}, 0) \cdot -) t^{-\boldsymbol{\zeta}^\mathbf{m} e_i} t^{\boldsymbol{\zeta}^\mathbf{m} e_i} \mathbf{x}_+^\mathbf{m} - \sum_{i=1}^n h_i(-) t^{-e_i} t^{e_i} \mathbf{x}_+^\mathbf{m}, \qquad \forall \mathbf{m} \in (\mathbb{Z}^+)^k.$$

Now for a fixed $\mathbf{m}$, since both sums involve finitely many terms, there is a unique largest positive exponent for t in both sums. For the two sums to be equal, either $e_i = 0$ for all i, or $\boldsymbol{\zeta}^\mathbf{m} = 1$. There are now two cases:

- If $\zeta_j = 1 \; \forall j$, then Z is easily seen to equal $(Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})) \cap H_0)[\{x_j^- x_j^+ : j \in J\}] \otimes_\mathbb{F} \mathrm{span}_\mathbb{F}\{t^{-e}t^e : e \in \mathbb{E} \cap [0, \infty)\}$. Moreover, it is not hard to show that $Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})) \cap H_0 = \mathbb{F}$, which proves this case.
- Otherwise there exists $\mathbf{m}$ such that $\boldsymbol{\zeta}^\mathbf{m} \neq 1$. In this case, the above computation must necessarily have one term, corresponding to $e_1 = 0$. But then we are once again reduced to computing $Z(\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})) \cap H_0$, which is $\mathbb{F}$.

(2) First note by rescaling the x_j^-, say, that $\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})$ is an associative $\mathbb{F}$-algebra that is isomorphic to the algebra $\mathcal{A}_\zeta(\mathbb{E}, \mathbf{d})$, where $d_j := 1 - \delta_{c_j, 0} \; \forall j$. Further observe that for any permutation $\sigma \in S_k$, we have an obvious isomorphism of algebras $\mathcal{A}_\zeta(\mathbb{E}, \mathbf{c}) \cong \mathcal{A}_{\sigma(\zeta)}(\mathbb{E}, \sigma(\mathbf{c}))$. It now remains to count the number of possible nondecreasing 0, 1-valued sequences of length k, and there are precisely $k + 1$ of them. Since $\boldsymbol{\zeta} \neq (1, \dots, 1)$, the previous part shows that the centers of these $k + 1$ algebras are polynomial rings with pairwise distinct transcendence degrees over $\mathbb{F}$. Thus the $k + 1$ algebras in question are pairwise non-isomorphic.

(3) That $B^{\pm}$ do not contain zerodivisors holds more generally by Theorem 4.9. The statement about Verma module embeddings is now standard.

(4) This part is proved similarly to Proposition 4.18.

(5) Note that $\langle \mathcal{Q}_0^+ \rangle = \mathbb{Z}^k \ltimes_\zeta \mathbb{E}$, which embeds into $\widehat{H_0}^{free}$ via the evaluation maps. Next, given $\lambda \in \widehat{H_0}^{free}$ and we first **claim** that every nonzero weight vector of the Verma module $M(\lambda)$ with weight in $(\mathbb{Z}^J \ltimes_\zeta \mathbb{E}) * \lambda$ is maximal. To show the claim, it suffices to show that $b_- m_\lambda$ is maximal, for every monomial word $b_- = t^{-e} \prod_{j \in J} (x_j^-)^{n_j} \in X_{irr}^-$. But now we compute using the algebra relations that $t^{e'} \cdot b_- m_\lambda = b_- t^{\prod_{j \in J} \zeta_j^{n_j} e'} m_\lambda = 0$; and similarly, $x_j^+ b_- m_\lambda = 0$ for all $1 \leq j \leq k$. This proves the claim. Finally, similar to Proposition 4.19 we obtain that $S^3(\lambda) \supset \mathbb{Z}^J \ltimes_\zeta \mathbb{E}$ and $S^1(\lambda) \supset (\mathbb{E} \cap (-\infty, 0)) * \lambda$ for all $\lambda \in \widehat{H_0}^{free}$.

$\square$

Remark 4.28. In [GGOR], one finds a homological treatment of Category $\mathcal{O}$ over a very general class of algebras A. We point out that the framework in the present paper cannot be subsumed under that paradigm, because of non-based examples such as $A = \mathcal{A}_\zeta(\mathbb{E}, \mathbf{c})$ above. In such non-based cases, there does not exist an inner grading by any subgroup of $\mathbb{R}$ (e.g. via taking the commutator with some element $\partial \in A$, as discussed in [GGOR]).

5. BASED AND NON-BASED LIE ALGEBRAS WITH TRIANGULAR DECOMPOSITION

The remainder of the paper focusses on applying the theory from Section 3 to a large class of algebras studied in the literature – as well as novel examples including stratified Virasoro algebras and certain triangular generalized Weyl algebras. The examples are presented in "decreasing order of familiarity" in the following sense: this section and the next discuss two "well-known" families of strict, based Hopf RTAs of finite rank: Lie algebras and quantum groups. The reader who wishes to skip these examples and focus immediately on non-strict or non-Hopf RTAs, can jump ahead to (a) non-based RTAs in Section 5.2; (b) infinitesimal Hecke algebras in Section 7 (rank one) and Section 10 (higher rank); or (c) generalized Weyl algebras in Sections 8 and 9.

We begin by discussing the case of $A = U\mathfrak{g}$ for $\mathfrak{g}$ a Lie algebra with regular triangular decomposition. Such Lie algebras are defined and explored in great detail in [RCW, MP], so this section is restricted to briefly mentioning some examples, after defining such Lie algebras. We also observe at the very outset that by Remark 3.7, it is possible to work with all of $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ when A is an HRTA. This is the case in the present section as well as the next two.

Definition 5.1. Assume char $\mathbb{F} = 0$. A Lie algebra $\mathfrak{g}$, together with the following data, is a *Lie algebra with triangular decomposition* (also called a *regular triangular Lie algebra* or *RTLA*):

(1) $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$, where all summands are nonzero Lie subalgebras of $\mathfrak{g}$, and $\mathfrak{h}$ is abelian.

(2) $\mathfrak{g}^+$ is an ad $\mathfrak{h}$-semisimple module with finite-dimensional $\mathfrak{h}$-weight spaces.

(3) All ad $\mathfrak{h}$-weights for $\mathfrak{g}^+$ lie in $Q^+ \setminus \{0\}$, where Q^+ denotes a free abelian monoid with finite basis $\Delta' := \{\alpha_j\}_{j \in J}$; this basis consists of linearly independent vectors in $\mathfrak{h}^*$.

(4) There exists an anti-involution ω of $\mathfrak{g}$ that sends $\mathfrak{g}^+$ to $\mathfrak{g}^-$ and preserves $\mathfrak{h}$ pointwise.

In contrast, a general, non-based RTA does not require $\mathcal{Q}_0^+$ to be $\mathbb{Z}^+ \Delta'$ for finite – or infinite – Δ'. Also note that we require char $\mathbb{F} = 0$ in order that the abelian monoid $\mathcal{Q}_0^+ = Q^+ = \mathbb{Z}^+ \Delta'$ is an RTM (i.e., satisfies Condition (RTM1)).

The following result summarizes the main (functorial) properties of such Lie algebras, and is not hard to show.

Proposition 5.2.

(1) If $\mathfrak{g}$ is an RTLA, then $U\mathfrak{g}$ is a strict, based Hopf RTA with base of simple roots Δ'.

(2) If $\mathfrak{g}_i$ is an RTLA for $1 \leq i \leq n$, and $\mathfrak{h}'$ is an abelian Lie algebra, then $\mathfrak{h}' \oplus \bigoplus_{i=1}^n \mathfrak{g}_i$ is an RTLA as well (with pairwise commuting summands).

(3) If $\mathfrak{g}$ is an RTLA and $V \subset Z(\mathfrak{g})$ is any subspace, then $\mathfrak{g}/V$ is an RTLA.

Note here that the adjoint action of $H_1 = H_0 = U\mathfrak{h} = \text{Sym}\,\mathfrak{h}$ is given by $\text{ad}\,h(x) = hx - xh$ for $x \in A = U\mathfrak{g}$. Moreover, Condition (RTA1) holds because of the Poincare-Birkhoff-Witt Theorem for $U\mathfrak{g}$, and $\widehat{H_1} = \widehat{H_0} = \mathfrak{h}^*$.

5.1. **Examples of RTLAs.** For completeness, we mention a large number of well-studied examples of RTLAs in the literature (which yield strict Hopf RTAs).

Example 5.3 (*Symmetrizable Kac-Moody Lie algebras*)**.** See [Kac2] for the definition and basic properties of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Note that if $\mathfrak{g}$ is complex semisimple (and finite-dimensional), then Harish-Chandra's theorem implies that $S^4(\lambda) = W \bullet \lambda \; \forall \lambda \in \mathfrak{h}^*$ (the twisted Weyl group orbit). Thus, all Conditions (S) hold by Theorem A, and all blocks $\mathcal{O}[S^3(\lambda)]$ of $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ are highest weight categories with BGG Reciprocity.

We now mention two generalizations of Kac-Moody Lie algebras, which are also RTLAs.

Example 5.4 (*Contragredient Lie algebras*)**.** These are a family of Lie algebras defined in [KK], which can be verified to be RTLAs (and for which Kac and Kazhdan proved the Shapovalov determinant formula).

Example 5.5 (*Some (symmetrizable) Borcherds algebras and central extensions*)**.** These Lie algebras are defined and studied in [Bo1, Bo2]; we remark that they are also RTLAs under certain additional assumptions, but not in general.

Example 5.6 (*(Centerless) Virasoro and Witt algebras*)**.** The Witt algebra is the centerless Virasoro algebra. Both of these Lie algebras are RTLAs; see [FeFu, KR], for example.

Example 5.7 (*Heisenberg algebras extended by derivations*)**.** Both these and the (centerless) Virasoro algebras can be found in [MP], for instance. It is not hard to show that all Conditions (S) fail to hold for (centerless) extended Heisenberg algebras if $V \neq 0$.

Example 5.8 (*Certain quotients of preprojective algebras of loop-free quivers*)**.** Let Q be a finite acyclic quiver (i.e., containing no loops or oriented cycles) with path algebra $\mathbb{F}Q = \oplus_{n \geq 0}(\mathbb{F}Q)_n$, where each summand has a basis consisting of (oriented) paths in Q of length n. Thus $(\mathbb{F}Q)_0$ and $(\mathbb{F}Q)_1$ have bases I of vertices e_i and E of edges a respectively. Assume $I, E \neq \emptyset$. Now construct the *double* $\overline{Q}$ of Q, by adding an "opposite" edge a^* for each $a \in E$.

The sub-quiver Q^* is defined with vertices I and edges a^*. Now define $\mathfrak{g} = \mathbb{F}\overline{Q}/(a'a^*, a^*a' : a' \in (\mathbb{F}Q)_1, a^* \in (\mathbb{F}Q^*)_1)$. This is an associative algebra, and a quotient of the *preprojective algebra* introduced in [GP], namely, $\mathbb{F}\overline{Q}/(\sum_{a \in E}[a, a^*])$. One uses the associative algebra structure to show that $\mathfrak{g}$ is an RTLA, using: $\mathfrak{g}^+ := \bigoplus_{n>0}(\mathbb{F}Q)_n$, $\mathfrak{h} := (\mathbb{F}Q)_0$, and $\mathfrak{g}^- := \bigoplus_{n>0}(\mathbb{F}Q^*)_n$. Moreover, $[\mathfrak{g}^+, \mathfrak{g}^-] = 0$, using which it can be shown that all Conditions (S) fail to hold.

Remark 5.9 (*Toroidal Lie algebras*)**.** These Lie algebras are defined (see [BeMo, Section 0]) to be the universal central extensions of $R_n \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a simply laced Lie algebra and $R_n = \mathbb{F}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$. The central extension is by $\mathfrak{z} := \Omega^1 R_n/dR_n$.

Clearly, the regularity condition fails here, so that toroidal Lie algebras are not RTLAs. We can, however, look at a related algebra, namely $U\mathfrak{g} \otimes R_n$. By the above result, this is a strict Hopf RTA. If the central extension above splits, then $U(\mathfrak{g} \oplus \mathfrak{z}) \otimes R_n$ is also a strict Hopf RTA.

5.2. **Non-based Lie algebras with triangular decomposition.** We now discuss examples of non-based RTAs arising from Lie algebras (which are necessarily not RTLAs). Such Lie algebras have emerged from mathematical physics and are the subject of active study.

Example 5.10 (*Generalized Virasoro algebras*)**.** A relatively modern construction (which is among the RTLAs not covered in [MP], say) consists of generalized Virasoro algebras $\mathrm{Vir}[G]$. These algebras were defined in [PZ] and have been the subject of a large body of literature; see e.g. [HWZ, LZ] and the references therein. They involve working over a field $\mathbb{F}$ of characteristic zero and a group $0 \neq G \subset (\mathbb{F}, +)$. Then $\mathrm{Vir}[G]$ is a G-graded Lie algebra with similar relations to the usual Virasoro algebra. Now suppose $\mathbb{F} = \mathbb{R} \supset G$. If $G = \alpha\mathbb{Z}$ for some $\alpha \neq 0$ then $\mathrm{Vir}[G]$ is discretely graded (and based); otherwise for $G \neq \alpha\mathbb{Z}$, the algebra is not discretely graded (and hence not based). In the case when the group G has a total ordering compatible with addition, $\mathrm{Vir}[G]$ has a triangular decomposition – in fact, $U(\mathrm{Vir}[G])$ turns out to be a (possibly non-based) strict Hopf RTA – and its Category $\mathcal{O}$ has been studied in great depth; see *loc. cit.*

Remark 5.11. Note in the theory developed above that the group $\langle \mathcal{Q}_r^+ \rangle$ usually does not equal the disjoint union $(\mathcal{Q}_r^- \setminus \{\mathrm{id}_{H_r}\}) \coprod \mathcal{Q}_r^+$ for $r = 0, 1$ (notation as in Lemma 2.9). For instance, this is the case for semisimple Lie algebras (and more generally, for all RTLAs) of rank at least 2. However, often in rank one situations or when the weight space is "one-dimensional", it does happen that $\langle \mathcal{Q}_r^+ \rangle = \mathcal{Q}_r^+ \cup \mathcal{Q}_r^-$. One example is precisely the higher rank/generalized Virasoro algebras over a totally ordered group G.

Example 5.12 (*Generalized Schrödinger-Virasoro algebras*)**.** Another modern construction of a strict RTA not found in [MP] is the Schrödinger-Virasoro algebra. This is a Lie algebra whose construction is motivated by the free Schrödinger equation in $(1+1)$-variables, and involves extending the centerless Virasoro Lie algebra by a 2-step nilpotent Lie algebra formed by bosonic currents. The larger class of generalized Schrödinger-Virasoro algebras $\mathfrak{gsv}[G]$ over totally ordered groups $G \subset (\mathbb{F}, +)$, as well as their Verma modules were studied in [TZ] (see also [LS]). Once again, their universal enveloping algebras provide examples of Hopf RTAs that are possibly non-based.

Example 5.13 (*Twisted Heisenberg-Virasoro algebra*)**.** This algebra was introduced and studied by Billig in [Bi]. It is not hard to show that its universal enveloping algebra is a Hopf RTA.

General construction: stratified Virasoro algebras. In light of the above "generalized" Virasoro-type examples, it is natural to ask if there is a unified framework of a general Lie algebra $\mathfrak{g}$, which encompasses all of the above examples (i.e., in Section 5.2). We now provide a positive answer to this question, over an arbitrary field $\mathbb{F}$ of characteristic zero:

(1) $\mathfrak{g}$ is a Lie algebra for which there exist nonnegative integers $M, N \in \mathbb{Z}^+$ such that

$$\mathfrak{g} = Z \oplus \bigoplus_{j=0}^{N} \mathfrak{g}_j, \qquad \mathfrak{g}_0 = \bigoplus_{k=0}^{M} \mathfrak{g}_0[k],$$

with all summands being vector spaces, and Z central in $\mathfrak{g}$.

(2) There exists an additive subgroup $G_k^0 \subset (\mathbb{F}, +)$ for each $0 \leq k \leq M$, such that $G_k^0 + G_{k'}^0 \subset G_{k+k'}^0$ whenever $k + k' \leq M$.

(3) For each $1 \leq j \leq N$, there exists a subset $G_j^+ \subset (\mathbb{F}, +)$ satisfying: (a) G_j^+ is an additive subgroup of $(\mathbb{F}, +)$, or else $\langle G_j^+ \rangle \setminus G_j^+$ is an additive subgroup of $(\mathbb{F}, +)$ and G_j^+ is a coset of it; (b) $G_j^+ + G_{j'}^+ \subset G_{j+j'}^+$ whenever $j + j' \leq N$; and (c) $G_k^0 + G_j^+ \subset G_j^+$ for all j, k.

(4) There exists a total ordering on the subgroup of $\mathbb{F}$ spanned by all G_k^0, G_j^+.

