THE ADDITIVE-MULTIPLICATIVE DISTANCE MATRIX OF A GRAPH, AND A NOVEL THIRD INVARIANT

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To the memory of Ronald L. Graham, with admiration

ABSTRACT. Graham showed with Pollak [*Bell Sys. Tech. J.* 1971] and Hoffman–Hosoya [*J. Graph Th.* 1977] that for any directed (additively weighted) graph G with strong blocks G_e, the determinant $\det(D_G)$ and cofactor-sum $\operatorname{cof}(D_G)$ of the distance matrix D_G can be computed from these same quantities for the blocks G_e. These phenomena have been generalized to trees – and in our recent work [*Eur. J. Combin.* 2024] to any graph – with multiplicative and q-distance matrices. For trees, we went further and unified all previous variants with weights in a unital commutative ring, into a distance matrix with both additive and multiplicative edge-data.

In the present paper, we extend the additive-multiplicative model to D_G for every graph G, and introduce a third, new invariant $\kappa(D_G)$. With these, we prove general formulas in the above vein connecting D_G to D_{G_e} that crucially require $\kappa(D_G)$; and we further refine the state-of-the-art in every setting to minors of D_G. (The simpler case of trees was in our recent work.) The proofs involve a novel application (to our knowledge) of Zariski density to this area. Thus, our results hold over an arbitrary commutative ring, and to our knowledge subsume all previous versions.

In greater detail: first, we introduce the additive-multiplicative distance matrix $D_{\mathcal{G}}$ of an arbitrary strongly connected graph G, using what we term the additive-multiplicative block-datum $\mathcal{G}$. This subsumes the previously studied additive, multiplicative, and q-distances for all graphs.

Second, we introduce an invariant $\kappa(D_{\mathcal{G}})$ that seems novel even from Graham's original works till now; and use it to prove "master" Graham–Hoffman–Hosoya (GHH) identities, which express $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}})$ in terms of the strong blocks G_e. We show how these imply all previous variants.

Third, we show that $\det(\cdot), \operatorname{cof}(\cdot), \kappa(\cdot)$ depend only on the block-data for not just $D_{\mathcal{G}}$, but also several minors of $D_{\mathcal{G}}$. This extension was not studied in any setting to date. We show it holds in the "most general" additive-multiplicative setting, hence in all previous settings.

Finally, we also compute in closed-form the inverse of $D_{\mathcal{G}}$. This again specializes to all known variants. In particular, we recover the explicit formula for D_T^{-1} for additive-multiplicative trees in our recent work [*Eur. J. Combin.* 2024] (which itself specializes to a result of Graham–Lovász [*Adv. Math.* 1978] and answers a question of Bapat–Lal–Pati [*Linear Algebra Appl.* 2006] in greater generality.) We show that not the Laplacian, but a closely related matrix is the "correct" one to use in $D_{\mathcal{G}}^{-1}$ – for the most general additive-multiplicative matrix $D_{\mathcal{G}}$, of an arbitrary graph G. As a sample example, we give closed-form expressions for $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}}), \kappa(D_{\mathcal{G}}), D_{\mathcal{G}}^{-1}$ for hypertrees.

CONTENTS

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Definition-Notation. *All graphs in this paper (see* [11] *for basics) are assumed to be finite, simple, directed, and* strongly connected*, i.e., for which there exist directed paths between any two distinct vertices. (This is assumed in order to be able to define a distance function between any two nodes; for an undirected graph, all edges are understood to be bidirected.) A* cut-vertex *is one whose removal disconnects the underlying undirected graph, and maximal subgraphs without cut-vertices are called* strong blocks*. Below, we always index by E the strong blocks of a graph G, not the edges (unless G is a tree). A* hypertree *is a graph whose strong blocks are all cliques/complete graphs.*

Unless otherwise specified we work over an arbitrary commutative unital ring R. For Zariski density arguments, we will first prove our desired equations over the field $\mathbb{Q}(\{a_e, m_{ij}\})$ generated by a set of say N variables, then observe that these equations in fact hold in the subring of polynomial functions $\mathbb{Z}[\{a_e, m_{ij}\}]$ *(using a Zariski dense subset of $\mathbb{A}^N_{\mathbb{Q}}$), and then specialize to arbitrary R.*

