MULTIVARIATE BLOWUP-POLYNOMIALS OF GRAPHS

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Abstract. In recent joint work (2021), we introduced a novel multivariate polynomial attached to every metric space – in particular, to every finite simple connected graph G – and showed it has several attractive properties. First, it is multi-affine and real-stable (leading to a hitherto unstudied delta-matroid for each graph G). Second, the polynomial specializes to (a transform of) the characteristic polynomial χ_{D_G} of the distance matrix D_G; as well as recovers the entire graph, where χ_{D_G} cannot do so. Third, the polynomial encodes the determinants of a family of graphs formed from G, called the blowups of G.

In this short note, we exhibit the applicability of these tools and techniques to other graph-matrices and their characteristic polynomials. As a particular case, we will see that the adjacency characteristic polynomial χ_{A_G} is in fact the shadow of a richer multivariate blowup-polynomial, which is similarly multi-affine and real-stable. Moreover, this polynomial encodes not only the aforementioned three properties, but also yields additional information for specific families of graphs. For instance, bipartite graphs are characterized by their adjacency blowup-polynomials being even; this extends a folklore 'univariate' characterization.

Throughout this work, $G = (V, E)$ denotes a finite simple connected graph (without self-loops or parallel edges).

1. Introduction

This work provides novel connections between various matrices obtained from graphs G, their spectra, and the geometry of real/complex polynomials. It is a follow-up to our recent work [10], where we were motivated by the problem of co-spectrality for (the characteristic polynomial of) the distance matrix D_G. In that work, we introduced a novel graph-invariant – a multi-affine, real-stable polynomial $p_G(\cdot)$ – which (a) specializes to a transformation of the usual characteristic polynomial of D_G, and (b) is able to recover the entire graph G up to isometry, where the univariate characteristic polynomial does not. Additionally, (c) $p_G(\cdot)$ encodes the determinant of $D_{G'}$ for G' every possible "blowup" of G:

Definition 1.1. Given a finite simple graph $G = (V, E)$, and a tuple $\mathbf{n} = (n_v : v \in V)$ of positive integers, the $\mathbf{n}$-*blowup* of G is defined to be the graph $G[\mathbf{n}]$ – with n_v copies of each vertex v – such that a copy of v and one of w are adjacent in $G[\mathbf{n}]$ if and only if $v \neq w$ are adjacent in G. (Blowups have previously been studied in the context of extremal graph theory among other areas; see e.g. [15, 16].)

In this short note, we apply the tools and techniques developed in [10] to explore other well-known matrices in spectral graph theory (and their corresponding characteristic polynomials). For each graph, we will introduce a multivariate polynomial for each matrix in a certain class, and prove that this polynomial has similarly attractive properties as in [10] (and mentioned above). As a prototypical example, we will briefly discuss the adjacency matrix later in this note.

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We now introduce the matrices of interest. Given a graph $G = (V, E)$, we study matrices of the form

$$\mathcal{A}_G := \Delta_G + M_G, \tag{1.2}$$

where:

- $M_G = (m_{vw})_{v,w \in V} \in \mathbb{R}^{V \times V}$ is a real symmetric matrix (which encodes a graph-property) that is "well-behaved" under blowups, in that $M_{G[\mathbf{n}]}$ is a block $V \times V$ matrix, with the (v, w) block equal to $m_{vw} \mathbf{1}_{n_v \times n_w}$.
- $\Delta_G = \operatorname{diag}(f_v)_{v \in V}$ is a nonsingular diagonal matrix that is well-behaved under blowups, in that $\Delta_{G[\mathbf{n}]}$ is a $V \times V$ block diagonal matrix, with the (v, v) block equal to $f_v \operatorname{Id}_{n_v}$.

Here are a few examples of well-known matrices that are subsumed by this paradigm:

(1) G is a finite simple connected graph, with $M_G = \mathcal{D}_G$ the modified distance matrix studied in [10]; and $\Delta_G = -2 \operatorname{Id}_V$. Thus $\mathcal{A}_G = \mathcal{D}_G - 2 \operatorname{Id}_V$ is precisely the 'usual' distance matrix $D_G = (d(v, w))_{v,w \in V}$, where $d(v, w)$ denotes the (edge-)length of a shortest path joining v, w in G. In fact the framework for an arbitrary finite metric space X is also subsumed by the present model: $M_X = \mathcal{D}_X$ and $\Delta_X = \operatorname{diag}(-2d_X(x, X \setminus \{x\}))_{x \in X}$, where $d_X(x, Y) := \min_{y \in Y} d_X(x, y)$ for a non-empty subset $Y \subset X$.

(2) G is a finite simple graph, and $M_G = A_G$ is its adjacency matrix. In this case, we fix a (nonzero) scalar $\lambda \in \mathbb{R}$ and let $\Delta_G = \lambda \operatorname{Id}_V$.

(3) G is a finite simple graph, and $\mathcal{A}_G$ is the *Seidel matrix*, also studied in spectral graph theory. In this case, $M_G = \mathbf{1}_{V \times V} - 2A_G$, and $\Delta_G = -\operatorname{Id}_V$.