(5) For all $0 < j \leq N$, the vector space $\mathfrak{g}_j$ is spanned by an $\mathbb{F}$-basis $\{L_{j,\alpha}^+ : \alpha \in G_j^+\}$. Similarly, for all $0 \leq k \leq M$, the vector space $\mathfrak{g}_0[k]$ is spanned by an $\mathbb{F}$-basis $\{L_{k,\alpha}^0 : \alpha \in G_k^0\}$.

Moreover, these basis vectors satisfy the relations:

$$[L_{j,\alpha}^+, L_{j',\beta}^+] = \mathbf{1}_{j+j'\leq N} f_{j,j'}^{++}(\alpha,\beta) L_{j+j',\alpha+\beta}^+ + \mathbf{1}_{\alpha+\beta=0} g_{j,j'}^{++}(\alpha,\beta) z_{j,j'}^{++},$$
$$[L_{k,\alpha}^0, L_{k',\beta}^0] = \mathbf{1}_{k+k'\leq M} f_{k,k'}^{00}(\alpha,\beta) L_{k+k',\alpha+\beta}^0 + \mathbf{1}_{\alpha+\beta=0} g_{j,j'}^{00}(\alpha,\beta) z_{k,k'}^{00}, \tag{5.14}$$
$$[L_{k,\alpha}^0, L_{j,\beta}^+] = f_{k,j}^{0+}(\alpha,\beta) L_{j,\alpha+\beta}^0 + \mathbf{1}_{\alpha+\beta=0} g_{k,j}^{0+}(\alpha,\beta) z_{k,j}^{0+},$$

for suitable functions $f_{j,j'}^{00}, f_{k,k'}^{++}, f_{k,j}^{0+}$ and similarly for the g-functions, and with (suitable) central elements $z_{j,j'}^{++}, z_{k,k'}^{00}, z_{k,j}^{0+} \in Z$.

The aforementioned construction yields a Lie algebra whose universal enveloping algebra is an RTA, provided the f, g-functions and central elements satisfy certain compatibility conditions arising for the following reasons:

- the anti-symmetry of the Lie bracket;
- the Jacobi identity; and
- the anti-involution i, which should send $L_{j,\alpha}^+$ to $L_{j,-\alpha}^+$ and $L_{k,\alpha}^0$ to $L_{k,-\alpha}^0$.

Call any Lie algebra $\mathfrak{g}$ satisfying these assumptions a **stratified Virasoro algebra**. Then $U(\mathfrak{g})$ is a Hopf RTA with Cartan subalgebra $U(\mathfrak{h})$, where $\mathfrak{h} = Z \oplus \bigoplus_{k=0}^{M} \mathbb{F}L_{k,0}^0 \oplus \bigoplus_{j:0\in G_j^+} \mathbb{F}L_{j,0}^+$. It is not hard to show that this construction of a stratified Virasoro algebra encompasses all of the variants of Virasoro-type algebras discussed above. For instance, the usual Virasoro algebra is a stratified Virasoro algebra with $\dim_{\mathbb{F}} Z = 1$ and $M = N = 0$, with $G_0^0 = \mathbb{Z}$.

6. EXTENDED QUANTUM GROUPS FOR SYMMETRIZABLE KAC-MOODY LIE ALGEBRAS

The next class of examples consists of quantum groups, which are also strict Hopf RTAs. Suppose C is a generalized Cartan matrix (GCM) corresponding to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(C)$. Our goal is to construct a family of quantum algebras associated to the generalized Cartan matrix C, which we term *extended quantum groups*. Examples of such algebras include quantum groups that use neither the co-root lattice $Q^\vee$ nor the co-weight lattice $P^\vee$, but some intermediate lattice, as well as possible torsion elements. Moreover, we also study conditions under which all of these algebras satisfy the various Conditions (S).

6.1. **The construction and triangular decomposition.** To define the aforementioned family of quantum groups, some notation is required. Recall that a GCM is a matrix $C = (c_{ij})_{i,j\in I}$ where I is finite, $c_{ii} = 2$, c_{ij} is a nonpositive integer for all $i \neq j \in I$, and $c_{ij} = 0$ if and only if $c_{ji} = 0$. We say that C is symmetrizable if there exist positive integers d_i such that $d_i c_{ij} = d_j c_{ji}$ for all $i, j \in I$. We will also use the *Gaussian integers and binomial coefficients* in the ground field $\mathbb{F}$: given $q \in \mathbb{F}^\times$ that is not a root of unity, and integers $0 \leq m \leq n$, define

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]_q! := \prod_{m=1}^n [m]_q, \qquad [0]_q! := 1, \qquad \binom{n}{m}_q := \frac{[n]_q!}{[m]_q![n-m]_q!}.$$

Definition 6.1. Fix a ground field $\mathbb{F}$ and a nonzero scalar $q \in \mathbb{F}^\times$ that is not a root of unity.

(1) In this section, an *extended Cartan datum* consists of the following data:
- A symmetrizable GCM $C := (c_{ij})_{i,j\in I}$ and a diagonal matrix D with positive integer diagonal entries d_i such that $d_i c_{ij} = d_j c_{ji}$.
- A free abelian group $Q^\vee \cong \mathbb{Z}^I$ with $\mathbb{Z}$-basis $\{K_i : i \in I\}$. (This is the "co-root lattice" inside $\mathfrak{h}$, in the symmetrizable Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(C)$.)
- An abelian group $\Gamma \supset Q^\vee$, as well as a finite set of characters $\Delta' := \{\nu_i : \Gamma \to \mathbb{F}^\times : i \in I\}$ such that $\nu_i|_{Q^\vee} = q^{\alpha_i}$. In other words, $\nu_i(K_j) = q^{d_j c_{ji}} = \nu_j(K_i)$ for all $i, j \in I$.

(2) Given an extended Cartan datum $(C, D, Q^\vee \subset \Gamma, \Delta' = \{\nu_i\})$, define the *extended quantum group* $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ to be the $\mathbb{F}$-algebra generated by Γ and $\{e_i, f_i : i \in I\}$, modulo the following relations:

$$ge_ig^{-1} = \nu_i(g)e_i, \quad gf_ig^{-1} = \nu_i(g^{-1})f_i \quad \forall i \in I, \ g \in \Gamma; \qquad [e_i, f_j] = \delta_{i,j}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},$$

$$\sum_{l=0}^{1-c_{ij}} (-1)^l \binom{1 - c_{ij}}{l}_{q^{d_i}} e_i^{1-c_{ij}-l} e_j e_i^l = 0, \qquad (q\text{-Serre-1})$$

$$\sum_{l=0}^{1-c_{ij}} (-1)^l \binom{1 - c_{ij}}{l}_{q^{d_i}} f_i^{1-c_{ij}-l} f_j f_i^l = 0. \qquad (q\text{-Serre-2})$$

(3) Define $B^\pm$ to be the subalgebras of $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ generated by the e_is and f_is respectively and $H_1 = H_0 := \mathbb{F}\Gamma$.

Remark 6.2. Extended quantum groups can be defined for Γ any intermediate lattice between $Q^\vee$ and $P^\vee$. For instance, for $\Gamma = Q^\vee$ we recover the usual quantum group $U_q(\mathfrak{g}) = \mathfrak{U}_{q,C}(Q^\vee, \{q^{\alpha_i} : i \in I\})$. This is the approach followed in [Ja, §4.2] (when A is of finite type). In what follows, we will freely identify α_i with q^{α_i}, since we only deal with quantum groups and q is not a root of unity.

On the other hand, [HK, Section 3.1] or [Jos, Section 3.2.10] work with $\Gamma = P^\vee$, the co-weight lattice inside $\mathfrak{h}$, and $\nu_i(q^h) = q^{\alpha_i(h)}$, for the simple roots $\alpha_i \in \mathfrak{h}^*$. Moreover, $K_i = q^{d_i h_i}$, where $h_i = [e_i, f_i]$ in $U\mathfrak{g}$. Note that all of these algebras are special cases of extended quantum groups.

In fact the family of extended quantum groups is more general than the above examples, because Γ is allowed to have torsion elements, in which case it does not embed into $\mathbb{Q} \otimes_{\mathbb{Z}} Q^\vee \subset \mathfrak{h}$. Thus there may not exist a bilinear form (and hence, a Hopf pairing) on Γ, as is used in the literature.

We now list some basic properties of extended quantum groups.

Proposition 6.3. *Fix an extended Cartan datum and define $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ as above.*

(1) $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ *has a Hopf algebra structure, with the comultiplication Δ, counit ε, and antipode S given on generators by*

$$\varepsilon(g) = 1, \qquad \varepsilon(e_i) = \varepsilon(f_i) = 0, \qquad \forall g \in \Gamma, \ i \in I$$

$$\Delta(g) = g \otimes g, \qquad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \qquad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i,$$

$$S(g) = g^{-1}, \qquad S(e_i) = -e_i K_i, \qquad S(f_i) = -K_i^{-1} f_i.$$

(2) $\mathfrak{U}_{q,C}(\Gamma, \nu)$ *has an involution T that sends $g \in \Gamma$ to g^{-1} and e_i to f_i for all $i \in I$. Restricted to B^+, T is an algebra isomorphism onto B^-.*

(3) $TST = S^{-1} \neq S$, *whence $ST \neq TS$ are anti-involutions i on $\mathfrak{U}_{q,C}(\Gamma, \nu)$, which restrict to the identity on H_1.*

(4) $A = \mathfrak{U}_{q,C}(\Gamma, \Delta')$, *together with the data $(B^\pm, H_1 = H_0 = \mathbb{F}\Gamma, \Delta', i = ST$ or $TS)$ forms a strict, based Hopf RTA (of finite rank) if and only if Δ' is $\mathbb{Z}$-linearly independent in $\widehat{H_1}$.*

Proof. The first three parts are shown by adopting the proofs and arguments found in [HK, Section 3.1] to $\mathfrak{U}_{q,C}(\Gamma, \Delta')$. Since both TST and S^{-1} are $\mathbb{F}$-algebra anti-automorphisms, the third part follows by checking that they agree on generators. For the fourth part, one implication is immediate from the axioms, and the converse is not hard to verify when Δ' is $\mathbb{Z}$-linearly independent. $\square$

Extended quantum groups are very similar in structure to the quantum groups that have been very well-studied in the literature. In some sense, they "quantize" the contragredient Lie algebras defined in Example 5.4 (i.e., in [KK]), after removing some of the assumptions therein. Thus the classical limit and representation theory (at least, for integrable modules) should be similar to the traditionally well-studied cases. We expect that the analysis in [HK, Chapter 3] should go through

for the algebras $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ as well, but do not proceed further along these lines, as it is not focus of the present paper.

We now show that extended quantum groups of finite type satisfy all of the Conditions (S). The following is the main result in this section.

Theorem 6.4. *Fix a ground field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2, 3$ and such that $\mathbb{F}^{\times}$ is a divisible group (e.g., $\mathbb{F} = \overline{\mathbb{F}}$). Also suppose $q \in \mathbb{F}^{\times}$ is not a root of unity, the matrix C is of finite type, and $[\Gamma : Q^{\vee}] < \infty$.*

Then there exist extensions ν_i of the characters q^{α_i} from $Q^{\vee}$ to Γ. For each such choice $\Delta' = \{\nu_i : i \in I\}$ of extensions, $A = \mathfrak{U}_{q,C}(\Gamma, \Delta')$ satisfies Condition (S4), and hence all other conditions (S). In particular, all blocks of $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ are highest weight categories with BGG reciprocity.

In particular, all Conditions (S) (and properties such as BGG Reciprocity) hold for all extended quantum groups with Γ "in between" the co-root and co-weight lattices, or containing additional finite-order torsion subgroups. We remark that a special case of our result was known for $\Gamma = P^{\vee}$ from [Jos, Lemma 8.3.2], which stated that $\chi_{\mu} = \chi_{\lambda}$ on the center of $\mathfrak{U}_{q,C}(P^{\vee}, \{q^{\alpha_i}\})$ if and only if $\mu \in (W \ltimes (\mathbb{Z}/2\mathbb{Z})^I) \bullet \lambda$.

Proof. As the proof is somewhat lengthy, we break it up in to steps for ease of exposition.

Step 1. We first extend the characters q^{α_i} from $Q^{\vee}$ to Γ. Consider the short exact sequence $0 \to Q^{\vee} \xrightarrow{\iota} \Gamma \to \Gamma/Q^{\vee} \to 0$ in the category of abelian groups. Since $\mathbb{F}^{\times}$ is divisible – i.e., injective – this yields:

$$0 \to \operatorname{Hom}_{group}(\Gamma/Q^{\vee}, \mathbb{F}^{\times}) \to \operatorname{Hom}_{group}(\Gamma, \mathbb{F}^{\times}) \xrightarrow{\iota^*} \operatorname{Hom}_{group}(Q^{\vee}, \mathbb{F}^{\times}) \to 0. \qquad (6.5)$$

Now think of the simple roots α_i as elements of $\operatorname{Hom}_{group}(Q^{\vee}, \mathbb{F}^{\times})$, via:

$$\alpha_i(q^h) := q^{\alpha_i(h)}.$$

Note that the subgroup generated by the α_i is free because q is not a root of unity in $\mathbb{F}$. It is then possible to lift q^{α_i}, via the injectivity of $\mathbb{F}^{\times}$, to any $\nu_i \in (\iota^*)^{-1}(q^{\alpha_i}) \subset \operatorname{Hom}_{group}(\Gamma, \mathbb{F}^{\times})$.

Step 2. The next claim is that *if $Q^{\vee} \subset \Gamma' \subset \Gamma$ are abelian groups with $\Delta' := \{\nu_i : i \in I\} \subset \operatorname{Hom}_{group}(\Gamma, \mathbb{F}^{\times})$ being $\mathbb{Z}$-linearly independent characters when restricted to Γ', then*

$$Z(\mathfrak{U}_{q,C}(\Gamma', \Delta'|_{\Gamma'})) = Z(\mathfrak{U}_{q,C}(\Gamma, \Delta')) \cap \mathfrak{U}_{q,C}(\Gamma', \Delta'|_{\Gamma'}). \qquad (6.6)$$

Indeed, the only nontrivial assertion in Equation (6.6) is to show that $Z(\mathfrak{U}_{q,C}(\Gamma', \Delta'|_{\Gamma'})) \subset Z(\mathfrak{U}_{q,C}(\Gamma, \Delta'))$. Suppose $z \in Z(\mathfrak{U}_{q,C}(\Gamma', \Delta'_{\Gamma'}))$; since z commutes with Γ', it has weight 0 in $\mathfrak{U}_{q,C}(\Gamma', \Delta'|_{\Gamma'})$, and hence also in $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ (since the weight space decompositions of $\mathfrak{U}_{q,C}(\Gamma', \Delta'|_{\Gamma'}) \hookrightarrow \mathfrak{U}_{q,C}(\Gamma, \Delta')$ agree). Thus, z commutes with Γ, and since it commutes with each e_i and f_i, z is central in $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ as well.

Step 3. For convenience, define $G\hat{} := \operatorname{Hom}_{group}(G, \mathbb{F}^{\times})$, for any group G. Thus $\Gamma\hat{} = \widehat{H_1}$ in our setting. Now to prove the result, fix $\lambda \in \Gamma\hat{}$ and suppose $\chi_{\mu} = \chi_{\lambda} : Z(\mathfrak{U}_{q,C}(\Gamma, \Delta')) \to \mathbb{F}$ for some $\mu : \Gamma \to \mathbb{F}^{\times}$. Then $\chi_{\mu}, \chi_{\lambda}$ agree when restricted (by the previous step) to $Z := Z(U_q(\mathfrak{g}))$, where $U_q(\mathfrak{g}) = \mathfrak{U}_{q,C}(Q^{\vee}, \{q^{\alpha_i}\})$. Thus $\mu \circ \xi = \lambda \circ \xi$ on Z. Now recall the following result from [Ja, Sections 4.2 and 6.25-6.26]: *If $\Gamma = Q^{\vee}$ and $\nu_i = \alpha_i$ are the "simple roots", then the Harish-Chandra map is an isomorphism*

$$\rho_{H_1}(q^{\theta}) \circ \xi : Z(U_q(\mathfrak{g})) \xrightarrow{\sim} \mathbb{F}[Q^{\vee} \cap 2P^{\vee}]^W.$$

Here, θ denotes the half-sum of positive roots, and ρ_{H_1} is the weight-to-root map that was studied in Proposition 2.15. (We identify $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$ via the Killing form.) It follows from above that $\mu \circ \rho_{H_1}(q^{-\theta}) = \lambda \circ \rho_{H_1}(q^{-\theta})$ on $\mathbb{F}[Q^{\vee} \cap 2P^{\vee}]^W$.