Given $n \geqslant 1$ and a set V of size n, define $\mathbf{e}_j$ to be the standard basis vector for $1 \leqslant j \leqslant n$; and

$$\mathbf{e} = \mathbf{e}(V) = \mathbf{e}(n) := (1, \dots, 1)^T = \sum_{j=1}^n \mathbf{e}_j \in R^n, \qquad J = J_n := \mathbf{e}(n)\mathbf{e}(n)^T, \qquad [n] := \{1, \dots, n\}.$$

Finally, given a matrix $A = (a_{ij}) \in R^{n \times n}$ with cofactors $c_{ij} = (-1)^{i+j} \det A_{ij}$, its adjugate matrix *and* cofactor-sum *are, respectively,*

$$\operatorname{adj}(A) := (c_{ji})_{i,j=1}^n, \qquad \operatorname{cof}(A) := \sum_{i,j=1}^n c_{ij} = \mathbf{e}(n)^T \operatorname{adj}(A) \mathbf{e}(n).$$

Now one has the following basic lemma (see e.g. [7]*) which is used below, possibly without reference:*

Lemma 0.1. *Suppose $n \geqslant 1$ is an integer and R is a commutative unital ring. If x is an indeterminate that commutes with R, and $A \in R^{n \times n}$ is any matrix, then* $\det(A + xJ_n) = \det(A) + x \operatorname{cof}(A)$*. Moreover,* $\operatorname{cof}(A) = \mathbf{e}(n)^T \operatorname{adj}(A)\mathbf{e}(n) = \operatorname{cof}(A + xJ_n)$*.*

1. The additive-multiplicative distance matrix of a graph

This paper contributes to the rich area of studying matrices associated to graphs (see e.g. [5, 6]). In it, we generalize and extend the results in our previous work [7], from trees to hypertrees and to arbitrary graphs. In particular, we subsume previous works in the literature that study distance matrices of graphs which are not trees. At the same time, this work provides a deeper, more conceptual understanding of the previously shown "explicit formulas" in the literature.

Classically, a distance function on a graph G was defined to be one that is additive across cut-vertices. Distance matrices of graphs (especially trees) and their invariants have been extensively studied in the literature, beginning with the seminal work of Graham with his coauthors [8, 9, 10]. In [10], the authors computed $\det(D_T)$ for $G = T$ a tree on $|E|$ edges (i.e. $|E| + 1$ nodes), and showed this depends only on $|E|$, not the structure of T. They also computed a second invariant, the *cofactor-sum* $\operatorname{cof}(D_T)$, and showed it has the same independence property. More generally [8], the pair $(\det(D_G), \operatorname{cof}(D_G))$ for an arbitrary (additively weighted) strongly connected directed graph depend only on the pairs $(\det(D_{G_e}), \operatorname{cof}(D_{G_e}))$ for the strong blocks $\{G_e : e \in E\}$ of G.

There have since been many variants of distance matrices proposed and studied, including with additive edgeweights a_e, multiplicative edgeweights m_e, and q-edgeweights $\frac{q^{\alpha_e}-1}{q-1}$ – and in all known cases, $\det(D_G)$ depends not on the graph structure but only on the edge-data of the blocks.

The additive-multiplicative distance matrix of a graph. In prior recent work [7], we unified all of these settings for *trees* into one common framework – that of a tree with additive and multiplicative edgeweights – and showed that the same independence property holds for the corresponding general form of the distance matrix. We further showed that allowing greater freedom in the parameters leads to dependence of the determinant on the tree structure. Additionally, we worked with distance matrices over an arbitrary unital commutative ring. In the present paper, our first novel contribution is to extend this "most general" setting for trees, to arbitrary graphs:

Definition 1.1. Given a graph G with strong blocks $\{G_e : e \in E\}$ on p_e vertices, an *additive-multiplicative block-datum* $\mathcal{G}$ attached to G is an E-tuple of pairs $\{\mathcal{G}_e := (a_e, D^*_{G_e}) : e \in E\}$ such that each $a_e \in R$, and each $D^*_{G_e} \in R^{p_e \times p_e}$ is a matrix with all diagonal entries 1.