As in [10], we are motivated by the well-studied problem of *co-spectrality* with respect to a graph-matrix (e.g. the distance matrix, or the adjacency/Seidel matrix). Recall that two graphs $G \not\cong H$ are said to be co-spectral with respect to a graph-matrix M if the spectra of M_G and M_H agree as multi-sets (equivalently, the characteristic polynomials of M_G, M_H agree). Thus, a longstanding problem in spectral graph theory is to understand, for a given graph-matrix M, which pairs of non-isomorphic graphs are M-cospectral. In particular, it is well-known that all three matrices above admit such graph pairs. In other words, none of these matrices M_G *detects* the underlying graph G – i.e., recovers the graph up to isomorphism. See Figure 1 for 'small graph' examples for the adjacency and Seidel matrices, and [13] for an example for distance matrices. (For completeness, we also refer the reader to the texts [11, 12] on spectral graph theory.)

FIGURE 1. Two non-isomorphic graphs on five vertices that are adjacency co-spectral; and two non-isomorphic graphs on three vertices that are Seidel co-spectral

The purpose of this note is to show that in such cases, one can nevertheless refine each such univariate polynomial – in a natural manner from multiple viewpoints: algebraic, spectral, and polynomial – to obtain a multivariate real polynomial with several interesting properties, listed presently. For instance, if M_G is the (modified) distance matrix or the adjacency matrix, then this polynomial does recover the graph, hence is truly a graph-invariant. This yields a novel family of multivariate graph-invariants for every graph $G = (V, E)$, which we will term $p_{\mathcal{A}_G}(\cdot)$ – see (1.2). (This was carried out for the distance matrix in previous work [10]; we now provide a general model that works for all graph-matrices $\mathcal{A}_G$ as in (1.2).) We also show that in addition to being polynomials, these invariants $p_{\mathcal{A}_G}(\cdot)$ have other attractive properties:

- They are multi-affine in their arguments $n_v,\ v \in V$.
- They are *real-stable* in the n_v.
- The univariate specialization of $p_{\mathcal{A}_G}(\cdot)$ is closely related to the characteristic polynomial of M_G, so that $p_{\mathcal{A}_G}(\cdot)$ is indeed related to the spectrum of M_G. Combined with the preceding point, this implies that the univariate specialization is also real-rooted.
- Unlike their univariate specializations (at least for the distance/adjacency/Seidel matrices), the polynomials $p_{\mathcal{A}_G}$ recover the matrix $M_G^{\circ 2}$, whence M_G if M_G has non-negative entries.
- The polynomials $p_{\mathcal{A}_G}$ simultaneously encode the determinants of the corresponding matrices $\mathcal{A}_{G[\mathbf{n}]}$ for *all* graph-blowups (defined above) of G. Thus, in addition to our original, 'spectral' motivation, these polynomials also carry algebraic information.

For the last-mentioned reason, we continue to adopt the notation in [10], and call this object the *$\mathcal{A}$-blowup-polynomial of G*. These polynomials are desirable in other ways as well. For example, it is well-known that the (adjacency) spectrum of a bipartite graph $B = (V, E)$ is always symmetric about the origin, as a multi-set. We show in this short note that the more general fact that the zero-locus of the corresponding adjacency blowup-polynomial of B is also symmetric around the origin in $\mathbb{C}^V$. Thus, the workings of [10] (and now of this paper) provide a broader recipe for a more refined study of graph-polynomials. We expect this line of investigation to lead to further multivariate refinements of known results.

In a sense, this short note conforms to the philosophy that univariate polynomials are special cases of multivariate ones, and these latter are the more natural and general objects to study – and they are more powerful too. A famous recent manifestation of this has been in the geometry of (the roots in $\mathbb{C}$ of) real and complex univariate polynomials, where Borcea–Brändén and other mathematicians have recently been extremely active in advancing the field (to cite a very few sources, [2, 3, 4, 20, 21, 28]), a century after the activity on the Laguerre–Pólya–Schur program [17, 22, 23]. This recent progress has extensively advanced the theory of (real) stable polynomials, with numerous applications including to negative dependence, constructing bipartite Ramanujan graphs, and the Kadison–Singer problem. Additionally, there are other several other examples. For instance, Wagner's involved proof of the univariate Brown–Colbourn conjecture [27] was shortly followed by a one-paragraph proof of its multivariate strengthening [24, 25]. Similarly, multivariate versions [18] of Lee–Yang type theorems for Ising models, have been very influential in e.g. the Borcea–Brändén program. We refer the reader to the excellent survey of Sokal [26] for more instances and details.

2. Algebraic results: polynomiality, coefficients, iterated blowup, symmetries

We now state and prove our results. The first set of assertions is algebraic in nature.