Step 4. The remainder of the proof studies the chain of algebras $\mathbb{F}[Q^{\vee} \cap 2P^{\vee}]^W \hookrightarrow \mathbb{F}[Q^{\vee} \cap 2P^{\vee}] \hookrightarrow \mathbb{F}[Q^{\vee}] \hookrightarrow \mathbb{F}[\Gamma]$. Note that $Q^{\vee} \cap 2P^{\vee}$ is a lattice, so $\operatorname{Spec} \mathbb{F}[Q^{\vee} \cap 2P^{\vee}] = (\mathbb{F}^{\times})^{rk(Q^{\vee} \cap 2P^{\vee})}$. Now recall the *Nagata-Mumford Theorem* from (a special case of) [Muk, Theorem 5.3]: *Suppose a finite group*

W *acts on an affine variety* X *(i.e., its coordinate ring* R*). Then the map* $\Phi : X = \operatorname{Spec}(R) \to$ $X/\!\!/W := \operatorname{Spec}(R^W)$ *(induced by the inclusion* $R^W \hookrightarrow R$*) is a surjection that factors through a bijection* $\overline{\Phi} : X/W \to X/\!\!/W$*, where* X/W *denotes the* W*-orbits in* X*.*

Applying this to $X := Q^\vee \cap 2P^\vee$, it follows that the set of possible extensions $\nu \in (Q^\vee \cap 2P^\vee)\hat{\ }$ of $\lambda \circ \rho_{H_1}(q^{-\theta}) : \mathbb{F}[Q^\vee \cap 2P^\vee]^W \to \mathbb{F}$ is a W-orbit, hence finite (thus, $\{\nu \circ \rho_{H_1}(q^{-\theta})\}$ is also finite).

Step 5. Finally, consider the map $: \Gamma\hat{\ } \to (Q^\vee \cap 2P^\vee)\hat{\ }$. By the injectivity of $\mathbb{F}^\times$ and an analogue of Equation (6.5) in this situation, it suffices to show that $\Gamma/(Q^\vee \cap 2P^\vee)$ is finite (for then ι^* is a surjection with finite fibers). But $\Gamma/(Q^\vee \cap 2P^\vee)$ is indeed finite, since

$$[\Gamma : Q^\vee \cap 2P^\vee] = [\Gamma : Q^\vee] \cdot [Q^\vee : Q^\vee \cap 2P^\vee] \leq [\Gamma : Q^\vee] \cdot [P^\vee : 2P^\vee] < \infty.$$

To conclude, $\{\mu \in \Gamma\hat{\ } = \widehat{H_1} : \chi_\mu = \chi_\lambda\} \subset \{\mu \in \Gamma\hat{\ } : \mu \circ \rho_{H_1}(q^{-\theta}) = \lambda \circ \rho_{H_1}(q^{-\theta}) \text{ on } (Q^\vee \cap 2P^\vee)^W\}$, and the latter is a finite set by the above analysis. Thus $\mathfrak{U}_{q,C}(\Gamma, \Delta')$ satisfies Condition (S4). $\qquad \square$

7. Further examples of strict, based Hopf RTAs

Before moving on to RTAs that are either not Hopf RTAs or not strict, we write down some more examples of strict, based Hopf RTAs of low rank. The first of these examples shows the need to use Condition (S3) instead of central characters/Condition (S4) in order to obtain a block decomposition of $\mathcal{O}$.

Example 7.1 (*Rank one infinitesimal Hecke algebras and their quantized analogues*). Suppose $\operatorname{char} \mathbb{F} = 0$. The (Lie) rank one infinitesimal Hecke algebra is defined to be a deformation $\mathcal{H}_z$ of $\mathcal{H}_0 := U(\mathfrak{sl}_2(\mathbb{F}) \ltimes \mathbb{F}^2)$, where $\mathbb{F}^2$ is spanned by a weight basis x, y (over the Cartan subalgebra of $\mathfrak{sl}_2$, which is spanned by h). The deformed relation is $[x, y] = z(C)$, where C is the quadratic Casimir element of $U(\mathfrak{sl}_2)$ and $z \in \mathbb{F}[T]$ is an arbitrary polynomial.

The family of algebras $\mathcal{H}_z$ was introduced in [Kh1] and extensively studied in [KT]. It can be seen from *loc. cit.* that $\mathcal{H}_z$ is a strict, based Hopf RTA of rank one with $H_1 = H_0 = \mathbb{F}[h]$ and $\Delta' = \{\frac{1}{2}\alpha\}$, where α is the root of $\mathfrak{sl}_2$. (We remind the reader that in the literature, *roots* of semisimple Lie algebras are assumed to lie in the dual space $\mathfrak{h}^*$ of the Cartan Lie subalgebra, via the weight-to-root map $\rho_{U(\mathfrak{h})}$.) In particular, $\widehat{H_1}^{free} = \widehat{H_1} = \mathbb{F}$. In [KT], it is also shown that similar to complex semisimple Lie algebras (e.g., $U(\mathfrak{sl}_2)$),

- The center $Z(A)$ is isomorphic to a polynomial algebra in one variable – the "quadratic" Casimir element.
- Condition (S4) holds for $\mathcal{H}_z$ if $z \neq 0$. (Thus $\mathcal{O}$ satisfies BGG Reciprocity.)
- Every central character is of the form χ_λ for some $\lambda \in \widehat{H_1}$, if $\mathbb{F}$ is algebraically closed of characteristic zero (see [H1, Exercise (23.9)]).
- If $z \neq 0$, there are at most finitely many pairwise non-isomorphic simple finite-dimensional objects in $\mathcal{O}$.

The algebras $\mathcal{H}_z$ possess quantizations $\mathcal{H}_{z,q}$ for $q \neq 0, \pm 1$, which were explored in detail in [GK]. The quantum algebras $\mathcal{H}_{z,q}$ turn out to be deformations of $U_q(\mathfrak{sl}_2(\mathbb{F})) \ltimes \mathbb{F}[x, y]$ whose classical limits as $q \to 1$ are once again $\mathcal{H}_z$; see [GK]. They have been found to possess very similar properties to $\mathcal{H}_z$, including a strict, based Hopf RTA structure. However, it was shown in [GK, Theorem 11.1] that if q is not a root of unity, and $z = qyx - xy \neq 0$, then $Z(\mathcal{H}_{z,q}) = \mathbb{F}$. Thus Condition (S4) clearly fails. Nevertheless, [GK, Propositions 8.2, 8.13] show that $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ is a highest weight category satisfying Condition (S3). Thus our framework allows us to prove that $\mathcal{O}$ is a direct sum of blocks with BGG Reciprocity, even though it has trivial center and Condition (S4) fails to hold.

Example 7.2. The next example is that of a strict Hopf RTA that was recently studied by Batra and Yamane [BY]. In that work, the authors defined "generalized quantum groups" $U(\chi, \Pi)$, which are a family of quantum algebras corresponding to a semisimple Lie algebra (akin to the algebras

$\mathfrak{U}_{q,C}(\Gamma, \Delta'))$. The (skew) centers of these algebras and Harish-Chandra type results were studied in *loc. cit.* We observe here that the algebra $U(\chi, \Pi)$ is a strict, based Hopf RTA when χ is non-degenerate and $\chi(\alpha_i, \alpha_j)$ is not a root of unity for any $i, j \in I$.

The following example is a degenerate one.

Example 7.3 (*Regular functions on affine algebraic groups*)**.** It is well-known that the category of commutative Hopf algebras is dual to the category of affine algebraic groups. Thus if $\mathbf{G}$ is any affine algebraic group, then $H_1 = H_0 = \mathbb{C}[\mathbf{G}]$ is a commutative Hopf algebra, and hence $A = H_1$ is a strict, based HRTA as well, with Δ' the empty set. Note that H_1 need not be cocommutative in general (since $\mathbf{G}$ need not be commutative).

In general, every commutative (Hopf) $\mathbb{F}$-algebra H_1 is a strict, based (Hopf) RTA of rank zero, via: $H_1 = H_0 = \mathbb{F} \otimes H_1 \otimes \mathbb{F} = Z(H_1)$. In this context, $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ trivially satisfies Condition (S4) (and hence Conditions (S1)–(S3)), and also is a semisimple (highest weight) category.

The final example in this section is stated for completeness, and is illustrative in showing how to combine both of the main theorems in Section 3.3, in order to study Category $\mathcal{O}$. (More generally, one can use Theorem B to create more examples of (strict) (based) (Hopf) RTAs by taking tensor products.)

Example 7.4. In [Zhi], Zhixiang studied the homological properties and representations of the "double loop quantum enveloping algebra" (DLQEA), which is a Hopf algebra isomorphic to $U_q(\mathfrak{sl}_2) \otimes \mathbb{F}[g^{\pm 1}, h^{\pm 1}]$ as an algebra (but not as Hopf algebras). Here $\mathbb{F}$ is an algebraically closed field of characteristic zero. The aforementioned algebra isomorphism and Theorem B shows that the DLQEA is a strict, based HRTA of rank one, and the representation theory of Category $\mathcal{O}$ reduces to that for $U_q(\mathfrak{sl}_2)$ and for $\mathbb{F}[g^{\pm 1}, h^{\pm 1}]$. Now use Theorems A and B, Example 7.3, and the results in Section 6 to conclude that the DLQEA satisfies Condition (S4), and hence, Theorem A.

8. RANK ONE RTAs: TRIANGULAR GENERALIZED WEYL ALGEBRAS

In the remainder of this paper we discuss more families of based RTAs, some of which are either not Hopf RTAs, or not strict. These examples further demonstrate the need to use the full power of our framework. All of the examples in this section and the next fall under the following setting.

Definition 8.1. Fix a field $\mathbb{F}$, an associative $\mathbb{F}$-algebra H, an $\mathbb{F}$-algebra map $\theta : H \to H$, and $z_0, z_1 \in H$. The *triangular generalized Weyl algebra (or triangular GWA)* associated to this data is the $\mathbb{F}$-algebra

$$\mathcal{W}(H, \theta, z_0, z_1) := H\langle d, u\rangle/(uh = \theta(h)u, \ hd = d\theta(h), \ ud = z_0 + dz_1 u \ \forall h \in H). \qquad (8.2)$$

As we will presently see, this construction is very general and incorporates a large number of algebras studied in the literature. We now briefly list the contents of this section. In Section 8.1 we discuss the structure and representation theory of $\mathcal{O}$ for triangular GWAs. Sections 8.2 and 8.3 discuss a large number of examples of triangular GWAs, many of them arising from mathematical physics. The examples are of two flavors - "classical" and "quantum". In Section 8.4 we explain how these two types of examples are related in a precise way. Our construction of the "classical limit" extends – to a large family of generalized down-up algebras – the relation between classical and quantum $\mathfrak{sl}_2$.

8.1. **Structure and block decomposition of $\mathcal{O}$ over triangular GWAs.** Henceforth we will assume that θ is an automorphism, as well as some other properties that we now discuss.

Lemma 8.3. *Suppose $\theta : H \to H$ is an automorphism. Then $\mathcal{W}(H, \theta, z_0, z_1)$ satisfies (RTA1) with $B^+ = \mathbb{F}[u]$, $B^- = \mathbb{F}[d]$, and $H_1 = H$, if and only if z_0, z_1 are central in H.*

Proof. Compute for all $h \in H$:

$$h(ud) = h(z_0 + dz_1 u) = hz_0 + d\theta(h)z_1 u,$$
$$(hu)d = u\theta^{-1}(h)d = (ud)h = (z_0 + dz_1 u)h = z_0 h + dz_1 \theta(h)u.$$

Now if $\mathcal{W}(H, \theta, z_0, z_1)$ satisfies (RTA1), then the equality between these two expressions for all $h \in H$ implies that $z_0, z_1 \in Z(H)$. To show the converse, use the Diamond Lemma from [Be] in a manner similar to the proof of Theorem 4.9. $\square$

Assumption 8.4. For the remainder of this section and the next, assume that H is commutative, and θ is an algebra automorphism of H of infinite order.

In order to discuss the structure of triangular GWAs, we now introduce a sequence $\widetilde{z}_n$ of distinguished elements in a triangular GWA (more precisely, in its subalgebra H).

Definition 8.5. Suppose $\theta : H \to H$ is an algebra automorphism. Given $n \in \mathbb{N}$, define

$$z'_n := \prod_{i=0}^{n-1} \theta^i(z_1), \qquad z'_0 := 1, \qquad \widetilde{z}_n := \sum_{j=0}^{n-1} \theta^j(z_0 z'_{n-1-j}), \qquad \widetilde{z}_0 := 0, \qquad \widetilde{z}_{-n} := \theta^{-n}(\widetilde{z}_n). \quad (8.6)$$

Now given a weight $\lambda : H \to \mathbb{F}$, define $[\lambda] := \{\theta^{-n} * \lambda : n \in \mathbb{Z}, \lambda(\widetilde{z}_n) = 0\} \subset \widehat{H}$.

We now discuss some basic properties of triangular GWAs, which concern central characters and the block decomposition of $\mathcal{O}$.

Theorem 8.7. *Suppose $A = \mathcal{W}(H, \theta, z_0, z_1)$ is a triangular GWA (over any field $\mathbb{F}$). Then A is a strict, based RTA of rank one (with $\Delta := \{\theta\}$ and $H_1 = H_0 := H$) if and only if A satisfies Assumption 8.4.*

Suppose henceforth that the triangular GWA A is a strict, based RTA of rank one.

(1) For all $m, n \geq 0$, the Shapovalov form of $\mathbb{F}[d]$ is given by $\langle d^m, d^n \rangle = \delta_{m,n} \prod_{j=1}^{n} \widetilde{z}_j$.

(2) $S^3(\lambda) = [\lambda]$ for all $\lambda \in \widehat{H}^{free}$.

(3) Suppose $z_1 = 1$ and $z_0 \in \mathrm{im}(\mathrm{id}_H - \theta)$. Define a quadratic Casimir operator to be $\Omega := du + \zeta$ for any $\zeta \in H$ satisfying: $(\mathrm{id}_H - \theta)(\zeta) = z_0$. Then,

$$Z(A) \cap H = \ker(\mathrm{id}_H - \theta), \qquad Z(A) = (Z(A) \cap H)[\Omega], \qquad S^4(\lambda) \cap (\mathbb{Z}\theta * \lambda) = S^3(\lambda) = [\lambda] \; \forall \lambda \in \widehat{H}^{free}.$$

Moreover, Ω is transcendental over $Z(A) \cap H$.

Consequently, A satisfies Condition (S3) if and only if $||[\lambda]|| < \infty$ for all $\lambda \in \widehat{H}^{free}$. The last part also says that the converse to Lemma 3.20 holds for triangular GWAs when $z_0 = 1$ and a quadratic Casimir exists.

Proof. Set $B^+ := \mathbb{F}[u]$, $B^- := \mathbb{F}[d]$, $H_1 = H_0 := H$, and $\Delta := \{\theta\}$. Now the first assertion is not hard to show, using the anti-involution that sends u to d and fixes H. To show the next result, we prove some intermediate equivalences that may be useful in their own right. First, specializing the analysis in Section 3 to A helps determine the structure of Verma modules:

For all weights $\mu \in \widehat{H}^{free}$, $M(\mu)$ is a uniserial module, with unique composition series:

$$M(\mu) \supset M(\theta^{-n_1} * \mu) \supset M(\theta^{-n_2} * \mu) \supset \cdots,$$

where $0 < n_1 \leq n_2 \leq \cdots$ comprise the set $\{n \in \mathbb{N} : \mu(\widetilde{z}_n) = 0\}$. Thus $\mathcal{O}$ is finite length if and only if $[\mu] \cap (-\mathbb{Z}^+ \Delta * \mu)$ is finite for every $\mu \in \widehat{H}^{free}$. Moreover, the following are equivalent, given $n \in \mathbb{Z}^+$ and $\mu \in \widehat{H}$: (a) The multiplicity $[M(\theta^n * \mu) : L(\mu)]$ is nonzero. (b) $[M(\theta^n * \mu) : L(\mu)] = 1$. (c) $(\theta^n * \mu)(\widetilde{z}_n) = 0$. (d) $\mu(\widetilde{z}_{-n}) = 0$.

The proof is straightforward, given that $M(\lambda) \cong \mathbb{F}[d]$ for all λ, and $d^n m_\lambda$ spans $M(\lambda)_{\theta^{-n} * \lambda}$ for all $\lambda \in \widehat{H}^{free}$ and $n \geq 0$. The key computation, which is straightforward but longwinded, is to show:

$$u^m d^n \;\in\; d^{n-m} \cdot \prod_{j=n-m}^{n-1} \widetilde{z}_{j+1} + A \cdot u, \qquad \forall 0 \leq m \leq n. \tag{8.8}$$

Setting $m = 1$ and applying (8.8) to the highest weight vector of $M(\theta^n * \mu)$ shows that (a) $\Leftrightarrow$ (c). The remaining equivalences are standard. We now sketch the proofs of the three assertions. The first part follows using Equation (8.8). Next, that $S^3(\lambda) = [\lambda]$ can be proved using the equivalences stated above.

It remains to prove part (3) about the center. That $Z(A) \cap H = \ker(\mathrm{id}_H - \theta)$ is easily verified. Now suppose $\omega \in Z(A)$ is central. Then ω commutes with H, whence $\omega \in H[du] = H[\Omega - \zeta]$. Consider such a central element $\omega := \sum_i (du)^i h_i$, where $h_i \in H \; \forall i \geq 0$. We then have

$$\omega = \sum_{i \geq 0} (\Omega - \zeta)^i h_i = \sum_{0 \leq j \leq i} \binom{i}{j} \Omega^j \zeta^{i-j} h_i = \sum_{j \geq 0} \Omega^j \sum_{i \geq j} \binom{i}{j} \zeta^{i-j} h_i = \sum_{j \geq 0} \Omega^j h'_j,$$

where $h'_j \in H \; \forall j$. Now if ω is central, we compute: $0 = [u, \omega] = \sum_j \Omega^j [u, h'_j]$, whence by the PBW property (RTA1), one checks that $[u, h'_j] = 0 \; \forall j$, whence $h'_j \in \ker(\mathrm{id}_H - \theta)$ from above. Thus $\omega \in (Z(A) \cap H)[\Omega]$ as claimed. Additionally, it is not hard to see using (RTA1) that Ω is transcendental over $Z(A) \cap H$.