Attached to this datum, we define two matrices for G (assuming G has vertex set $[n]$):

(1) The *multiplicative distance matrix* $D^*_G := (m_{ij})_{i,j=1}^n \in R^{n\times n}$ is the matrix with principal submatrices $D^*_{G_e}$ corresponding to the nodes of each strong block; and for $i, j \in [n]$ not in the same block, if v is any cut-vertex along a directed path from i to j, then we inductively define $m_{ij} := m_{iv}m_{vj}$. (Note: all diagonal entries in D^*_G are still 1.)

(2) The *additive-multiplicative distance matrix* $D_{\mathcal{G}} := (d_{ij})_{i,j=1}^n \in R^{n\times n}$ has principal submatrices $D_{\mathcal{G}_e} := a_e(D^*_{G_e} - J_{p_e})$ over the nodes of each strong block G_e; and for $i, j \in [n]$ not in the same block, if v is any cut-vertex along a directed path : $i \to j$, then inductively define $d_{ij} := d_{iv} + m_{iv}d_{vj}$.[1] (Note: d_{ij} – as also m_{ij} in (1) – does not depend on choice of v.)

Remark 1.2 (Special cases)**.** We briefly explain how the above framework encompasses (to our knowledge) all known variants of distance matrices of graphs studied in the literature over unital commutative rings. We refer the reader to [7] for a detailed history and survey of the literature.

(1) Our framework includes all known distance matrices of trees (i.e. with additive, multiplicative, and q edgeweights), because it includes as a special case the general additive-multiplicative distance matrix for trees, introduced in our prior work [7] over any unital commutative ring. This is the special case of $D_{\mathcal{G}}$ where each G_e is a single edge $e = \{i, j\}$, and E denotes the set of edges of the tree. In particular, $D^*_{G_e} = \begin{pmatrix} 1 & m_{ij} \\ m_{ji} & 1 \end{pmatrix}$, so that the datum $\mathcal{G}$ precisely equals $\{\mathcal{G}_e := (a_e, m_{ij}, m_{ji}) : e \in E\}$ as in [7]. As sample references we mention [1, 3, 4, 14, 15, 16]; for a comprehensive list, see the bibliography in [7].

(2) As explained in [7] (mostly but not only for trees), the above general framework also includes the special case in which all $a_e = 1$ – for all graphs. In other words, $D_{\mathcal{G}} = D^*_G - J$ is a rank-one update of the multiplicative distance matrix D^*_G. In particular, our data of interest $(\det(\cdot), \operatorname{cof}(\cdot))$ for $D_{\mathcal{G}}$ and for D^*_G can be recovered from each other via Lemma 0.1.

(3) Now if we set $m_{ij} = q^{\alpha_{ij}}$ for $i, j \in V(G_e)$ (with $\alpha_{ii} = 0\ \forall i \in [n]$), and reset $a_e := 1/(q-1)$ for all $e \in E$, then we recover the *q-distance matrix* $D_q(G)$ as a special case of the additive-multiplicative distance matrix. In particular, further setting $q \to 1$ recovers the classical distance matrix of a possibly directed graph, with additive edgeweights.[2] Note that the classical and q-distance matrices are precisely the variants studied in the literature for non-tree graphs: unicyclic, bicyclic, polycyclic, cactus, and cycle-clique graphs; as well as hypertrees (i.e. graphs whose strong blocks are all complete).

[1] Alternately, one can work with the "dual" definition $d_{ij} = d_{iv}m_{vj} + d_{vj}$ – but this essentially means that one deals with $(D^*_G)^T$ and $D^T_{\mathcal{G}}$. We will not proceed further along these lines in the present work. Also, $d_{ii} = 0\ \forall i$.

[2] Observe, naively setting all $m_{ij} = 1$ instead, yields a special case with trivial additive edgeweights: $D_{\mathcal{G}} = 0$.

(4) We also remark that in the case of hypertrees, the special case in which all off-diagonal entries m_{ij} in a given block G_e equal q, and each a_e equals $w_e/(q-1)$, was studied in [12], and is not a q-distance matrix. In fact, to our knowledge the closed-form expression for $\det(D_{\mathcal{G}})$ here is perhaps the one *known* formula[3] that our previous work [7] could not handle. This is addressed by the present paper, whose results specialize to closed-form expressions for $\det(D_{\mathcal{G}})$ as well as $\operatorname{cof}(D_{\mathcal{G}})$ for arbitrary hypertrees – and in the more general additive-multiplicative setting. See Section 2.2.