Theorem 2.1. *We retain the notation in* (1.2) *and the lines immediately thereafter. Thus $M_G = M_G^T$, and $f_v \in \mathbb{R} \setminus \{0\}$ for all $v \in V$.*

(1) *There exists a polynomial $p_{\mathcal{A}_G} : \mathbb{R}^V \to \mathbb{R}$ such that for all integer tuples $\mathbf{n} \in \mathbb{Z}_{>0}^V$, we have:*

$$\det \mathcal{A}_{G[\mathbf{n}]} = \prod_{v \in V} f_v^{n_v - 1} \cdot p_{\mathcal{A}_G}(\mathbf{n}), \qquad \forall \mathbf{n} \in \mathbb{Z}_{>0}^V.$$

In fact, the polynomial is (unique, and) given by:

$$p_{\mathcal{A}_G}(\mathbf{n}) = \det(\Delta_G + \Delta_{\mathbf{n}} M_G), \qquad \textit{where} \qquad \Delta_{\mathbf{n}} := \operatorname{diag}(n_v)_{v \in V}.$$

In particular, $p_{\mathcal{A}_G}$ is multi-affine in $\mathbf{n}$, with constant and linear terms respectively equal to

$$p_{\mathcal{A}_G}(\mathbf{0}) = \det \Delta_G = \prod_{v \in V} f_v \qquad \textit{and} \qquad (1, \dots, 1) \cdot \Delta_{\mathbf{n}} \cdot \nabla p_{\mathcal{A}_G}(\mathbf{0}) = p_{\mathcal{A}_G}(\mathbf{0}) \sum_{v \in V} \frac{m_{vv}}{f_v} n_v.$$

(2) *More generally, for each $I \subset V$ the coefficient in $p_{\mathcal{A}_G}(\mathbf{n})$ of $\prod_{v\in I} n_v$ equals*

$$\det(M_G)_{I\times I} \prod_{v\in V\setminus I} f_v.$$

(3) *The $\mathcal{A}$-blowup-polynomial of the blowup $G[\mathbf{n}]$ has a closed-form expression. More precisely, given an integer tuple $\mathbf{n} \in \mathbb{Z}_{>0}^V$ and variables $\{m_{vi} : v \in V,\ 1 \leq i \leq n_v\}$, define $n'_v := \sum_i m_{vi}$ and $\mathbf{n}' := (n'_v)_{v\in V}$. Then:*

$$p_{\mathcal{A}_{G[\mathbf{n}]}}(\mathbf{m}) \equiv p_{\mathcal{A}_G}(\mathbf{n}') \prod_{v\in V} f_v^{n_v-1}.$$

As an immediate consequence of the final part, the polynomials $p_{\mathcal{A}_{G[\mathbf{n}]}}(\cdot)$ all have total degree at most $|V|$, regardless of the integer tuple $\mathbf{n}$.

Proof. The second part follows easily from the first (see e.g. [10, Section 2]). The first part was shown in *loc. cit.* using arguments from commutative algebra (specifically, Zariski density), and these arguments also apply here, so that the result holds over an arbitrary unital commutative ring. In the interest of variance, we now provide an alternate, more direct proof using real numbers. Begin by noticing that in the desired assertion

$$\det \mathcal{A}_{G[\mathbf{n}]} = \prod_{v\in V} f_v^{n_v-1} \det(\Delta_G + \Delta_{\mathbf{n}} M_G),$$

both sides are polynomial functions, whence continuous, in the entries of (the real symmetric matrix) M_G. In particular, if M_G is singular, we may replace it by $M_G + \epsilon \operatorname{Id}_V$ for small $\epsilon > 0$, and let $\epsilon \to 0^+$. Thus, without loss of generality, we may assume $\det M_G \neq 0$.

We now proceed to the proof. Define the integers $0 < k \leq K$ and the matrix $\mathcal{W}$ via:

$$k := |V|, \qquad K := \sum_{v\in V} n_v, \qquad \mathcal{W}_{K\times k} := \begin{pmatrix} \mathbf{1}_{n_1\times 1} & 0_{n_1\times 1} & \cdots & 0_{n_1\times 1} \\ 0_{n_2\times 1} & \mathbf{1}_{n_2\times 1} & \cdots & 0_{n_2\times 1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times 1} & 0_{n_k\times 1} & \cdots & \mathbf{1}_{n_k\times 1} \end{pmatrix},$$

where $(n_1, \dots, n_k)$ is a fixed enumeration of the integers n_v. Now compute, with a repeated use of Schur complements, and using that $\Delta_G = \operatorname{diag}(f_v)_{v\in V}$ is invertible:

$$\begin{aligned} \det \mathcal{A}_{G[\mathbf{n}]} =&\ \det(\Delta_{G[\mathbf{n}]} + \mathcal{W} M_G \mathcal{W}^T) = \det \begin{pmatrix} \Delta_{G[\mathbf{n}]} & -\mathcal{W} \\ \mathcal{W}^T & M_G^{-1} \end{pmatrix} \det(M_G) \\ =&\ \det(\Delta_{G[\mathbf{n}]}) \det(M_G^{-1} + \mathcal{W}^T \Delta_{G[\mathbf{n}]}^{-1} \mathcal{W}) \det(M_G), \end{aligned}$$

where we label Δ_G, M_G compatibly with the enumeration $(n_j)_{j=1}^k$. Now since $\mathcal{W}^T \Delta_{G[\mathbf{n}]}^{-1} \mathcal{W} = \operatorname{diag}(n_v f_v^{-1})_{v\in V}$, we continue:

$$= \prod_{v\in V} f_v^{n_v} \cdot \det(\operatorname{Id}_V + \Delta_G^{-1} \Delta_{\mathbf{n}} M_G) = \prod_{v\in V} f_v^{n_v-1} \cdot \det(\Delta_G + \Delta_{\mathbf{n}} M_G).$$

This proves the first part, except for the uniqueness of the polynomial – but this follows from the Zariski density of $\mathbb{Z}_{>0}^V$ in $\mathbb{R}^V$, which simply means that any polynomial on $\mathbb{R}^V$ that vanishes on $\mathbb{Z}_{>0}^V$ is identically zero. (There is some more work to do if one wants to prove this result over an arbitrary commutative ring, as was done in [10].)