Finally, by a previous part and Lemma 3.20, it suffices to show that $S^4(\lambda) \cap (\mathbb{Z}\theta * \lambda) \subset S^3(\lambda)$ for all $\lambda \in \widehat{H}^{free}$. Moreover, it further suffices to show the **claim** that $\chi_{\theta^{-n} * \lambda} \equiv \chi_\lambda$ for some $n \geq 0$ if and only if $[M(\lambda) : L(\theta^{-n} * \lambda)] > 0$. By the proof of Theorem 8.7, this is equivalent to showing that $\lambda(\widetilde{z}_n) = 0$. Now compute using any quadratic Casimir element and Proposition 3.3:

$$\chi_\lambda(\Omega) - \chi_{\theta^{-n} * \lambda}(\Omega) = \lambda(\zeta) - \lambda(\theta^n(\zeta)) = \lambda \circ (\mathrm{id}_H - \theta^n)(\zeta) = \lambda \circ (\mathrm{id}_H + \theta + \cdots + \theta^{n-1})(z_0) = \lambda(\widetilde{z}_n),$$

since $z_1 = 1$. Thus the above claim follows, completing the proof. $\square$

8.2. **Examples: generalized down-up algebras.** We now discuss a family of examples of triangular GWAs, which has been extensively studied in many papers in the literature. These are the "generalized down-up algebras" introduced by Cassidy and Shelton in [CS], and they are strict, based RTAs of rank one, with

$$H = \mathbb{F}[h], \qquad \theta = \theta_{r,\gamma}(h) := r^{-1}(h + \gamma), \qquad z_1 = s^{-1}, \qquad z_0 = s^{-1} f(h), \tag{8.9}$$

where $r, s \in \mathbb{F}^\times$, $\gamma \in \mathbb{F}$, and $f(h) \in H$ is a fixed polynomial in h. (Note that if $r = 1$ then $\mathcal{W}(\mathbb{F}[h], \theta_{1,\gamma}, s^{-1} f(h), s^{-1})$ is a strict, based Hopf RTA of rank one.) The operators d, u in $\mathcal{W}(\mathbb{F}[h], \theta_{r,\gamma}, s^{-1} f(h), s^{-1})$ are thought of as "lowering" and "raising" operators respectively (hence the name of "down-up" algebras). Examples of such algebras occur in many different settings in the literature:

(1) In representation theory, Smith [Smi] introduced and studied a family of triangular GWAs (more precisely, generalized down-up algebras) that are deformations of $U(\mathfrak{sl}_2)$. Smith showed that these algebras satisfy Condition (S4), as well as an analogue of Duflo's theorem for primitive ideals and annihilators of simple modules $L(\lambda)$.

(2) In mathematical physics, Witten [Wi] introduced a 7-parameter family of deformations of $U(\mathfrak{sl}_2)$ that include a large sub-family of GWAs. Witten's motivations arose from vertex models and duality in conformal field theory. Witten's family of deformations was later studied by Kulkarni [Ku1], and a three-parameter subfamily $U_{abc}(\mathfrak{sl}_2)$ was studied by Le Bruyn [LeB] under the name of "conformal $\mathfrak{sl}_2$-algebras".

(3) In the comprehensive paper [Kac1] studying Lie superalgebras, Kac studied the "dispin Lie superalgebra" $B[0,1]$. In this case,

$$U(B[0,1]) = \mathcal{W}(\mathbb{C}[h], \theta = \theta_{1,-1}, h, 1), \qquad \theta(h) = h - 1.$$

(4) In [Wo], Woronowicz introduced and studied the algebra $\mathcal{W}(\mathbb{F}[h], \theta, \nu^{-1}h, \nu^{-2})$ in the context of quantum groups. This algebra is a generalized down-up algebra where $\nu \in \mathbb{F} \setminus \{0, \pm 1\}$ and $\theta(h) = \nu^{-4}h + 1 + \nu^{-2}$.

(5) These algebras also occur in combinatorics, in certain cases when "down" and "up" operators are defined on the span of a partially ordered set. These were the original "down-up" algebras, studied by Benkart and Roby in [BR], and they are a special case of generalized down-up algebras with $z_0 = h$ and $z_1 \in \mathbb{F}$. They have been the subject of continuing interest – see [CM, Jo2, KM, Ku2, LL] among others.

(6) The algebras studied by Jing and Zhang, as discussed in Example 3.4. In this case one can show that $\mathcal{O}[\widehat{H}^{free}]$ satisfies Condition (S3) if q is not a root of unity and char $\mathbb{F} \neq 2, 3$.

Note that in a large number of examples mentioned in the above list, the generalized down-up algebras of interest are described by (8.9) with parameters $r = 1, \gamma \neq 0, f \not\equiv 0$, and char $\mathbb{F} = 0$. In such settings it is possible to describe when the algebra satisfies Condition (S3). Thus the following result deals with block decompositions of $\mathcal{O}$, for all of the above examples at once.

Theorem 8.10. *Under the setting of* (8.9), *and identifying the weights* $\lambda_a : h \mapsto a$ *of* $\mathbb{F}[h]$ *with the corresponding scalars* $a \in \mathbb{F}$, *we have:*

$$\widehat{H}^{free} = \begin{cases} \mathbb{F} \setminus \{\gamma r^{-1}/(1 - r^{-1})\}, & \text{if } r \notin \sqrt{1}; \\ \mathbb{F}, & \text{if } \gamma \neq 0 = \text{char}(\mathbb{F}), \ r = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $r = s = 1$ *and* char $\mathbb{F} = 0$, *a quadratic Casimir operator* Ω *always exists, and the center of* A *is the polynomial algebra* $\mathbb{F}[\Omega]$.

Now suppose $r = 1$ *and* $\gamma \neq 0$, $f \not\equiv 0$.

(1) If $s = 1$, *then* $[\lambda]$ *is finite for one (equivalently, all) weights* λ *if and only if* char $\mathbb{F} = 0$.

(2) If s *is not a root of unity and* char $\mathbb{F} = 0$, *then* $[\lambda]$ *is finite for all* λ.

In particular, we conclude via Theorem 8.7 that if char $\mathbb{F} = 0$ and part (1) or (2) holds, then A satisfies Condition (S3) and hence $\mathcal{O}[\widehat{H}^{free}]$ has BGG Reciprocity.

Theorem 8.10 and its proof are similar in flavor to a subsequent result for "quantum" down-up algebras (see Theorem 8.14). The proofs of both of these results are deferred to Section 8.5.

8.3. **Further examples: quantum triangular GWAs.** Another well-studied and important class of algebras in the literature is similar in structure and has many properties in common with down-up algebras. These algebras have a "quantum" flavor; a prominent example is $U_q(\mathfrak{sl}_2)$. We now introduce the general notion of a *quantum triangular GWA*. This is a strict, based Hopf RTA of rank one, which includes as examples several algebras studied in the literature, and also resembles generalized down-up algebras.

To define a quantum triangular GWA, suppose Γ is an arbitrary abelian group equipped with a fixed character (or weight) $\alpha : \Gamma \to \mathbb{F}^\times$, and $H = \mathbb{F}\Gamma$ is its group algebra. Now define the associated quantum triangular GWA to be

$$\mathcal{W}(\Gamma) := \mathcal{W}(\mathbb{F}\Gamma, \theta = \rho_H(\alpha), z_0, z_1), \quad z_0, z_1 \in H, \tag{8.11}$$

where the weight-to-root map ρ_H was studied in Proposition 2.15. Note by Lemma 8.3 that $\mathcal{W}(\Gamma)$ satisfies Conditions (RTA1) and (RTA3). Moreover, quantum triangular GWAs do not fall under the framework of generalized down-up algebras, since H is now no longer a polynomial ring but a group algebra. The present work unites these two settings via triangular GWAs (from Definition

8.1). Moreover, quantum triangular GWAs encompass many families of quantum algebras studied in the literature:

(1) *Quantum $\mathfrak{sl}_2$:* A motivating and fundamental example is $U_q(\mathfrak{sl}_2)$. This is obtained by setting $\Gamma = \mathbb{Z}$ (more precisely, $\Gamma = K^{\mathbb{Z}}$ for some variable K), and $\alpha(K) = q^2, z_1 = 1, z_0 = \frac{K - K^{-1}}{q - q^{-1}}$ for some $q \neq 0, \pm 1$. More generally, Ji et. al. [JWZ] and Tang [Ta2] studied the quantum triangular GWAs with arbitrary $z_0 \in H = \mathbb{F}[K^{\pm 1}]$.

(2) The Drinfeld quantum double of the positive part of $U_q(\mathfrak{sl}_2)$ is a special case of a family of quantum algebras studied by Ji et. al. [JWY] as well as Tang-Xu [TX]. These algebras are also quantum triangular GWAs, where $H = \mathbb{F}[K^{\pm 1}, h^{\pm 1}]$ and $\alpha(K) = q^2, \alpha(h) = q^{-2}, z_1 = 1$.

(3) *Double loop quantum enveloping algebras:* This construction was discussed in Example 7.4.

(4) *Quantized Weyl algebras:* This is a degenerate example that we mention for completeness. Namely, when $H = \mathbb{F}$, α is the (constant) counit map on Γ, and $z_0 = 1, z_1 \neq 0$, one obtains the quantized Weyl algebras (and in particular, the first Weyl algebra A_1 if $z_1 = 1$).

Remark 8.12. Recall that Crawley-Boevey and Holland studied noncommutative deformations of Kleinian singularities in [CBH]. These are algebras associated with finite subgroups of $SL_2(\mathbb{C})$. In Type A, these algebras are triangular GWAs with $H = \mathbb{F}[\mathbb{Z}/n\mathbb{Z}]$ for $n \in \mathbb{N}$, together with $z_1 = 1$ and $\alpha(m + n\mathbb{Z}) := \varepsilon^m$ (where $\varepsilon \in \mathbb{F}^{\times}$ is a primitive nth root of unity). In general, one replaces $\mathbb{Z}/n\mathbb{Z}$ by a finite subgroup of $SL_2(\mathbb{F})$. In contrast, we will work with subgroups Γ of the torus $\mathbb{F}^{\times} \subset SL_2(\mathbb{F})$ (which we assume to be infinite in order to obtain a strict, based Hopf RTA structure).

Remark 8.13 (Ambiskew polynomial rings). All of the examples discussed above in this section have in common that $z_1 \in \mathbb{F}$. Triangular GWAs where $z_1 \in \mathbb{F}^{\times}$ are known as *ambiskew polynomial rings*. The study of ambiskew polynomial rings was initiated and carefully developed by Jordan (see [Jo1, Jo2] for more details). The subject continues to attract much interest – see for instance [BrMa, Ha, JW] and the references therein. We also remark that the level of generality in defining an ambiskew polynomial ring has varied throughout the literature. The current – and most general – version of an ambiskew polynomial ring can be found in [JW, Definition 2.2].

We now state a similar result to Theorem 8.10 for quantum triangular GWAs algebras, which characterizes when Condition (S3) holds for such algebras. In the following result, as in Theorem 8.10, we will assume that $z_1 \in H^{\times}$ is a unit.

Theorem 8.14. *In the setting of* (8.11), *the orders of θ, α, and $\Gamma / \ker(\alpha) \cong \alpha(\Gamma)$ are either all infinite, or all equal. Thus $\widehat{H} = \widehat{H}^{free}$ if and only if $\widehat{H}^{free}$ is nonempty, if and only if $\alpha(\Gamma) \subset \mathbb{F}^{\times}$ is infinite.*
Now suppose $z_1 = s \cdot [1_{\Gamma}] = s \in \mathbb{F}^{\times}$. Define $\sqrt{1}$ to be the roots of unity in $\mathbb{F}^{\times}$, and define

$$\Gamma_1 := \{g \in \Gamma : \alpha(g) = s^{-1}\}, \qquad \Gamma_2 := \{g \in \Gamma : \alpha(g)s \in \sqrt{1} \setminus \{1\}\}, \qquad \Gamma_3 := \{g \in \Gamma : \alpha(g)s \notin \sqrt{1}\}.$$

Also write $g \in \Gamma$ to denote $[g]$, and write

$$z_0 = \sum_{g \in \Gamma_1 \cup \Gamma_2} a_g g + \sum_{i,j} a_{ij} g_{ij} \in \mathbb{F}\Gamma$$

with $a_g, a_{ij} \in \mathbb{F}$, and $g_{ij} \in \Gamma_3$ such that $\alpha(g_{ij}^{-1} g_{kl})$ has finite order if and only if $j = l$.

(1) *Suppose there exists $\mu \in \widehat{H}^{free}$ such that at least one of the following equations holds:*

$$\sum_i \frac{a_{ij}\mu(g_{ij})}{1 - (\alpha(g_{ij})s)^{-1}} = 0 = \sum_{g \in \Gamma_1} a_g \mu(g), \qquad \forall j, \tag{8.15}$$

$$or \qquad \sum_i \frac{a_{ij}\alpha(g_{ij})\mu(g_{ij})}{1 - (\alpha(g_{ij})s)^{-1}} = 0 = \sum_{g \in \Gamma_1} a_g \mu(g), \qquad \forall j. \tag{8.16}$$

Then $[\mu]$ is infinite.

(2) *Conversely, if* $\operatorname{char} \mathbb{F} = 0$ *and* $[\lambda]$ *is infinite for at least one* $\lambda \in \widehat{H}^{free}$*, then at least one of*
(8.15) *and* (8.16) *holds for some* $\mu \in \mathbb{Z}\Delta * \lambda = \mathbb{Z}\theta * \lambda \subset \widehat{H}^{free}$.

As in the case of Theorem 8.10, the proof of Theorem 8.14 is deferred to Section 8.5. We also observe that quantum triangular GWAs with $z_1 = 1$ have certain similarities in structure and center, to symplectic reflection algebras (which were discussed in [EG, Eti]). We do not elaborate further on this point in the present paper.

8.4. **Quantization of generalized down-up algebras.** We now describe a concrete connection between a distinguished class of quantum triangular GWAs and generalized down-up algebras, which to our knowledge is not explored in the literature even though both families have been extensively studied (as indicated by the numerous references in this section). More precisely, recall that $U_q(\mathfrak{sl}_2)$ is a quantization of $U(\mathfrak{sl}_2)$, in the sense of taking a "classical limit" as $q \to 1$ to obtain $U(\mathfrak{sl}_2)$. Given the family of deformations of $U(\mathfrak{sl}_2)$ studied in [Smi], it is natural to ask if these triangular GWAs also admit quantizations, which are themselves then flat/PBW deformations of $U_q(\mathfrak{sl}_2)$. We now introduce a family of quantum triangular GWAs that provides a positive answer to this question for Smith's family of algebras, and more generally, for a large class of generalized down-up algebras.

Example 8.17 (Deformations of quantum $\mathfrak{sl}_2$ = quantization of generalized down-up algebras). Consider a generalized down-up algebra given by (8.9), with $\operatorname{char} \mathbb{F} = 0 \neq \gamma$ and $r = 1$. By Theorems 8.7 and 8.10, $\mathcal{W} = \mathcal{W}(\mathbb{F}[h], \theta_{1,\gamma}, s^{-1}f(h), s^{-1})$ is a strict, based HRTA of rank one, with $\widehat{H} = \widehat{H}^{free} = \mathbb{F}$.

Let q be an indeterminate over $\mathbb{F}$. We now propose a hitherto new family of triangular GWAs $\mathcal{W}_q$ over the $\mathbb{F}(q)$-algebra $H_q := \mathbb{F}(q)[K, K^{-1}]$, such that $\mathcal{W}$ is the "$q \to 1$" quasi-classical limit of the algebra $\mathcal{W}_q$. First define a more general family of $\mathbb{F}(q)$-algebras $\mathcal{W}(H_q = \mathbb{F}(q)[K^{\pm 1}], \theta, z_0', z_1')$ with $z_0', z_1' \in H_q$ and $\theta : H_q \to H_q$ an $\mathbb{F}(q)$-algebra automorphism of infinite order. As above, these algebras are strict, based RTAs of rank one. Now for the desired special case: given $l, m, n \in \mathbb{Z}$ with $l \neq 0$, define the $\mathbb{F}(q)$-algebra $\mathcal{W}_q(l, m, n)$ to be:

$$\mathcal{W}_q(l, m, n) := \mathcal{W}(\mathbb{F}(q)[K^{\pm 1}], \ \theta : K \mapsto q^{-l}K, \ s^{-1}q^m K^n f(\tfrac{-\gamma}{l} \cdot \tfrac{K-1}{q-1}), \ s^{-1}). \tag{8.18}$$

Observe that for various special cases of parameters, $\mathcal{W}_q(l, m, n)$ was studied earlier in the literature (but not in general). Namely, Ji et. al. [JWZ] and Tang [Ta2] studied the sub-family of algebras $\mathcal{W}_q(2, 0, 0)$ with $s = 1$ and $\theta(K) = q^{-2}K$.