It remains to show the third part. Once again, we avoid Zariski density arguments as in [10], and work with the real symmetric matrix M_G. As above, we may assume M_G is invertible (as is

Δ_G). We compute using the first part, and Schur complements:

$$\begin{aligned}
p_{\mathcal{A}_{G[\mathbf{n}]}}(\mathbf{m}) =&\ \det(\Delta_{G[\mathbf{n}]} + \Delta_{\mathbf{m}} \mathcal{W} M_G \mathcal{W}^T)\\
=&\ \det(\Delta_{\mathbf{m}}) \det(\Delta_{\mathbf{m}}^{-1} \Delta_{G[\mathbf{n}]} + \mathcal{W} M_G \mathcal{W}^T)\\
=&\ \det(\Delta_{\mathbf{m}}) \det \begin{pmatrix} \Delta_{\mathbf{m}}^{-1} \Delta_{G[\mathbf{n}]} & -\mathcal{W} \\ \mathcal{W}^T & M_G^{-1} \end{pmatrix} \det(M_G)\\
=&\ \det(\Delta_{\mathbf{m}}) \det(\Delta_{\mathbf{m}}^{-1}) \det(\Delta_{G[\mathbf{n}]}) \det(M_G^{-1} + \mathcal{W}^T \Delta_{\mathbf{m}} \Delta_{G[\mathbf{n}]}^{-1} \mathcal{W}) \det(M_G).
\end{aligned}$$

But $\mathcal{W}^T \Delta_{\mathbf{m}} \Delta_{G[\mathbf{n}]}^{-1} \mathcal{W} = \operatorname{diag}(f_v^{-1} n'_v)_{v \in V}$, where $n'_v := \sum_i m_{vi}$ as above. Hence we continue:

$$= \prod_{v \in V} f_v^{n_v} \cdot \det(M_G^{-1} + \Delta_G^{-1} \Delta_{\mathbf{n}'}) \det(M_G) = \prod_{v \in V} f_v^{n_v - 1} \cdot \det(\Delta_G + \Delta_{\mathbf{n}'} M_G),$$

which proves the third part. □

A consequence of the preceding result is a linear delta-matroid that arises from the $\mathcal{A}$-blowup-polynomial:

Corollary 2.2. *Setting as in Theorem 2.1. The set of monomials with nonzero coefficients in* $p_{\mathcal{A}_G}(\mathbf{n})$ *forms the linear delta-matroid* $\mathcal{M}_{M_G}$.

Recall that delta-matroids were defined and studied by Bouchet [6], and generalize the notion of a matroid. Given a symmetric matrix M over a field, Bouchet showed in [7] that the subsets of indices corresponding to non-vanishing principal minors, form a (linear) delta-matroid $\mathcal{M}_M$. This explains how the corollary follows immediately from Theorem 2.1(2).

The next result shows that the $\mathcal{A}$-blowup-polynomial $p_{\mathcal{A}_G}$, together with the scalars f_v, m_{vv}, determine the rest of the matrix M_G – or more precisely, its Hadamard square $M_G^{\circ 2} = (m_{ij}^2)$.

Proposition 2.3. *The homogeneous quadratic part of* $p_{\mathcal{A}_G}$*, i.e. its Hessian at the origin, equals*

$$\mathcal{H}(p_{\mathcal{A}_G}) := ((\partial_{n_v} \partial_{n_{v'}} p_{\mathcal{A}_G})(\mathbf{0}))_{v,v' \in V} = \prod_{v \in V} f_v \cdot \Delta_G^{-1} (\mathbf{m}_G \mathbf{m}_G^T - M_G^{\circ 2}) \Delta_G^{-1},$$

where $\mathbf{m}_G := (m_{vv})_{v \in V} \in \mathbb{R}^V$ *is the (column) vector containing the diagonal entries of* M_G*. Thus if the scalars* f_v, m_{vv} *are known, then the* $\mathcal{A}$*-blowup-polynomial* $p_{\mathcal{A}_G}(\mathbf{n})$ *detects the matrix* $M_G^{\circ 2}$.

For instance, if one works with the distance or adjacency matrix of G, then all entries in M_G are non-negative, and the entries f_v, m_{vv} are also known (see above), so Proposition 2.3 recovers the entire matrix M_G, and hence the graph G.

As the proof of Proposition 2.3 is based on a direct computation using Theorem 2.1(2), and is similar to the corresponding proof in our recent paper [10], we omit it here. (Note however that the formula here is more general than its counterpart in [10].)

3. Results on real-stability

We now move to results connecting the blowup-polynomial with spectral graph theory and the geometry of (real) polynomials. The next result provides a sufficient condition for the real-stability of $p_{\mathcal{A}_G}$ in the above paradigm (and hence the real-rootedness of its univariate specialization):

Theorem 3.1. *Setting as in Theorem 2.1. Define* $u_{\mathcal{A}_G}(x) := p_{\mathcal{A}_G}(x, x, \ldots, x)$.

1. *Suppose all scalars* f_v *are (nonzero and) of the same sign. Then* $p_{\mathcal{A}_G}(\cdot)$ *is real-stable. (This means that if all arguments* $\mathbf{n}$ *lie in the open upper half-plane* $\Im(z) > 0$*, then* $p_{\mathcal{A}_G}(\mathbf{n}) \neq 0$*.) In particular,* $u_{\mathcal{A}_G}$ *is real-rooted.*
2. *Suppose in fact that all scalars* f_v *are equal, say to* $\lambda \neq 0$*. Then* $x \in \mathbb{R}$ *is a root of* $u_{\mathcal{A}_G}$ *if and only if* $x \neq 0$ *and* $-\lambda / x$ *is an eigenvalue of* M_G.

We remark here that these hypotheses are indeed satisfied when one studies distance matrices of graphs [10], or the adjacency or Seidel matrices as above.

Proof.

(1) Suppose $\varepsilon \in \{\pm 1\}$ is the sign of every f_v, so that the diagonal matrix $\varepsilon \Delta_G$ is positive definite. We compute, allowing for the n_v to now be *complex* variables:

$$p_{\mathcal{A}_G}(\mathbf{n}) = \det(\Delta_{\mathbf{n}})(-\varepsilon)^{|V|} \det(-\varepsilon M_G - \Delta_{\mathbf{n}}^{-1}(\varepsilon \Delta_G)).$$

Let E_{vv} denote the elementary $V \times V$ matrix, with 1 in the (v, v) entry and all other entries zero. Then,

$$p_{\mathcal{A}_G}(\mathbf{n}) = \det(\Delta_{\mathbf{n}})(-\varepsilon)^{|V|} \det\left(-\varepsilon M_G + \sum_{v \in V} (-n_v^{-1})(\varepsilon f_v) E_{vv}\right). \tag{3.2}$$

Ignoring the scalar $(-\varepsilon)^{|V|}$, it suffices to show that the determinant-expression times $\det(\Delta_{\mathbf{n}})$ is real-stable in the n_v. Here, we recall a result of Borcea and Brändén [2, Proposition 2.4], which says that the determinantal polynomial $\det(B + \sum_{j=1}^{m} z_j A_j)$ is real-stable in the z_j if all A_j are positive semidefinite and B is real symmetric. As an application, since the matrices $\varepsilon f_v E_{vv}$ are positive semidefinite for all $v \in V$, the determinant in (3.2) (without the extra factor of $\prod_v n_v$) is real-stable, provided that one replaces each $(-n_v^{-1})$ by n_v. Now use that stability is preserved under 'inversion': if a polynomial $p(\{n_v\})$ with n_w-degree d is stable (for some fixed $w \in V$), then so is $n_w^d p(\{n_v : v \neq w\}, -n_w^{-1})$. Applying this for each variable n_v in turn, the first assertion follows.

(2) By Theorem 2.1(1), $u_{\mathcal{A}_G}(x) = \det(\lambda \operatorname{Id}_V + x M_G)$, and this does not vanish if $x = 0$, since $\lambda \neq 0$. Thus, x is a root here, if and only if $x \neq 0$ and

$$0 = x^{-|V|} u_{\mathcal{A}_G}(x) = \det(\lambda x^{-1} \operatorname{Id}_V + M_G),$$

and the result is immediate from here. □

The next result provides necessary conditions and sufficient conditions for when a graph is a blowup, in terms of the matrix $\mathcal{A}_G$:

Proposition 3.3. *Notation as above; also suppose that all scalars f_v are equal, say to $\lambda \in \mathbb{R}$. Then each of the following statements implies the next:*

(1) *G is a nontrivial blowup. In other words, G is a blowup of a graph H with $|V(H)| < |V(G)|$.*
(2) *There exist two vertices $v \neq w$ in G which share the same set of neighbors. (Thus, v, w are not adjacent.)*
(3) *λ is an eigenvalue of the matrix $\mathcal{A}_G$.*
(4) *The blowup-polynomial $p_{\mathcal{A}_G}$ has total degree at most $|V| - 1$.*

In fact the first two assertions are equivalent (and do not depend on $\mathcal{A}_G$), and so are the last two.

Proof. The first two assertions are taken from [10]; we reproduce that proof here. If (1) holds then there are two distinct vertices which are copies of one another, proving (2). Conversely, if (2) holds then G is a blowup of the smaller graph where one of these two vertices is removed, proving (1).

We next show (2) $\implies$ (3): if (2) holds, then M_G contains two identical rows, hence is singular. But then $\mathcal{A}_G = M_G + \lambda \operatorname{Id}_V$ has λ as an eigenvalue, proving (3). Finally, (3) holds if and only if M_G is singular, if and only if (by Theorem 2.1(2)) the unique monomial in $p_{\mathcal{A}_G}(\mathbf{n})$ of top degree is zero, proving that (3) $\iff$ (4). □

Our final result in this section takes an in-depth look at the real-stability of the polynomial $p_{\mathcal{A}_G}(\mathbf{n})$. Notice by Theorem 2.1(1) that these polynomials are not homogeneous; nor are their coefficients *a priori* all of the same sign. Real-stable polynomials with either of these two properties are the subjects of active research: the former fall in the broader family of *Lorentzian polynomials* [9], as well as *strongly/completely log-concave polynomials* [1, 14]; and the latter are termed *strongly Rayleigh polynomials*, and are important in probability and the theory of negative dependence (see [5] and the references therein). We first provide the necessary definitions.