We now prove that the algebras $\mathcal{W}_q(l, m, n)$ are indeed quantum analogues of Smith's family of deformations of $U(\mathfrak{sl}_2)$ – and more generally, the quantizations of a large class of generalized down-up algebras (8.9). Note that if such a result is to hold, then highest weight modules over $\mathcal{W}_q(l, m, n)$ should also "specialize" to highest weight modules over the classical limit. It is natural to ask how the corresponding highest weights are related.

To answer these questions, a natural procedure to follow is that in [HK, Chapter 3] (see also [Lu]) – although several of the steps therein need to be modified, as explained presently. Let R be the local subring of $\mathbb{F}(q)$, of rational functions that are regular at the point $q = 1$. Also define

$$(K^n; m)_q := \frac{q^m K^n - 1}{q - 1}, \qquad m, n \in \mathbb{Z}.$$

Now let $\mathcal{W}_q^R(l, m, n)$ denote the (unital) R-subalgebra of $\mathcal{W}_q(l, m, n)$ generated by $U, D, K^{\pm 1}$, and $(K; 0)_q = (K - 1)/(q - 1)$. Then the following result holds.

Theorem 8.19 (Deformation-quantization equals quantization-deformation). *Suppose* $\mathbb{F}$ *is a field of characteristic zero,* $\gamma \in \mathbb{F}$*, and* $\theta_{1,\gamma} \in \operatorname{Aut}_{\mathbb{F}-alg} \mathbb{F}[h]$ *sends* h *to* $h + \gamma$*. Fix* $f \in \mathbb{F}[h]$*,* $r = 1$*, and* $s \in \mathbb{F}^\times$ *not a root of unity. Now define* $\mathcal{W}_q(l, m, n)$ *as in* (8.18)*, with* $z_1 = s^{-1}$ *and* $z_0 =$

$s^{-1}q^m K^n f(-\gamma(K;0)_q/l)$ *for some* $l \neq 0, m, n \in \mathbb{Z}$. *Then,*

$$\mathcal{W}_1 := \mathcal{W}_q^R(l, m, n)/(q-1)\mathcal{W}_q^R(l, m, n) \cong \mathcal{W}(\mathbb{F}[h], \theta_{1,\gamma}, s^{-1}f(h), s^{-1}). \tag{8.20}$$

Now fix a scalar $\lambda \in \mathbb{F}(q)^\times$ *such that* $\dfrac{\lambda - 1}{q - 1} \in R$, *and a highest weight module* $M_q(\lambda) \twoheadrightarrow \mathbb{V}_q^\lambda$ *over* $\mathcal{W}_q(l, m, n)$, *where we identify* λ *with the* $\mathbb{F}(q)$*-weight of* H_q *sending* K *to* λ. *If* $v_\lambda \in (\mathbb{V}_q^\lambda)_\lambda$ *generates* $\mathbb{V}_q^\lambda$, *then*

$$\mathbb{V}_1^\lambda := \mathcal{W}_q^R(l, m, n)v_\lambda/(q-1)\mathcal{W}_q^R(l, m, n)v_\lambda \tag{8.21}$$

is a highest weight module over $\mathcal{W}_1 \cong \mathcal{W}(\mathbb{F}[h], \theta_{1,\gamma}, s^{-1}f(h), s^{-1})$ *with highest* $\mathbb{F}[h]$*-weight given by* $h \mapsto \dfrac{-\gamma}{l} \cdot \dfrac{\lambda(K) - 1}{q - 1}\Big|_{q \to 1}$, *and with the same graded character as* $\mathbb{V}_q^\lambda$ *(up to modification of the highest weight).*

In particular when $s = 1$, the family of algebras studied by Smith [Smi] are indeed "classical limits" (as $q \to 1$) of triangular GWAs. Note that these algebras also provide deformations of $U_q(\mathfrak{sl}_2)$ (for $s = 1$).

Proof. We follow the approach in [HK, Chapter 3], developing the results for both $\mathcal{W}_1$ and $\mathbb{V}_1^\lambda$ simultaneously. We outline the steps, omitting the proofs when they are similar to those in *loc. cit.* The meat of the (new) proof lies in Step 5.

(1) Set $\mathcal{W}_\pm^R$ to be $R[U], R[D]$ respectively, and $\mathcal{W}_0^R$ to be the R-subalgebra of $H_q = \mathbb{F}(q)[K^{\pm 1}]$ that is generated by $K^{\pm 1}$ and $(K;0)_q$. Then all elements of the form $(K^n; m)_q$ and $\frac{\beta K - \beta^{-1}K^{-1}}{q - q^{-1}}$ lie in $\mathcal{W}_0^R$, where $m, n \in \mathbb{Z}$ and $\beta \in R^\times$ such that $1 = \beta|_{q \to 1} := \beta \mod (q-1)R$.

(2) The multiplication map $: \mathcal{W}_-^R \otimes_R \mathcal{W}_0^R \otimes_R \mathcal{W}_+^R \to \mathcal{W}_q^R(l, m, n)$, induced from the triangular decomposition of $\mathcal{W}_q(l, m, n)$, is an isomorphism of R-algebras.

(3) Henceforth, fix a weight $\lambda \in \widehat{H_q}$ such that $\frac{\lambda(K)-1}{q-1} \in R$, as well as a highest weight module $M_q(\lambda) \twoheadrightarrow \mathbb{V}_q^\lambda$. The R-*form* of $\mathbb{V}_q^\lambda$ is defined to be $\mathbb{V}_R^\lambda := \mathcal{W}_q^R(l, m, n)v_\lambda$, where v_λ is the image of 1 under the map $\mathcal{W}_q(l, m, n) \twoheadrightarrow M_q(\lambda) \twoheadrightarrow \mathbb{V}_q^\lambda$. Via the previous step, we claim:

$$\mathbb{V}_R^\lambda = \mathcal{W}_-^R v_\lambda = \bigoplus_{\mu \leq \lambda} (\mathbb{V}_R^\lambda)_\mu.$$

More precisely, we assert that the R-form $\mathbb{V}_R^\lambda$ is $\mathcal{W}_0^R$-semisimple, with each weight space a free rank one R-module with R-basis $D^n v_\lambda$ for (unique) $n \geq 0$. Moreover, $\mathbb{F}(q) \otimes_R \mathbb{V}_R^\lambda = \mathbb{V}_q^\lambda$.

In this step, we only explain why $\mathbb{V}_R^\lambda$ is $\mathcal{W}_0^R$-semisimple. First note that the weights of $\mathbb{V}_R^\lambda$ are of the form $\theta^{-n} * \lambda$ for $n \geq 0$. Thus suppose $v = \sum_{j=1}^k v_j \in \mathbb{V}_R^\lambda$ with v_j of weight $\theta^{-n_j} * \lambda$ for $0 \leq n_1 < n_2 < \cdots$. The first claim is that for each fixed j, the "interpolating polynomial"

$$I_j := \prod_{k \neq j} \frac{\lambda(K)^{-1}q^{ln_k}K - 1}{q^{l(n_k - n_j)} - 1}$$

lies in $\mathcal{W}_0^R$. Indeed, we show that each factor lies in $\mathcal{W}_0^R$ by computing for any $r, 0 < s \in \mathbb{Z}$:

$$\frac{\lambda(K)^{-1}q^r K - 1}{q^s - 1} = \frac{\lambda(K)^{-1}q^r}{1 + \cdots + q^{s-1}}(K;0)_q + \lambda(K)^{-1}\frac{q^r - 1}{q^s - 1} - \frac{\lambda(K)^{-1}}{1 + \cdots + q^{s-1}} \cdot \frac{\lambda(K) - 1}{q - 1},$$

and this is indeed in $\mathcal{W}_0^R$ by assumption. Now apply the quantity I_j (defined above) to v_λ to obtain v_j. Thus $v_j \in \mathbb{V}_R^\lambda \, \forall j$, proving the $\mathcal{W}_0^R$-semisimplicity of $\mathbb{V}_R^\lambda$.

(4) Define $\mathfrak{m} := (q-1)R \subset R$ to be the unique maximal ideal of the local ring R, and $\mathcal{W}_1 := \mathcal{W}_q^R(l,m,n)/\mathfrak{m}\mathcal{W}_q^R(l,m,n)$, $\mathbb{V}_1^\lambda := \mathbb{V}_R^\lambda/\mathfrak{m}\mathbb{V}_R^\lambda$. These are called the *classical limits* of $\mathcal{W}_q(l,m,n)$ and $\mathbb{V}_q^\lambda$ respectively. Also define $(\mathbb{V}_1^\lambda)_\mu := (R/\mathfrak{m}) \otimes_R (\mathbb{V}_R^\lambda)_\mu$. Then $\mathbb{V}_1^\lambda$ is a $\mathcal{W}_1$-module, and each weight space is one-dimensional with $\mathbb{F}$-basis $\overline{D^n v_\lambda}$ for some integer $n \geq 0$.

(5) There exists a surjection of algebras $\pi : \mathcal{W} = \mathcal{W}(\mathbb{F}[h], \theta_{1,\gamma}, s^{-1}f(h), s^{-1}) \twoheadrightarrow \mathcal{W}_1$, which sends u, d, h to the images of $U, D, -\gamma(K;0)_q/l$ respectively, under the quotient map : $\mathcal{W}_q(l,m,n) \twoheadrightarrow \mathcal{W}_1$. To see why, first note that the image of $(q-1)(K;0)_q = K - 1$ is zero in $\mathcal{W}_1$, whence $\overline{K} = 1$ in $\mathcal{W}_1$. This shows the surjectivity of the map π if we show that π is an algebra map. We verify one of the relations; the others are similar. Namely, $\pi(u)\pi(h)$ is the image in $\mathcal{W}_1$ of

$$U \cdot \frac{-\gamma}{l} \frac{K-1}{q-1} = \frac{-\gamma}{l} \frac{Kq^{-l}-1}{q-1}U = \frac{-\gamma}{l} \cdot K \cdot \frac{q^{-l}-1}{q-1}U + \frac{-\gamma}{l}\frac{K-1}{q-1}U,$$

and the image of the right-hand side in $\mathcal{W}_1$ is precisely $(-\gamma/l) \cdot 1 \cdot (-l)U + \pi(h)U = (\pi(h) + \gamma)\pi(u)$, as desired.

The meat of the proof lies in showing that the surjection π is an isomorphism of algebras. We now describe an argument that utilizes the GWA structure in our setting, as opposed to the symmetries under the Weyl group in the setting of [HK, Chapter 3].

Note that $\pi : \mathcal{W} \twoheadrightarrow \mathcal{W}_1$ restricts to a surjection of algebras on the respective factors in the two triangular decompositions. We first claim that π is an isomorphism of Cartan subalgebras. Indeed, given $0 \neq p(h) \in \mathbb{F}[h] = \mathcal{W}_0$, choose $x \in \mathbb{F}$ such that $p(x) \neq 0$ (since $\mathbb{F}$ is infinite). Define $\lambda \in \widehat{H_q}$ via:

$$\lambda : K \mapsto 1 - xl(q-1)/\gamma \in 1 + (q-1)\mathbb{F} \subset 1 + \mathfrak{m} \subset R^\times,$$

since R is a commutative local ring. Then the above analysis of $\mathbb{V}_q^\lambda$ (in steps (3) and (4)) holds, and $\pi(p(h))$ acts on the highest weight space of $\mathbb{V}_1^\lambda$ by the scalar $p(x) \neq 0$. Therefore $\pi(p(h)) \neq 0$, whence $\pi|_{\mathcal{W}_0}$ has zero kernel, and hence is an isomorphism of Cartan subalgebras.

We now claim that $\pi|_{\mathcal{W}_-}$ also has trivial kernel. To prove the claim, first fix any field extension $\mathbb{F}_u$ of $\mathbb{F}$, with $\mathbb{F}_u$ an uncountable field. Since $\mathcal{W}_q(l,m,n)$ is the quotient of the tensor algebra $T_{\mathbb{F}(q)}(\text{span}_{\mathbb{F}(q)}(K, K^{-1}, U, D))$ by an ideal, it is possible to tensor this construction with $\mathbb{F}_u$ to obtain the same algebra over $\mathbb{F}_u(q)$. Label these algebras $\mathcal{W}_q^\mathbb{F}$ and $\mathcal{W}_q^{\mathbb{F}_u}$ respectively, and similarly for the other algebras considered in the previous steps. Now reconsider the entirety of the above procedure over $\mathbb{F}_u(q)$ instead of $\mathbb{F}(q)$. We then make the *sub-claim* that $\pi|_{\mathcal{W}^{\mathbb{F}_u}}$ has trivial kernel. To see why, note that $\mathcal{W}_-^{\mathbb{F}_u} \cong \mathbb{F}_u[d] \twoheadrightarrow (\mathcal{W}_1^{\mathbb{F}_u})_-$, and this in turn surjects onto every highest weight module. Thus it suffices to produce an infinite-dimensional Verma module over $\mathcal{W}_1^{\mathbb{F}_u}$.

Now recall from Definition 8.5 that $\widetilde{z}_n = \sum_{i=0}^{n-1} s^{-(n-i)} f(h + i\gamma)$ is a nonzero polynomial in h of degree $\deg(f)$ (since s is not a root of unity). Thus it has finitely many roots for each n. Since $\mathbb{F}_u$ is uncountable, choose $x \in \mathbb{F}_u$ that is not a root of $\widetilde{z}_n$ for any $n \geq 0$. It follows that the Verma module $M_1^{\mathbb{F}_u}(\lambda_x)$ is simple over $\mathcal{W}_1^{\mathbb{F}_u}$. In particular, $(\mathcal{W}_1^{\mathbb{F}_u})_-$ is infinite-dimensional over $\mathbb{F}_u$. Finally, $(\mathcal{W}_1^{\mathbb{F}_u})_- = \mathbb{F}_u \otimes_\mathbb{F} (\mathcal{W}_1^\mathbb{F})_-$, so we obtain that $(\mathcal{W}_1)_- = (\mathcal{W}_1^\mathbb{F})_-$ is also infinite-dimensional over $\mathbb{F}$. Thus $\pi|_{\mathcal{W}_-}$ is also an algebra isomorphism as claimed.

Having shown the claim for $\pi|_{\mathcal{W}_-}$, one shows the same result for $\pi|_{\mathcal{W}_+}$, either by a similar argument using lowest weight theory, or directly via the anti-involutions in both settings from (RTA3). Thus $\pi : \mathcal{W} \twoheadrightarrow \mathcal{W}_1$ is an isomorphism of algebras using (RTA1).

(6) It follows using the previous step that $\mathbb{V}_1^\lambda$ is a $\mathcal{W}$-module (since it is a $\mathcal{W}_1$-module), with the same weight bases for both module structures.

One now shows that $\mathbb{V}_1^\lambda$ is a highest weight module over $\mathcal{W}$, with the same formal character as $\mathbb{V}_q^\lambda$. Moreover, the highest h-weight for $\mathbb{V}_1^\lambda$ is precisely as claimed, since h acts on the highest weight space via $\pi(h)$, i.e. by the scalar $\left. \dfrac{-\gamma}{l} \cdot \dfrac{\lambda(K) - 1}{q - 1} \right|_{q \to 1}$ as claimed. We also remark that if $\mathbb{V}_1^\lambda$ is simple but $\mathbb{V}_q^\lambda$ has a maximal vector of weight $\theta^{-n} * \lambda < \lambda$, then since the two graded characters are equal, the corresponding vector in $\mathbb{V}_1^\lambda$ would also be maximal, which is impossible. It follows that $\mathbb{V}_q^\lambda$ is a simple $\mathcal{W}_q(l, m, n)$-module if $\mathbb{V}_1^\lambda$ is a simple $\mathcal{W}_1$-module.

$\square$

8.5. Solutions of polynomial-exponential equations.

We conclude this section by showing Theorems 8.10 and 8.14. The proofs use a result on "polynomial-exponential equations" over a general field. We begin with a result by Schlickewei [Sch] that was proved for number fields. Namely, Schlickewei showed that a special family of equations (with argument $n \in \mathbb{Z}$) has only finitely many integer solutions.

Theorem 8.22 (Schlickewei [Sch, Theorem 1.1]). *Given a field* $\mathbb{F}$ *of characteristic zero, consider the* polynomial-exponential equation *(with argument* $n \in \mathbb{Z}$*):*

$$F_n := \sum_{j=1}^{m} p_j(n)\alpha_j^n = 0, \qquad n \in \mathbb{Z}, \tag{8.23}$$

where $m \in \mathbb{N}$, $0 \not\equiv p_j \in \mathbb{F}[X], \alpha_j \in \mathbb{F}^\times \ \forall j \le m$, *and* α_i/α_j *is not a root of unity for all* $i \ne j$.
If $\mathbb{F}$ *is an algebraic number field, then* (8.23) *has only finitely many solutions in* $\mathbb{Z}$.

It turns out that Theorem 8.22 is true in all fields of characteristic zero; as we are unsure if this is mentioned in the literature, we write down a proof for completeness. (The proof does not use Theorem 8.22.)

Theorem 8.24. *The conclusion of Theorem 8.22 holds over any field* $\mathbb{F}$ *of characteristic zero.*

Proof. We prove the result in various steps. The first step is to claim that every such polynomial-exponential function gives rise to a *linear recurrence sequence* $\{F_n : n \in \mathbb{Z}\}$ (with suitable initial values); this has essentially been shown for any field in [MvP, Section 2].

Now suppose F_n vanishes infinitely often in $\mathbb{Z}$, say on the set T. (We will prove that $p_i \equiv 0 \ \forall i$.) If $T \subset \mathbb{Z}$ is the set of zeros, then we restrict to $T' = T \cap \mathbb{N}$ if this is an infinite set. Otherwise $T' \cap -\mathbb{N}$ is infinite, and changing every α_i to α_i^{-1} and p_i to a new polynomial $q_i(X) := p_i(-X)$ if necessary, we may assume that $F_n = 0$ for all n in an infinite set $T' \subset \mathbb{N}$. (Note that $q_i \equiv 0 \Leftrightarrow p_i \equiv 0$, so we may work with the new setup now.)

Since $\operatorname{char} \mathbb{F} = 0$, we conclude by the Skolem-Mahler-Lech Theorem [Lech] that F_n vanishes for all n in an infinite arithmetic progression, say $r + \mathbb{N}d$. But then

$$\sum_j (p_j(r + dn)\alpha_j^r)(\alpha_j^d)^n = 0 \ \forall n \in \mathbb{N}.$$

Once again, we fix $d \ne 0, r$ and call the new polynomial $q_j(X) := p_j(r + dX)$; then $q_j \equiv 0$ if and only if $p_j \equiv 0$. Also set $\beta_j := \alpha_j^d$; these are pairwise distinct, and we are left to prove the following

Claim. Fix pairwise distinct $\beta_i \in \mathbb{F}$ and polynomials $q_i(T) \in \mathbb{F}[T]$, for a field $\mathbb{F}$ of characteristic zero. If $G(n) := \sum_i q_i(n)\beta_i^n = 0 \ \forall n \in \mathbb{N}$, then all the polynomials q_i are identically zero.

We prove this claim by assuming it to be false and obtaining a contradiction. If the claim is false, then $D := \sum_{i : q_i \not\equiv 0} \deg(q_i)$ is defined (and nonnegative). Now obtain a contradiction by induction on D. (The base case $D = 0$ is treated using the Vandermonde determinant from $G(1), \ldots, G(n)$; for the general case, consider $H(n) := G(n+1) - \beta_i G(n)$, where $\deg q_i > 0$.) $\square$

It is now possible to show that a large number of "classical" and "quantum" generalized down-up algebras satisfy Condition (S3).

Proof of Theorem 8.10. Throughout this proof we use θ instead of $\theta_{r,\gamma}$. Fix an algebra map $\lambda : H \to \mathbb{F}$. First suppose that $r = 1$; then $\lambda \circ \theta^n(h) = \lambda(h) + n\gamma$. Thus $\lambda \in \widehat{H}^{free}$ if and only if $n\gamma$ is never zero for $n > 0$, i.e., char $\mathbb{F} = 0 \neq \gamma$. Next, if $r \neq 1$, then compute:

$$\lambda \circ \theta^n(h) = r^{-n}\lambda(h) + r^{-1}\gamma\frac{1 - r^{-n}}{1 - r^{-1}}.$$

It is clear that if r is a root of unity, then this expression equals $\lambda(h)$ for all h, for infinitely many n. On the other hand, if $r \notin \sqrt{1}$, then it is clear for any $n > 0$ that

$$\lambda \equiv \lambda \circ \theta^n \quad \Longleftrightarrow \quad \lambda(h) = \frac{r^{-1}\gamma}{1 - r^{-1}},$$

and this completes the proof of the first part. Next when $r = s = 1$, it is not hard to show that $Z(A) \cap \mathbb{F}[h] = \mathbb{F}$. Moreover, a quadratic Casimir operator always exists because of the identity $\binom{X}{k} = \binom{X-1}{k-1} + \binom{X-2}{k-1} + \cdots$, which helps show that power sums $\sum_{i=1}^n i^k$ are polynomials in n of degree $k + 1$ with rational coefficients.

Finally, to study the sets $[\lambda]$ we first compute for $n > 0$ and $r = 1$:

$$\widetilde{z}_n = \sum_{i=0}^{n-1} s^{-1}\theta^i(f(h))s^{-i} = \sum_{i=0}^{n-1} s^{-1-i}f(\theta^i(h)) = \sum_{i=0}^{n-1} s^{-1-i}f(h + i\gamma).$$

If $f \equiv 0$ then clearly $\widetilde{z}_n = 0$ and $[\lambda]$ is infinite for every weight λ. Now suppose $\gamma \neq 0$ and $f \not\equiv 0$ is of the form $f(h) = \sum_{j=1}^k c_j h^{m_j}$ for integers $0 \leq m_1 < \cdots < m_k$, with $c_k \in \mathbb{F}^\times$. We first assume that $r = 1$ and compute:

$$\widetilde{z}_n = \sum_{i=0}^{n-1} s^{-1-i}\sum_{j=1}^k c_j(h+i\gamma)^{m_j} = \sum_{i=0}^{n-1}\sum_{j=1}^k\sum_{l=0}^j s^{-1-i}c_j\binom{j}{l}h^{j-l}\gamma^l i^l = \sum_{j=1}^k\sum_{l=0}^j c_j\binom{j}{l}h^{j-l}\gamma^l \sum_{i=0}^{n-1} s^{-1-i}i^l.$$

If moreover $s = 1$, then it is clear that $\widetilde{z}_n = 0$ if char$(\mathbb{F})|n$ (since for every prime $p > 0$ and all integers $l \geq 0$, $\sum_{i=0}^{p-1} i^l$ is divisible by p, by using the primitive generator of $\mathbb{Z}/p\mathbb{Z}$). Now if char $\mathbb{F} = 0$, then $\lambda(\widetilde{z}_n)$ is a polynomial in n of degree at most $1 + m_k$, so it has only finitely many roots $n > 0$. A similar argument for $n < 0$ shows that $[\lambda]$ is always finite if $r = s = 1$ and char $\mathbb{F} = 0$. On the other hand, if char $\mathbb{F} > 0$ and $r = s = 1$, then $[\lambda]$ is always infinite.

Now suppose char $\mathbb{F} = 0$, $\gamma \neq 0$, $r = 1$, and $s \notin \sqrt{1}$. First assume by a change of variables that $\gamma = 1$, without loss of generality; since char $\mathbb{F} = 0$, one can then write the polynomial $f(h)$ as a linear combination of the basis elements $t_{n,s}(h) := s^{-1}\binom{h+1}{n} - \binom{h}{n}$ of $\mathbb{F}[h]$. Now if $f \equiv \sum_{j\geq 0} a_j t_{j,s}$ (finite sum), then define $\widetilde{f}(h) := s^{-1}\sum_{j\geq 0} a_j\binom{h}{j}$. Then for $n \geq 0$,

$$\widetilde{z}_n = \sum_{i=0}^{n-1} s^{-1-i}f(h + i) = \sum_{i=0}^{n-1} s^{-i}(s^{-1}\widetilde{f}(h + i + 1) - \widetilde{f}(h + i)) = s^{-n}\widetilde{f}(h + n) - \widetilde{f}(h),$$

$$\widetilde{z}_{-n} = \theta^{-n}(\widetilde{z}_n) = s^{-n}(\widetilde{f}(h) - s^n\widetilde{f}(h - n)).$$

Now given any weight λ, applying Theorem 8.24 to the nontrivial polynomial-exponential equation (in $n \in \mathbb{Z}$) given by

$$F_n := \lambda(\widetilde{z}_n) = \widetilde{f}(\lambda(h))1^n + (-\widetilde{f}(\lambda(h) + n))(s^{-1})^n = 0$$

shows that there are only finitely many integer solutions, whence $[\lambda]$ is finite for every λ. $\square$

Finally, we show the analogous result (to Theorem 8.10) for quantum triangular GWAs.

Proof of Theorem 8.14. The first part is standard (noting that if $\alpha(\Gamma) \subset \mathbb{F}^\times$ is a finite group, then it is cyclic.) In order to show the next two parts, we first define N_0 to be the least common multiple of the orders of the roots of unity $\{\alpha(g)s : g \in \Gamma_1 \cup \Gamma_2\} \subset \sqrt{1}$, as well as of the orders of $\alpha(g_{ij}^{-1}g_{kj})$ over all i,j,k. Now compute for any $\mu \in \widehat{H}^{free}$ that $\mu(z_n') = z_n' = s^n$ for all $n \geq 0$. Therefore we obtain for $n > 0$:

$$\mu(\widetilde{z}_n) = \sum_{i=0}^{n-1} s^{n-1-i} \sum_{g \in \Gamma} a_g \alpha(g)^{-i} \mu(g) = ns^{n-1} \sum_{g \in \Gamma_1} a_g \mu(g) + s^{n-1} \sum_{g \in \Gamma_2 \cup \Gamma_3} a_g \mu(g) \frac{1 - (\alpha(g)s)^{-n}}{1 - (\alpha(g)s)^{-1}},$$

$$\mu(\widetilde{z}_{-n}) = \mu(\theta^{-n}(\widetilde{z}_n)) = \sum_{i=0}^{n-1} s^{n-1-i} \sum_{g \in \Gamma} a_g \alpha(g)^{n-i} \mu(g) = \sum_{g \in \Gamma} \sum_{i=0}^{n-1} a_g \alpha(g) \mu(g) (\alpha(g)s)^i \qquad (8.25)$$

$$= ns^{-1} \sum_{g \in \Gamma_1} a_g \mu(g) + \sum_{g \in \Gamma_2 \cup \Gamma_3} a_g \alpha(g) \mu(g) \frac{1 - (\alpha(g)s)^n}{1 - (\alpha(g)s)}.$$

We now show the two remaining parts in this result.

(1) If (8.15) holds, then we claim that $\mu(\widetilde{z}_{mN_0}) = 0$ for all $m \in \mathbb{N}$. Indeed, the sum in (8.25) over $g \in \Gamma_1$ vanishes by assumption, and we are left with:

$$\mu(\widetilde{z}_{mN_0}) = s^{mN_0-1} \sum_{g \in \Gamma_2} a_g \mu(g) \frac{1 - (\alpha(g)s)^{-mN_0}}{1 - (\alpha(g)s)^{-1}} + s^{mN_0-1} \sum_j \sum_i a_{ij} \mu(g_{ij}) \frac{1 - (\alpha(g_{ij})s)^{-mN_0}}{1 - (\alpha(g_{ij})s)^{-1}}.$$

By construction, each summand of the sum over $g \in \Gamma_2$ vanishes, and moreover, the element $(\alpha(g_{ij})s)^{-mN_0}$ is independent of i for each fixed j. Thus, we obtain:

$$\mu(\widetilde{z}_{mN_0}) = s^{mN_0-1} \sum_j (1 - (\alpha(g_{1j})s)^{-mN_0}) \sum_i \frac{a_{ij} \mu(g_{ij})}{1 - (\alpha(g_{1j})s)^{-1}},$$

which vanishes by assumption, proving the claim.

Similarly, one shows using (8.25) that if (8.16) holds, then $\mu(\widetilde{z}_{-mN_0}) = 0$ for all $m \in \mathbb{N}$.

(2) Conversely, suppose $[\lambda]$ is infinite for $\lambda \in \widehat{H}^{free}$. Then at least one of $[\lambda] \cap (\pm \mathbb{N}\theta * \lambda)$ is infinite. Suppose first that the former case holds. Define N_0 as above; then there exists $n_0 \in \mathbb{N}$ such that $[\lambda] \cap ((n_0 + N_0\mathbb{N})\theta * \lambda)$ is infinite. Thus, fix $0 < n_1 < n_2 < \dots$ such that $(n_0 + N_0 n_k)\theta * \lambda \in [\lambda]$ for all $k > 0$. Then using (8.25),

$$0 = s^{1-n_0-N_0 n_k} \lambda(\widetilde{z}_{n_0+N_0 n_k}) = (n_0 + N_0 n_k) \sum_{g \in \Gamma_1} a_g \lambda(g) + \sum_{g \in \Gamma_2 \cup \Gamma_3} a_g \lambda(g) \frac{1 - (\alpha(g)s)^{-n_0-N_0 n_k}}{1 - (\alpha(g)s)^{-1}},$$

for all $k \in \mathbb{N}$. Rearranging this expansion, we obtain that

$$\left(p_{00} + N_0 \sum_{g \in \Gamma_1} a_g \lambda(g) X \right) 1^X + \sum_j p_j \beta_j^X = 0, \qquad \forall X = n_1, n_2, \dots$$

where

$$p_{00} := \sum_{g \in \Gamma_2} a_g \lambda(g) \frac{1 - (\alpha(g)s)^{-n_0}}{1 - (\alpha(g)s)^{-1}} + \sum_{g \in \Gamma_3} \frac{a_g \lambda(g)}{1 - (\alpha(g)s)^{-1}} + n_0 \sum_{g \in \Gamma_1} a_g \lambda(g) \in \mathbb{F},$$

$$p_j := -s^{-n_0} \sum_i \frac{a_{ij} \lambda(g_{ij})}{1 - (\alpha(g_{ij})s)^{-1}} \alpha(g_{ij})^{-n_0}, \qquad (8.26)$$

$$\beta_j := (\alpha(g_{1j})s)^{-N_0}.$$

Now note that 1 and the β_j are distinct, and the ratio of no two of these is a root of unity. Since $\operatorname{char} \mathbb{F} = 0$, Theorem 8.24 now implies that

$$p_{00} = \sum_{g \in \Gamma_1} a_g \lambda(g) = p_j = 0 \qquad \forall j.$$

Finally, define $\mu := \theta^{n_0} * \lambda$. Then,

$$0 = -s^{n_0} p_j = \sum_i \frac{a_{ij} \mu(g_{ij})}{1 - (\alpha(g_{ij}s))^{-1}} \qquad \forall j,$$

$$0 = s^{n_0} \sum_{g \in \Gamma_1} a_g \lambda(g) = \sum_{g \in \Gamma_1} a_g \lambda(g) \alpha(g)^{-n_0} = \sum_{g \in \Gamma_1} a_g \mu(g),$$

and (8.15) follows. A similar analysis shows using (8.25) and Theorem 8.24 that if $[\lambda] \cap (-\mathbb{N}\theta * \lambda)$ is infinite, then (8.16) holds, which concludes the proof.

$\square$

9. Non-Hopf examples of RTAs

Note that all of the previous examples of triangular GWAs in Section 8 – with the exception of generalized down-up algebras (8.9) with $r \neq 1$ (such as Example 3.4) – were strict Hopf RTAs. We now provide examples of triangular GWAs that are not Hopf RTAs. The Hopf structure in the examples gets increasingly weaker, in the following precise sense:

- As a first example, consider Example 3.4, in which $H = \mathbb{F}[h]$ is a Hopf algebra, but the Hopf structure is (necessarily) not used.
- In the second example – see Example 9.1 – H is a topological Hopf algebra but not a Hopf algebra.
- In the final example – see Example 9.4 – H is not even a topological Hopf algebra.

Example 9.1 (*Continuous Hecke algebra of $\mathfrak{gl}_1$*). Let $\mathbb{F}$ be any field, and $H = \mathcal{O}(\mathbb{F}^\times)^* = \mathbb{F}[T^{\pm 1}]^* = \mathbb{F}[[t^{\pm 1}]]$, the algebra of "Fourier series" or distributions on the unit circle (if $\mathbb{F} = \mathbb{C}$). This is a topological Hopf algebra with coordinatewise multiplication, and other Hopf operations given by

$$\eta(1) = \sum_{n \in \mathbb{Z}} t^n, \quad \Delta(t^m) = \sum_{n \in \mathbb{Z}} t^n \otimes t^{m-n} \in H \widehat{\otimes} H, \quad \varepsilon\left(\sum_{n \in \mathbb{Z}} a_n t^n\right) = a_0, \quad S\left(\sum_{n \in \mathbb{Z}} a_n t^n\right) = \sum_{n \in \mathbb{Z}} a_n t^{-n}.$$

The corresponding triangular GWA with $z_1 = 1$ is the *continuous Hecke algebra* of $GL(1)$ and $\mathbb{F} \oplus \mathbb{F}^*$, where θ is (non-coordinatewise) multiplication by t, i.e.,

$$\mathsf{H}_\kappa(GL(1), \mathbb{F} \oplus \mathbb{F}^*) := \mathcal{W}(\mathcal{O}(\mathbb{F}^\times)^*, \theta, \kappa, 1), \quad \kappa \in \mathcal{O}(\mathbb{F}^\times)^*, \quad \theta\left(\sum_{n \in \mathbb{Z}} a_n t^n\right) := \sum_{n \in \mathbb{Z}} a_n t^{n+1} = \sum_{n \in \mathbb{Z}} a_{n-1} t^n.$$

Continuous Hecke algebras were introduced in [EGG] as "continuous" generalizations of Drinfeld's family of degenerate affine Hecke algebras. The family of algebras under discussion is in some sense the simplest special case, of "Lie rank zero". Higher (Lie) rank examples of infinitesimal Hecke algebras are discussed in the following section. In this section and the next, we differentiate between the Lie rank of an infinitesimal Hecke algebra (which equals the rank of the underlying reductive Lie algebra $\mathfrak{g}$) and the "(RTA) rank" of a strict, based RTA given in Definition 2.5. In fact, the based Hopf RTAs considered in Section 10 are not strict, hence we will only talk about their Lie rank, but not their RTA-rank.