Definition 3.4. Fix a real polynomial of complex variables $p(z_1, \dots, z_k) \in \mathbb{R}[z_1, \dots, z_k]$.

(1) A real symmetric matrix is *Lorentzian* if it is nonsingular with only one positive eigenvalue.
(2) Following [9], we say p is *Lorentzian* if p is homogeneous of some degree $d \geq 2$, has non-negative coefficients, and given indices $1 \leq j_1, \dots, j_{d-2} \leq k$, if we define

$$g(z_1, \dots, z_k) := \left(\partial_{z_{j_1}} \cdots \partial_{z_{j_{d-2}}} p\right)(z_1, \dots, z_k),$$

then its Hessian matrix $\mathcal{H}_g := (\partial_{z_i} \partial_{z_j} g)_{i,j=1}^k \in \mathbb{R}^{k \times k}$ is Lorentzian.
(3) Following [14], we say that p is *strongly log-concave* if p has all coefficients non-negative, and for all tuples $\alpha \in \mathbb{Z}_{\geq 0}^k$, either the derivative $\partial^\alpha(p) := \prod_{i=1}^k \partial_{z_i}^{\alpha_i} \cdot p$ is identically zero, or $\log(\partial^\alpha(p))$ is defined and concave on $(0, \infty)^k$.
(4) Following [1], we say that p is *completely log-concave* if p has all coefficients non-negative, and for all integers $m \geq 0$ and matrices $A = (a_{ij}) \in [0, \infty)^{m \times k}$, either the derivative $\partial_A(p) := \prod_{i=1}^m \left(\sum_{j=1}^k a_{ij} \partial_{z_j} \right) \cdot p$ is identically zero, or $\log(\partial_A(p))$ is defined and concave on $(0, \infty)^k$.
(5) We say p is *strongly Rayleigh* if p is multi-affine and real-stable in the z_j, and has all coefficients non-negative and summing to 1.

We now explore when the homogenized blowup-polynomial of $p_{\mathcal{A}_G}$ is Lorentzian – or its normalization is strongly Rayleigh – in the spirit of a result proved in [10] for distance matrices. The following adapts that result to the current setting.

Theorem 3.5. *Setting as in Theorem 2.1, with $k := |V| \geq 2$. Also suppose that all scalars $f_v \in \mathbb{R}$ are nonzero and have the same sign $\varepsilon \in \{\pm 1\}$. Define the homogenized polynomial*

$$\widetilde{p}_{\mathcal{A}_G}(z_0, z_1, \dots, z_k) := (\varepsilon z_0)^k p_{\mathcal{A}_G}(\varepsilon z_1 / z_0, \dots, \varepsilon z_k / z_0) \in \mathbb{R}[z_0, z_1, \dots, z_k],$$

with possibly complex arguments. Then the following statements are equivalent.

(1) $\widetilde{p}_{\mathcal{A}_G}(z_0, z_1, \dots, z_k)$ *is real-stable.*
(2) $\widetilde{p}_{\mathcal{A}_G}(z_0, z_1, \dots, z_k)$ *is Lorentzian (equivalently, strongly / completely log-concave).*
(3) *All coefficients of* $\widetilde{p}_{\mathcal{A}_G}(z_0, z_1, \dots, z_k)$ *are non-negative.*
(4) *We have* $\varepsilon^k p_{\mathcal{A}_G}(\varepsilon, \dots, \varepsilon) > 0$, *and the following polynomial is strongly Rayleigh:*

$$(z_1, \dots, z_k) \quad \mapsto \quad \frac{p_{\mathcal{A}_G}(\varepsilon z_1, \dots, \varepsilon z_k)}{p_{\mathcal{A}_G}(\varepsilon, \dots, \varepsilon)}.$$

(5) *The matrix* M_G *is positive semidefinite.*

Before proving Theorem 3.5, we note that it is a 'negative' result for the graph-properties discussed in this paper. For example, for the distance matrix the only graphs for which $M_G = D_G + 2\,\mathrm{Id}_V$ is positive semidefinite, are complete (multipartite) graphs, by [19]. For the adjacency matrix, the only graphs for which M_G is positive semidefinite are the graphs with no edges. Nevertheless, the family of matrices $\mathcal{A}_G$ as in (1.2) can contain other examples for which the matrix M_G is positive semidefinite.