Remark 9.2. Observe that $H = \mathbb{F}[[t^{\pm 1}]]$ is the $\mathbb{F}$-algebra of functions on $\mathbb{Z}$. Thus if $\kappa = 0$, then the triangular GWA $\mathsf{H}_\kappa(GL(1), \mathbb{F} \oplus \mathbb{F}^*)$ also equals $\mathcal{A}(\mathbb{Z}^+)$, where $\mathcal{A}(Q_0^+)$ was defined in Theorem 4.6, with $\theta_1 \ltimes \theta_2^{-1} := \theta_2^{-1}$ for $\theta_1, \theta_2 \in Q_0^+ = \mathbb{Z}^+$.

We now list some of the properties of (Lie) rank zero continuous Hecke algebras, which are triangular GWAs from above.

Proposition 9.3. *Suppose $\kappa = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathcal{O}(\mathbb{F}^\times)^*$ and $\mathsf{H}_\kappa := \mathsf{H}_\kappa(GL(1), \mathbb{F} \oplus \mathbb{F}^*)$.*

(1) H_κ is a strict, based RTA of rank one, but not a Hopf RTA.

(2) The set of weights is $\widehat{H} = \{\mu_m : m \in \mathbb{Z}\}$ with $\mu_m(t^n) := \delta_{m,n}$. Moreover, $\theta^n * \mu_m = \mu_{n+m}$ for $m, n \in \mathbb{Z}$, so $\widehat{H}^{free} = \widehat{H}$.

(3) For all $m \in \mathbb{Z}$, define $s_{m,n}(\kappa) := a_m + a_{m-1} + \cdots + a_{m+n+1}$ for $n < 0$, $s_{m,0}(\kappa) := 0$, and $s_{m,n}(\kappa) := a_{m+1} + \cdots + a_{m+n}$ for $n > 0$. Then the Verma module $M(\mu_m)$ is uniserial, with

$$l(M(\mu_m)) = \#\{n \leq 0 : s_{m,n} = 0\}, \qquad |S^3(\mu_m)| = |[\mu_m]| = \#\{n \in \mathbb{Z} : s_{m,n} = 0\}.$$

We remark that $s_{m,n}(\kappa) = \sum_{j=1+\min(m,m+n)}^{\max(m,m+n)} a_j$ for all $m, n \in \mathbb{Z}$.

Proof. The first part follows from Theorem 8.7, and the second holds since $\{t^n : n \in \mathbb{Z}\}$ is a complete set of primitive idempotents in H. To show the third part, we compute using Definition 8.5 and that $z_1 = 1$:

$$\widetilde{z}_n = \sum_{i=0}^{n-1} \theta^i(\kappa) = \sum_{m \in \mathbb{Z}} \sum_{i=0}^{n-1} a_m t^{m+i} = \sum_{m \in \mathbb{Z}} t^m \sum_{i=0}^{n-1} a_{m-i}, \qquad \widetilde{z}_{-n} = \sum_{m \in \mathbb{Z}} t^m \sum_{i=0}^{n-1} a_{m+n-i}, \qquad \forall n > 0.$$

The third part now follows from results on the uniseriality of Verma modules, as discussed in the proof of Theorem 8.7. $\qquad \square$

Next is an example of an RTA in which the Cartan subalgebra is not a (topological) Hopf algebra.

Example 9.4 (*GWA arising from geometry*)**.** Suppose X is an object in some category $\mathscr{C}$ of topological spaces containing the real line, and $T : X \to X$ is an automorphism in $\mathscr{C}$ such that $X^\vee := \text{Hom}_{\mathscr{C}}(X, \mathbb{R})$ is an $\mathbb{R}$-algebra containing the constant map $: X \to 1$, which is stable under pre-composition with T. We now construct a "first approximation" to a GWA. Consider the subalgebra $A' \subset \text{End}_{\mathbb{R}}(X^\vee)$ generated by the operators $H_X := \{M_f : f \in X^\vee\}$, and two additional operators U, D, where:

- M_f corresponds to multiplication by f in $X^\vee$;
- $U(f) := f \circ T$ and $D(f) := f \circ T^{-1}$ for $f \in X^\vee$.

Then U, D "count" the dynamics of applying $T^{\pm 1}$ to X, i.e., the following equations hold in $\text{End}_{\mathbb{R}}(X^\vee)$:

$$U^n f(-) = f(T^n(-))U^n, \qquad D^n f(-) = f(T^{-n}(-))D^n.$$

Moreover, $UD = DU = 1_{H_X}$ in $\text{End}_{\mathbb{R}}(X^\vee)$, $H_X \cong X^\vee$, and $T^* : H_X \to H_X$ is indeed an algebra automorphism. Thus $A' = \mathcal{W}(H_X, \theta = T^*, 0, 1)/(UD - 1_{H_X}, DU - 1_{H_X})$.

We now define an associated family of triangular GWAs as follows. Suppose $T : X \to X$ is an automorphism in $\mathscr{C}$ of infinite order that stabilizes $X^\vee$. For each $z_0, z_1 \in H_X$, define $A := \mathcal{W}(H_X, T^*, z_0, z_1)$. This is a strict, based RTA of rank one, but not necessarily a (topological) Hopf RTA, since $H_X \cong X^\vee$ is not a (topological) Hopf algebra for every topological space X.

We conclude with a conjectural example involving twisted generalized Weyl algebras.

Example 9.5 (*Twisted generalized Weyl algebras*)**.** We follow the treatment in the paper [FH]. Given a TGW datum (R, σ, t), define the twisted GWA $A := \mathcal{A}_\mu(R, \sigma, t)$, constructed as the quotient of $\mathcal{C}_\mu(R, \sigma, t)$ by the ideal $\mathcal{I}_\mu(R, \sigma, t)$, as in [FH, Definition 2.3]. (These algebras were originally defined by Mazorchuk and Turowska [MT].) We further **assume** that the algebra A satisfies three additional conditions:

- The parameter matrix (μ_{ij}) with diagonals removed, is symmetric.
- The "middle" subalgebra R is isomorphic to a polynomial algebra $H[t_1, ..., t_n]$ over some commutative $\mathbb{F}$-algebra H. (Then t_i equals $y_i x_i$ as in the defining algebra relations.)

- The algebra A satisfies [FH, Definition 2.5 and Theorem 2.7] of "μ-consistency".

In this case, a natural question to ask is if the algebra A is an RTA. That (RTA3) holds is not hard to show, but the other two RTA axioms are not known to hold in this degree of generality. Specifically, are the subalgebras B_x, B_y generated by the x_i and the y_i respectively, isomorphic as vector spaces to polynomial algebras in these variables? Does the condition (RTA1) hold?

Another question of interest is to verify whether or not the type A_1^n case of a multiparameter twisted GWA (defined in [FH, Theorem 4.1]) is an RTA.

10. Non-strict RTAs: higher Lie rank infinitesimal Hecke algebras

In the final section we address yet another motivation for this paper – to construct a framework that includes RTAs that are not strict. In this section we consider *infinitesimal Hecke algebras* $\mathcal{H}_\beta(\mathfrak{g}, V)$, which are deformations of $H_0(\mathfrak{g}, V) := U(\mathfrak{g} \ltimes V)$, with $\mathfrak{g}$ a reductive Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Note that these algebras include reductive Lie algebras, for which $V = 0$. In this section we work over a ground field $\mathbb{F}$ of characteristic zero.

The first example of infinitesimal Hecke algebras is over $\mathfrak{sl}_2$. A family of these algebras was described in Example 7.1 and studied in detail in [Kh1, KT], and they are strict, based Hopf RTAs of rank one. The next two classes of examples discussed in this section, were introduced in [EGG].

10.1. **Partial examples.** Before discussing specific families of infinitesimal Hecke algebras, we first mention a general framework for such algebras, in which one can show that Condition (HRTA2) is related to Ginzburg's *Generalized Duflo Theorem* [Gi, Theorem 2.3].

Proposition 10.1. *Suppose an $\mathbb{F}$-algebra A is generated by an abelian Lie algebra $\mathfrak{h}_1$ and a finite-dimensional $\mathfrak{h}_1$-semisimple module M, with $M_0 = 0 = \operatorname{char} \mathbb{F}$. The following are equivalent:*

(1) *"HRTA2" holds; in other words, there exist*
   - *a Lie subalgebra $\mathfrak{h}_0 \subset \mathfrak{h}_1$,*
   - *a decomposition $M = M^+ \oplus M^-$ into $\mathfrak{h}_1$-semisimple submodules, and*
   - *an $\mathbb{F}$-linearly independent set $\Delta' \subset \mathfrak{h}_0^*$,*
   such that $M^\pm = \bigoplus_{\mu \in \pm\mathbb{Z}^+ \Delta'} M_\mu^\pm$. (In particular, the subalgebras generated by $M^\pm$ are $\mathfrak{h}_1$-semisimple, with finite-dimensional weight spaces, and one-dimensional zero weight space spanned by the unit.)
(2) *There exists a codimension d subspace $K \subset \mathfrak{h}_1^*$ (for some d), such that modulo K, and up to a change of basis, $\overline{\operatorname{wt}(M)} := \operatorname{wt}(M) + K \subset \mathbb{Q}^d \setminus \{0\}$.*
(3) *There exists $\delta \in \mathfrak{h}_1$ such that $\operatorname{wt}(M)(\delta) \subset \mathbb{Z} \setminus \{0\}$.*

Remark 10.2.
(1) For example, for the infinitesimal Hecke algebras associated to $(\mathfrak{g}, V) = (\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ or $(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ (which were characterized in [EGG]), the second condition is easily verified, for $M = V \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$, $K = 0$, and the basis consisting of the fundamental weights (and one additional weight in $Z(\mathfrak{g})^*$ for $\mathfrak{gl}_n$).
(2) The third condition indicates that for infinitesimal algebras $H_\beta(\mathfrak{g}, V)$ with $V_0 = 0$, one can always take Δ' to be a singleton. (In particular, this also holds for semisimple Lie algebras $\mathfrak{g}$.) This is why the present paper discusses the "Lie rank" of non-strict based RTAs, but does not define the RTA rank for such algebras.
(3) The first of the three equivalent conditions is what is needed to show that A is an HRTA; the second is what typically comes as "given data" for A; and the third is needed to apply Ginzburg's Generalized Duflo Theorem [Gi].
(4) Note that the conditions in (HRTA2) are stated in terms of $B^\pm$, unlike the first statement above. However, in the case of infinitesimal Hecke algebras $\mathcal{H}_\beta(\mathfrak{g}, V)$, the spaces $M^\pm$ are typically Lie algebras if $\beta = 0$, and $B^\pm$, which are the subalgebras generated by $M^\pm$ inside

$\mathcal{H}_\beta(\mathfrak{g}, V)$, are deformations of $U(M^\pm) \subset \mathcal{H}_0(\mathfrak{g}, V)$. In particular, given (RTA1), a suitable version of the PBW property yields the regularity conditions inside (HRTA2).

Proof. We prove a series of cyclic implications.

$(1) \Rightarrow (2)$: Since wt M is finite, choose a finite subset $\Delta_0 \subset \Delta'$ such that $M = \bigoplus_{\mu \in \pm \mathbb{Z}\Delta_0} M_\mu$. Now define $d := |\Delta_0|$, $\mathfrak{h}_{00} := \operatorname{span}_\mathbb{F}(\Delta_0)$, and $K := \mathfrak{h}_{00}^\perp \subset \mathfrak{h}_1^*$. Then (2) follows.

$(2) \Rightarrow (3)$: Since $\mathbb{Q}$ is an infinite field and $0 \notin \overline{\operatorname{wt}(M)}$, choose a hyperplane $K_1 \subset \mathbb{Q}^d \setminus \overline{\operatorname{wt}(M)}$, and consider $0 \neq h_0 \in (K_1 + K)^\perp = (\overline{K_1})^\perp$. Since these weights all lie in a $\mathbb{Q}$-vector space, there exists $c \in \mathbb{F}^\times$ such that
$$\alpha(h_0) \in \mathbb{Q}^\times \cdot c \; \forall \alpha \in \operatorname{wt}(M) \subset \mathfrak{h}_1^*.$$
Now rescale h_0 using that char $\mathbb{F} = 0$, to obtain δ such that $\alpha(\delta) \in \pm\mathbb{N} \; \forall \alpha \in \operatorname{wt}(M)$.

$(3) \Rightarrow (1)$: Set $\mathfrak{h}_0 = \mathbb{F} \cdot \delta$, $M^\pm := \bigoplus_{n \in \pm\mathbb{N}} M_n$ with respect to ad δ, and $\alpha \in \mathfrak{h}_0^*$ via: $\alpha(\delta) = 1$. Now set $\Delta' := \{\alpha\}$. $\square$

10.2. **The general linear case.** We now show that all infinitesimal Hecke algebras of the form $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ are based Hopf RTAs. First recall the definition of these algebras from [EGG, Section 4.1.1]: Set $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $V = \mathbb{F}^n \oplus (\mathbb{F}^n)^*$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{F}: (A, B) \mapsto \operatorname{tr}(AB)$, and identify $U\mathfrak{g}$ with $\operatorname{Sym}\mathfrak{g}$ via the symmetrization map. Then for any $x \in (\mathbb{F}^n)^*$, $y \in \mathbb{F}^n$, $A \in \mathfrak{g}$, one writes
$$(x, (1 - TA)^{-1}y) \det(1 - TA)^{-1} = r_0(x, y)(A) + r_1(x, y)(A)T + r_2(x, y)(A)T^2 + \cdots$$
where $r_i(x, y)$ is a polynomial function on $\mathfrak{g}$, for all i. Now for each polynomial $\beta = \beta_0 + \beta_1 T + \beta_2 T^2 + \cdots \in \mathbb{F}[T]$, the authors define in [EGG] the algebra $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ as a quotient of $T(\mathbb{F}^n \oplus (\mathbb{F}^n)^*) \rtimes U\mathfrak{g}$ by the relations
$$[x, x'] = 0, \qquad [y, y'] = 0, \qquad [y, x] = \beta_0 r_0(x, y) + \beta_1 r_1(x, y) + \cdots$$
for all $x, x' \in (\mathbb{F}^n)^*$ and $y, y' \in \mathbb{F}^n$. It is proved in [EGG] that these algebras are infinitesimal Hecke algebras (so the "PBW property" holds). Also note that if $\beta \equiv 0$, then $\mathcal{H}_0(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*) = U(\mathfrak{gl}_n \ltimes (\mathbb{F}^n \oplus (\mathbb{F}^n)^*))$.

The algebras $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ provide us with the first examples of RTAs for which one needs to use a non-strict structure to analyze them.

Proposition 10.3. *If* char $\mathbb{F} = 0$, *then* $A = \mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ *is a based Hopf RTA with* $B^+ = \mathfrak{n}^+ \oplus (\mathbb{F}^n)^*$. *Moreover,* $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ *is not strict for any* $n \geq 2$ *and polynomial* β.

Proof. We first make the necessary identifications: set $\mathfrak{h}_1$ to be the Cartan subalgebra of $\mathfrak{gl}_n$, B^+ as above, and $B^- := \mathfrak{n}^- \oplus (\mathbb{F}^n)$. Then this algebra satisfies (RTA1) by [EGG], where $B^\pm \cong U(\mathfrak{n}^+ \ltimes V), U(\mathfrak{n}^- \ltimes V^*)$ respectively. Moreover, the verification of (HRTA2) is the same as what is done in proving Proposition 10.1. In particular, $H_0 = U(\mathfrak{h}_0)$ can be chosen with $\mathfrak{h}_0 = \mathbb{F} \cdot \delta$ one-dimensional. We now claim that for $n > 1$, the Hopf RTA structure is necessarily not strict. This is because if all of $\mathbb{F}^n$ is "positive" (i.e., with H_1-roots in $\mathcal{Q}_1^+$), then so is the sum of the $\mathfrak{h}_1$-weights in it. But this sum is over an integrable $\mathfrak{sl}_n$-module, hence W-invariant, hence has zero projection when restricted to the Cartan subalgebra of $\mathfrak{sl}_n$, while the eigenvalue with respect to the central element $\operatorname{diag}(1, \ldots, 1)$ is constant on all of $\mathbb{F}^n \oplus (\mathbb{F}^n)^*$. This contradicts the RTA axioms.