Proof of Theorem 3.5. That (1) $\implies$ (2) was shown in [9], and that a Lorentzian polynomial satisfies (3) follows from the definitions. The equivalences inside assertion (2) were shown in [9, Theorem 2.30]. We next show that (1) $\implies$ (4) $\implies$ (3). Given (1) (and hence (3)), we see that the sum of all coefficients in $\widetilde{p}_{\mathcal{A}_G}$ equals its value at $(1,1,\dots,1)$, and so using (3):

$$\varepsilon^k p_{\mathcal{A}_G}(\varepsilon,\dots,\varepsilon) = \widetilde{p}_{\mathcal{A}_G}(1,1,\dots,1) \geq \widetilde{p}_{\mathcal{A}_G}(1,0,\dots,0) = \prod_{v\in V}(\varepsilon f_v) > 0.$$

Moreover, the coefficients of the normalized polynomial

$$\frac{p_{\mathcal{A}_G}(\varepsilon z_1,\dots,\varepsilon z_k)}{p_{\mathcal{A}_G}(\varepsilon,\dots,\varepsilon)} = \frac{\widetilde{p}_{\mathcal{A}_G}(1,z_1,\dots,z_k)}{\widetilde{p}_{\mathcal{A}_G}(1,1,\dots,1)} \tag{3.6}$$

are non-negative and add up to one. Finally, the left-hand side is real-stable because the right-hand side is, by (1) and by specializing at $z_0 \mapsto 1$ (which preserves real-stability). This shows (4). Conversely, if (4) holds, then Equation (3.6) implies (3), as desired.

Finally, if (3) holds, then Theorem 2.1(2) (and the hypotheses that $\varepsilon f_v > 0\ \forall v \in V$) implies that every principal minor of M_G is non-negative. This is because the coefficient of $z_0^{k-|J|}\prod_{v\in J} z_v$ equals $\prod_{v\in V\setminus J}(\varepsilon f_v)\cdot \det(M_G)_{J\times J}$, and this is to be non-negative for every subset J. This shows (5). Conversely, if (5) holds, then we use the positive semidefinite matrix

$$\mathcal{C}_G := (\varepsilon\Delta_G)^{-1/2} M_G (\varepsilon\Delta_G)^{-1/2}$$

in the following computation:

$$\begin{aligned}\widetilde{p}_{\mathcal{A}_G}(z_0,z_1,\dots,z_k) &= \det(\varepsilon\Delta_G)^{1/2}\det(z_0\operatorname{Id}_k + \Delta_{\mathbf{z}}\mathcal{C}_G)\det(\varepsilon\Delta_G)^{1/2}\\ &= \det(\varepsilon\Delta_G)\det(z_0\operatorname{Id}_k + \sqrt{\mathcal{C}_G}\Delta_{\mathbf{z}}\sqrt{\mathcal{C}_G}) = \det(z_0\operatorname{Id}_k + \sum_{v\in V} z_v\sqrt{\mathcal{C}_G}E_{vv}\sqrt{\mathcal{C}_G}),\end{aligned}$$

where the second equality comes from expanding $\det\begin{pmatrix} z_0\operatorname{Id}_k & -\sqrt{\mathcal{C}_G} \\ \Delta_{\mathbf{z}}\sqrt{\mathcal{C}_G} & \operatorname{Id}_k\end{pmatrix}$ in two ways, both via Schur complements; and where the matrix E_{vv} was defined prior to (3.2). Now the final expression is indeed real-stable by [2, Proposition 2.4] (see the discussion following (3.2)), so (1) holds. ☐

4. The adjacency blowup-polynomial

We conclude this note by illustrating several of the above results in the general case, by specializing them to the adjacency matrix of a graph and its blowup-polynomial, as a particular example.

Suppose $M_G = A_G$ is the adjacency matrix (so $m_{vv} = 0\ \forall v \in V$), and $\lambda \in \mathbb{R}$ is any fixed nonzero scalar. Defining $\mathcal{A}_{G,\lambda} = \lambda\operatorname{Id}_V + A_G$, we obtain a real-stable polynomial $p_{\mathcal{A}_{G,\lambda}}(\mathbf{n})$. Now the coefficient in $p_{\mathcal{A}_{G,\lambda}}(\mathbf{n})$ of any monomial $\mathbf{n}^I$ (where $I \subset V$) is a scalar times the determinant of $(A_G)_{I\times I}$; note this principal submatrix is itself block diagonal, with components corresponding to the connected components of the induced subgraph on I (this makes $\det(A_G)_{I\times I}$ 'easier' to compute). This observation and others lead to the following result.

Proposition 4.1. *Suppose $M_G = A_G$, the adjacency matrix of a graph G; and $\Delta_G = \lambda\operatorname{Id}_V$ for a fixed nonzero scalar $\lambda \in \mathbb{R}$.*

1. *Suppose $H \subset G$ is an induced subgraph. Then*
$$p_{\mathcal{A}_{H,\lambda}}(\{n_v : v \in V(H)\}) = p_{\mathcal{A}_{G,\lambda}}(\mathbf{n})|_{n_v=0\ \forall v\in V(G)\setminus V(H)} \cdot \lambda^{|V(H)|-|V(G)|}.$$
2. *If H, H' are isomorphic subgraphs inside G, then the coefficients in $p_{\mathcal{A}_{G,\lambda}}$ of the monomials corresponding to $V(H)$ and $V(H')$ are equal.*
3. *Suppose for some non-empty subset $J \subset V(G)$ that the induced subgraph on J contains a connected component which is a tree without a perfect matching. Then the coefficient of $\mathbf{n}^J$ in $p_{\mathcal{A}_{G,\lambda}}(\mathbf{n})$ is zero.*

(4) *If* $\lambda = -1$*, then up to a scalar, the univariate specialization* $u_{\mathcal{A}_{G,-1}}(x)$ *is precisely the 'inversion' of the characteristic polynomial of* A_G*:* $u_{\mathcal{A}_{G,-1}}(z) = z^{|V|}\chi_{A_G}(z^{-1})$.

The final part allows one to interpret the eigenvalues of A_G in terms of the roots of $u_{\mathcal{A}_{G,-1}}$. (E.g. the second largest eigenvalue is important for studying d-regular bipartite Ramanujan graphs.)

Proof. The key fact used in these results is that the adjacency matrix of any subgraph on $J \subset V(G)$ is the principal $J \times J$ submatrix of A_G. This fact, combined with Theorem 2.1(2) and that Δ_G is a scalar matrix, immediately yields the second part. The 'key fact' also yields the first part via setting all other variables $n_v, v \notin V(H)$ to zero, since this yields a matrix with only the diagonal entry nonzero in each row not indexed by $V(H)$. The first part follows easily from here.

The third part holds because of the observation immediately preceding this proposition, combined with the classical fact (see e.g. [11]) that the adjacency matrix of a tree is nonsingular if and only if the tree has a perfect matching. The final part follows immediately from Theorem 3.1(2). □

The multivariate blowup-polynomial also has other attractive properties; we mention one that is crucially used in studying bipartite graphs (including in constructing bipartite Ramanujan expanders in [20] and its precursors). A folklore result is that G is bipartite if and only if its adjacency spectrum is symmetric (as a multiset) around the origin. In fact, this extends to the zero-locus of the blowup-polynomial – now yielding an even polynomial since $u_{\mathcal{A}_{G,-1}}$ is the 'inversion' of the adjacency characteristic polynomial:

Proposition 4.2. *Suppose* G *is a graph,* $\lambda \in \mathbb{R}$ *is nonzero, and we set* $\mathcal{A}_{G,\lambda} = \lambda \operatorname{Id}_V + A_G$*. The following are equivalent:*

(1) $p_{\mathcal{A}_{G,\lambda}}$ *is even in* $\mathbf{n}$.
(2) G *is bipartite.*

In particular, the zeros of $p_{\mathcal{A}_{G,\lambda}}$ for any bipartite graph G are symmetric around the origin, and there are no odd-degree monomials in $p_{\mathcal{A}_{G,\lambda}}$.

Proof. First suppose G is bipartite. Write the adjacency matrix as $A_G = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$, and compute:

$$\begin{aligned} p_{\mathcal{A}_{G,\lambda}}(-\mathbf{n}) =&\ \det \begin{pmatrix} -\operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} \det(\lambda \operatorname{Id}_V - \Delta_{\mathbf{n}} A_G) \det \begin{pmatrix} -\operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} \\ =&\ \det \left(\lambda \operatorname{Id}_V - \Delta_{\mathbf{n}} \begin{pmatrix} -\operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} -\operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} \right) \\ =&\ \det(\lambda \operatorname{Id}_V + \Delta_{\mathbf{n}} A_G) = p_{\mathcal{A}_{G,\lambda}}(\mathbf{n}), \end{aligned}$$

as desired.

Conversely, suppose (1) $p_{\mathcal{A}_{G,\lambda}}(\mathbf{n}) \equiv p_{\mathcal{A}_{G,\lambda}}(-\mathbf{n})$. Now it is well-known that the adjacency matrix A_{C_m} for any cycle graph C_m on m vertices, is circulant, and has eigenvalues $2\cos(2\pi j/m)$ for $0 \leq j < m$. If m is odd then no eigenvalue is zero and A_{C_m} is non-singular. Returning to the proof of the converse, if G is not bipartite then it has an induced odd cycle, say with vertices $v_1, \dots, v_m$ for m odd. In particular, the coefficient of $n_{v_1} \cdots n_{v_m}$ in $p_{\mathcal{A}_{G,\lambda}}$ is nonzero, by Theorem 2.1(2). But this contradicts (1). The converse follows. □

We conclude with two directions of future investigation. In one direction, as we just saw, for particular classes of graphs (e.g., bipartite graphs), one can obtain additional information that refines and enriches the univariate picture. Another direction involves the adjacency blowup-polynomial and blowup delta-matroid, and exploring their connections to previously studied notions and invariants arising from combinatorics. For example, Bouchet showed in [8] that for any graph $G = (V, E)$, the subsets $I \subset V$ for which the induced subgraph on I has a perfect matching, comprise a delta-matroid. Now by Corollary 2.2, the monomials with nonzero coefficients in $p_{\mathcal{A}_{G,\lambda}}$ form the adjacency-blowup delta-matroid of G (independent of $\lambda \neq 0$). Notice that these constructions agree whenever G is a tree. It is thus natural to ask if these two constructions agree in general; but even for small graphs G, this is not the case. For example, for $G = C_5$, the subset $V(G)$ cannot have a perfect matching, yet occurs in the blowup delta-matroid $\mathcal{M}_{\mathcal{A}_{C_5},\lambda}$ (defined in the paragraph after Corollary 2.2); and for $G = C_4$, the 'reverse' holds: $V(G)$ has a perfect matching, but does not occur in $\mathcal{M}_{\mathcal{A}_{C_4},\lambda}$ (e.g. by Proposition 3.3, since C_4 is a blowup).

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