To conclude the proof, we now present a map from [KT], which we **claim** is an anti-involution satisfying (RTA3) for general n, β: j takes $X \in \mathfrak{gl}_n$ to X^T, and $v_i \leftrightarrow -v_i^* \; \forall i$. To show the claim, first observe that j is an anti-involution on $\mathfrak{gl}_n$. Next, $[e_{ij}, v_k] = \delta_{jk}v_i$ and $[e_{ji}, v_k^*] = -\delta_{jk}v_i^*$ are clearly interchanged by j, so these relations are also preserved. Third, $[v_1, v_2] \equiv [v_1^*, v_2^*] \equiv 0$ are also j-stable relations (for $v_i \in \mathbb{F}^n$, $v_i^* \in (\mathbb{F}^n)^*$).

It remains to consider the relations: $[v_l, v_k^*] = \sum_{i \geq 0} \beta_i r_i(v_k^*, v_l)$. Note that each $r_i(v^*, v)$ is in $U\mathfrak{g}$ - and at the same time, identified with a function $r_i(v^*, v)(-) : \mathfrak{g} \to \mathbb{F}$, via the symmetrization

map. Now first analyze the left side: $j([v_l, v_k^*]) = [v_k, v_l^*] = \sum_{i \geq 0} \beta_i r_i(v_l^*, v_k)$. Recall how the r_k were defined. Treating $v \in \mathfrak{h}$ and $v^* \in \mathfrak{h}^*$ as column and row vectors respectively, the inner product (v^*, Av) is merely matrix multiplication $v^* A v$. Thus, we compute (inside our algebra):

$$\sum_{i \geq 0} r_i(v_l^*, v_k)(A) T^i = v_l^T (1 - TA)^{-1} v_k \cdot \det(1 - TA)^{-1} = v_k^T (1 - TA^T)^{-1} v_l \cdot \det(1 - TA^T)^{-1}$$

$$= (v_k^*, (1 - TA^T)^{-1} v_l) \det(1 - TA^T)^{-1} = \sum_{i \geq 0} r_i(v_k^*, v_l)(A^T) T^i.$$

Finally, use Proposition 10.10 below to show that $j(r_i(v_k^*, v_l)(A)) = r_i(v_k^*, v_l)(A^T)$ for all i, k, l. Then using the above computation of power series equality,

$$j\left(\sum_{i \geq 0} \beta_i r_i(v_k^*, v_l)(A) \right) = \sum_{i \geq 0} \beta_i r_i(v_k^*, v_l)(A^T) = \sum_{i \geq 0} \beta_i r_i(v_l^*, v_k)(A) = [v_k, v_l^*] = j([v_l, v_k^*]),$$

which shows that j does indeed preserve these last relations. $\qquad\square$

Remark 10.4. The based HRTA structure in Proposition 10.3 is not unique. For instance, one checks that taking δ to be the matrix $\mathrm{diag}(2n - 1, 2n - 5, \ldots, 3 - 2n)$ works for $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ for all n and all linear $\beta = \beta_0 + \beta_1 T$.

Higher rank continuous and infinitesimal Hecke algebras continue to be the focus of much recent and ongoing research – see e.g. [DT, Tik1, Tik2, Tsy] for more results and references. In particular, Category $\mathcal{O}$ has been defined and studied over $A = \mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ for all β. Using Proposition 10.3 and the theory developed in Section 3, we now claim:

Theorem 10.5. *Suppose $\mathbb{F}$ is algebraically closed of characteristic zero. For all n, β, the category $\mathcal{O} = \mathcal{O}[\widehat{H_1}^{free}]$ over $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ splits into a direct sum of highest weight categories.*

This is because in [Tik1], Tikaradze computed the center of this algebra, and showed that it satisfies Condition (S4).

10.3. **The symplectic case.** These algebras are generated by $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{F})$ and its natural representation, $V = \mathbb{F}^{2n}$. The bases for these that we use are e_i, e_{i+n} for $\mathbb{F}^{2n}$ with $1 \leq i \leq n$, and

$$u_{jk} := e_{jk} - e_{k+n, j+n}, \quad v_{jk} := e_{j, k+n} + e_{k, j+n}, \quad w_{jk} := e_{j+n, k} + e_{k+n, j}, \qquad 1 \leq j, k \leq n.$$

As discussed in [KT], given a scalar parameter β_0, the algebras $\mathcal{H}_{\beta_0}(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ are generated by $\mathfrak{sp}_{2n} \oplus V$, modulo the usual Lie algebra relations for $\mathfrak{g} = \mathfrak{sp}_{2n}$, the "semidirect product" relations $[X, v] = X(v)$ for all $X \in \mathfrak{g}, v \in V$, and the relations $[e_i, e_j] = \beta_0 \delta_{|i-j|, n}(i - j)/n$.

Proposition 10.6. *The algebras $\mathcal{H}_{\beta_0}(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ are based Hopf RTAs (assuming $\mathrm{char}\, \mathbb{F} = 0$).*

There are other based Hopf RTAs of "symplectic" type – e.g., (Lie) rank one infinitesimal Hecke algebras $\mathcal{H}_\beta(\mathfrak{sl}_2, \mathbb{F}^2)$ for any β, which were discussed in Example 7.1 above. Moreover, for all n and "all possible" β, we show below that $H_\beta(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ always has an anti-involution as in (RTA3).

Proof. Define $h_0 := \mathrm{diag}(n, n-1, \ldots, 1, -n, -(n-1), \ldots, -1)$, and consider the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h}_1 \oplus \mathfrak{n}^+$. Then $\mathfrak{g} \oplus V$ has a basis of eigenvectors for $\mathfrak{h}_1$, and in particular, for h_0 (with eigenvalues in $\mathbb{Z}$). Write $\mathfrak{g} \oplus V = \mathfrak{n}'^- \oplus \mathfrak{h}' \oplus \mathfrak{n}'^+$, a decomposition into spans of eigenvectors with negative, zero, and positive eigenvalues respectively. Then $\mathfrak{h}'$ is indeed the Cartan subalgebra $\mathfrak{h}_1 \subset \mathfrak{g}$, and $\mathfrak{n}'^\pm = \mathfrak{n}^\pm \oplus V^\pm$ are Lie subalgebras in H_β, where $V^\pm$ are the spans of $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2n}\}$ respectively.

Next, define $\mathfrak{h}_0 := \mathbb{F} h_0$, $H_r := \mathrm{Sym}\, \mathfrak{h}_r$ for $r = 0, 1$, and $B^\pm := U(\mathfrak{n}'^\pm)$. Then $\mathcal{H}_{\beta_0}(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ has the required triangular decomposition by [EGG], and H_1 is a commutative Hopf algebra with sub-Hopf algebra H_0. Moreover, $\widehat{H_1} = \widehat{H_1}^{free} = \mathfrak{h}_1^*$ surjects onto $\widehat{H_0} = \mathfrak{h}_0^* \cong \mathbb{F}$. Define $\mathcal{Q}'^+_0 :=$

$\mathbb{Z}^+\Delta' := \mathbb{Z}^+\{\alpha\}$, where $\alpha(h_0) = 1$. Then $\mathbb{Z}\Delta'$ is generated by the $\operatorname{ad} h_0$-weights of $\mathfrak{g} \oplus V$. The remaining part of (HRTA2) is shown as for RTLAs. Finally, that there exists an anti-involution was shown in [KT]:

$$j : u_{kl} \leftrightarrow u_{lk}, \ v_{kl} \leftrightarrow -w_{lk}, \ e_i \leftrightarrow e_{i+n}. \tag{10.7}$$

$\square$

10.4. **The symmetrization map and anti-involutions.** We end this section by studying anti-involutions in infinitesimal Hecke algebras. The first result is that all algebras $\mathcal{H}_\beta(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ possess an anti-involution as in (RTA3), which generalizes a part of Proposition 10.6. To see why, we first define these algebras for general n, β as in [EGG]. Denote by ω the symplectic form on $V = \mathbb{F}^{2n}$; one then identifies $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{F})$ with $\mathfrak{g}^*$ via the pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$, $(A, B) \mapsto \operatorname{tr}(AB)$, and $\operatorname{Sym} \mathfrak{g}$ with $U\mathfrak{g}$ via the symmetrization map. Write

$$\omega(x, (1 - T^2 A^2)^{-1} y) \det(1 - TA)^{-1} = l_0(x, y)(A) + l_2(x, y)(A)T^2 + \cdots$$

where $x, y \in V, A \in \mathfrak{g}$, and $l_i(x, y) \in \operatorname{Sym} \mathfrak{g} \cong U\mathfrak{g}$ is a polynomial in $\mathfrak{g}$ for all i. For each polynomial $\beta = \beta_0 + \beta_2 T^2 + \cdots \in \mathbb{F}[T]$, the algebra $\mathcal{H}_\beta(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ is the quotient of $TV \rtimes U\mathfrak{g}$ by the relations

$$[x, y] = \beta_0 l_0(x, y) + \beta_2 l_2(x, y) + \cdots$$

for all $x, y \in V$. We now show:

Proposition 10.8. *For all n, β, the map $j : \mathcal{H}_\beta(\mathfrak{sp}_{2n}, \mathbb{F}^{2n}) \to \mathcal{H}_\beta(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$ defined in Equation (10.7) is an anti-involution that fixes $H_1 = \operatorname{Sym} \mathfrak{h}_1$ (the Cartan subalgebra of $U(\mathfrak{g})$). Moreover, the conditions of Proposition 10.1 are satisfied.*

Proof. The first step is to show the following facts via straightforward computations:

(1) The map j on $\mathfrak{sp}_{2n}$ can be extended to all of $\mathfrak{gl}_{2n}$, via: $j(C) = \tau C^T \tau$ – where $\tau = \tau^{-1} = \begin{pmatrix} \operatorname{Id}_n & 0 \\ 0 & -\operatorname{Id}_n \end{pmatrix} \in GL(2n)$.

(2) One has $\omega(x, Cy) = \omega(j(y), j(C)j(x))$, for all $x, y \in \mathbb{F}^{2n}$, $C \in \mathfrak{gl}_{2n}$ (using j as in the previous part).

(3) $j\left((1 - T^2 A^2)^{-1}\right) = (1 - T^2 j(A)^2)^{-1}$.

Now note that the conditions of Proposition 10.1 hold here, if one defines $\delta := h_0$, the special element from the proof of Proposition 10.6. As for the proposed anti-involution, it is not hard to check that j is an anti-involution on $\mathfrak{sp}_{2n}$, which preserves the relations $[X, v] = X(v)$ for $X \in \mathfrak{sp}_{2n}$ and $v \in \mathbb{F}^{2n}$. We are left to consider the relations $[x, y]$. Now compute using the above facts:

$$\sum_{i \geq 0} l_{2i}(x, y)(A) T^{2i} = \omega(x, (1 - T^2 A^2)^{-1} y) \det(1 - TA)^{-1}$$

$$= \omega(j(y), (1 - T^2 j(A)^2)^{-1} j(x)) \det(1 - Tj(A))^{-1} = \sum_{i \geq 0} l_{2i}(j(y), j(x))(j(A)) T^{2i},$$

where the second equality is not hard to show. In particular, replacing A by $j(A)$ and equating coefficients of T, it follows that

$$l_{2i}(x, y)(j(A)) = l_{2i}(j(y), j(x))(A) \quad \forall x, y \in \mathbb{F}^{2n}, \ i \geq 0. \tag{10.9}$$

Now compute:

$$j([x, y]) = j\left(\sum \beta_i l_{2i}(x, y)(A)\right) = \sum \beta_i l_{2i}(x, y)(j(A)) = \sum_i \beta_i l_{2i}(j(y), j(x))(A) = [j(y), j(x)],$$

where the first and last equalities are by definition, the second uses Proposition 10.10 below (via the trace form), and the third follows from Equation (10.9). $\square$

We finally mention a result that was used in proving that every infinitesimal Hecke algebra over $\mathfrak{gl}_n$ has an anti-involution that is required to make it a (based) Hopf RTA.

Proposition 10.10. *Suppose $\mathfrak{g}$ is any Lie algebra, and we identify $\mathrm{Sym}\,\mathfrak{g}$ with $U\mathfrak{g}$ via the symmetrization map*

$$\mathrm{sym} : X_1 \ldots X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \ldots X_{\sigma(n)}.$$

Suppose j is a Lie algebra anti-involution of $\mathfrak{g}$. Then the automorphism j of $\mathrm{Sym}\,\mathfrak{g}$ is transferred to $U\mathfrak{g}$ via sym.

Proof. Observe that the symmetrization map commutes with j, and both composite maps are $\mathbb{F}$-vector space isomorphisms. $\qquad\square$

Applying this result to infinitesimal Hecke algebras over $(\mathfrak{gl}_n, \mathbb{F}^n \oplus (\mathbb{F}^n)^*)$ in the proof of Proposition 10.3, we get (via a further identification of $\mathfrak{g} \leftrightarrow \mathfrak{g}^*$ by the trace form):

$$r_i(v, v^*)(A^T) = r_i(v, v^*)(j(A)) = j(r_i(v, v^*)(A)),$$

as desired. A similar application yields the anti-involution mentioned above for infinitesimal Hecke algebras over $(\mathfrak{sp}_{2n}, \mathbb{F}^{2n})$.

Concluding example. Recall that the construction in Section 4.3 provided a setting that could not be studied using previous theories of $\mathcal{O}$, because the "root lattice" $\mathcal{Q}_0^+$ is not abelian. Using the above results on $\mathcal{O}$ for general RTAs, as well as the examples studied above, we now present a second example of a regular triangular algebra, whose study requires the full generality of our axiomatic framework and not a more specialized setting. The following example, combined with the Existence Theorems in Section 4, reinforces the viewpoint that our theory is not merely abstract, but is required in its totality in applications to specific regular triangular algebras.

Example 10.11 (A non-strict, non-Hopf, RTA)**.** Suppose $q \in \mathbb{C}^\times$ is not a root of unity, $\beta \in \mathbb{C}[T]$, $n \geq 2$, and z is a nonzero polynomial in the quantum Casimir in $U_q(\mathfrak{sl}_2)$. Also suppose X is a topological space with the algebra of continuous functions $C(X, \mathbb{R})$ not a Hopf algebra, and T is a homeomorphism of X of infinite order. Now define

$$A := U_q'(\mathfrak{sl}_2) \otimes \mathcal{H}_{z,q} \otimes (\mathbb{C} \otimes_{\mathbb{R}} \mathcal{W}(C(X, \mathbb{R}), T^*, 1, 1)) \otimes \mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*), \qquad (10.12)$$

where the individual tensor factors were studied in Examples 3.4, 7.1, 9.4, and Section 10.2 respectively. We now claim that A is an RTA satisfying BGG Reciprocity, and that the study of Category $\mathcal{O}$ over A requires the full scope of our general framework and no less.

Theorem 10.13. *The algebra A defined in (10.12) has the following properties:*

  (1) *A is a based Regular Triangular Algebra but not a strict one.*
  (2) *Neither of the algebras $H_1 \supsetneq H_0$ is a Hopf algebra, so A is not an HRTA.*
  (3) *The simple roots Δ are not weights for H_0.*
  (4) *$\mathcal{O}[\widehat{H_1}^{free}] \subsetneq \mathcal{O}$, because $\widehat{H_1}^{free} \subsetneq \widehat{H_1}$. Thus, not all Verma modules are in $\mathcal{O}$.*
  (5) *Condition (S4) is not satisfied because the center is not "large enough". Thus, central characters cannot be used to obtain a block decomposition of $\mathcal{O}[\widehat{H_1}^{free}]$ into blocks with finitely many simple objects.*

Nevertheless, the algebra A satisfies Condition (S3). Hence Theorem A holds and $\mathcal{O}[\widehat{H_1}^{free}]$ decomposes into a direct sum of finite length, self-dual blocks. Each block has finitely many simples and enough projectives/injectives, and is a highest weight category satisfying BGG Reciprocity.

Proof. That the algebra A is a based RTA follows from the analysis in the aforementioned examples by using Theorem B, since each individual tensor factor is a based RTA. Next, the RTA is not strict because $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ is necessarily not a strict RTA by Proposition 10.3. The algebras H_0, H_1 are not Hopf algebras because $\mathbb{C} \otimes_\mathbb{R} C(X, \mathbb{R})$ is not a Hopf algebra by assumption. Properties (3),(4) hold in A because they hold respectively in $\mathbb{C} \otimes_\mathbb{R} \mathcal{W}(C(X, \mathbb{R}), T^*, 1, 1)$ and in $U_q'(\mathfrak{sl}_2)$. Finally, Condition (S4) is not satisfied in $\mathcal{H}_{z,q}$ by [GK], since $Z(\mathcal{H}_{z,q}) = \mathbb{C}$. Hence A also does not satisfy (S4), by Theorem B.

Next, to show that A satisfies Condition (S3) it suffices to verify the same property for each of the tensor factors. As stated in Section 8.2, if q is not a root of unity then $U_q'(\mathfrak{sl}_2)$ satisfies Condition (S3) over $\mathbb{C}$. That $\mathcal{H}_{z,q}$ satisfies Condition (S3) was shown in [GK] (also see Example 7.1); and that $\mathcal{H}_\beta(\mathfrak{gl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ satisfies Condition (S4) (and hence (S3)) was shown in [Tik1]. Finally, $\mathbb{C} \otimes_\mathbb{R} \mathcal{W}(C(X, \mathbb{R}), T^*, 1, 1)$ satisfies Condition (S3) by Theorem 8.7(2), since $z_0 = z_1 = 1$. $\qquad\square$

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