(5) Note in the previous example of [12] that the inverse of the q-weighted distance matrix was not computed for hypertrees. In this paper we are able to achieve this goal as well – and for an arbitrary *additive-multiplicative* graph G (in terms of $D_{\mathcal{G}_e}^{-1}$ for the strong blocks G_e of G). This also subsumes our formula in [7] for $D_{\mathcal{T}}^{-1}$ for additive-multiplicative trees.

Remark 1.3. If one varies the additive edgeweight inside a block, to try and work with an even more general additive-multiplicative distance matrix, then $\det(D_{\mathcal{G}})$ and $\operatorname{cof}(D_{\mathcal{G}})$ depend on these additive edgeweights vis-a-vis the geometry of the blocks. This dependence was shown via an example in [7, Section 1] for trees. Similarly, in [7] we defined an invariant $\kappa(D_{\mathcal{G}}, v_0)$ which depends on the location of the vertex v_0 when the additive edgeweight is allowed to vary inside a block – see the example in [7, Section 7] of a triangle. Below, we show that for all additive-multiplicative matrices, $\kappa(D_{\mathcal{G}}, v_0)$ is independent of v_0 – and in fact, provides the key ingredient to computing $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}})$ from the strong blocks G_e of G. In this sense also, is our additive-multiplicative model "most general", at least to date.

Remark 1.4 (Zariski density)**.** A quick word on a *novel technique* in this subfield, which we introduced for trees in [7], and which is equally effective and powerful for general graphs: Zariski density. This affords the freedom to assume that several quantities that are not identically zero, are "always" nonzero – in fact invertible. This helps significantly in computing determinants and inverses, and is stated more precisely in Remark 2.3 and (in greater detail) in previous work [7] (cf. [2, Lemma 2.1]).

Organization. We conclude by briefly listing the main results of this paper: one theorem in each section below. In Section 2, we show our first main result: a general set of Graham–Hoffman–Hosoya type identities, which furthermore explain a novel graph invariant $\kappa(D_{\mathcal{G}})$, introduced for trees in our recent work [7]. The identities also have a host of applications, explained below.

In Section 3, we go even beyond computing the invariants $\det(\cdot), \operatorname{cof}(\cdot), \kappa(\cdot)$ for arbitrary additive-multiplicative $D_{\mathcal{G}}$, to computing them for *minors* of $D_{\mathcal{G}}$ – and explain several cases when they too depend only on the block-data. In Section 4, we compute $D_{\mathcal{G}}^{-1}$ in closed-form, and show that the traditionally used "Laplacian matrix" should be corrected in the present, more general setting.

2. Theorem A: The Master Graham–Hoffman–Hosoya identities

The Graham–Hoffman–Hosoya (GHH) identities [8] are the reason why – for any directed additively weighted tree T – the quantities $\det(D_T), \operatorname{cof}(D_T)$ are independent of the tree structure and depend only on the set of edgeweights (as mentioned in Section 1). As shown in *loc. cit.*, these "classical" GHH identities hold for all (finite simple directed strongly connected, as above) graphs G with *additive* edgeweights, using their strong blocks.

The GHH identities in [8] were extended to a q-variant in [13]. In our previous work [7] we obtained generalizations of the original GHH identities that held for all multiplicative distance matrices and all q-distance matrices (and in particular specialized to the identities in [8, 13]). We also provided at the end of [7], a new invariant $\kappa(D_G, v_0)$ for additive-multiplicative trees and the corresponding GHH identities, for general graphs G rooted at a fixed cut-vertex v_0.

[3]This work strictly extends all known settings, so its results were of course not known or conjectured either.

Now a natural question is if analogous identities hold in the present, general additive-multiplicative setting. This turns out to be true, as is now shown; moreover, a new invariant $\kappa(D_{\mathcal{G}})$ is crucially required. Thus, our first result provides "master" Graham–Hoffman–Hosoya identities, with the word "master" used for the following reasons:

(1) The identities below explain the new invariant $\kappa(D_G, v_0)$ introduced in prior work [7].
(2) These (novel) identities apply to all additive-multiplicative distance matrices of all graphs. Previous versions in [7, 8, 13] were applicable only to restricted versions – the multiplicative, q, and classical distance matrices – but not to the general version $D_{\mathcal{G}}$ (which was not even defined). For identities in our general additive-multiplicative setting, $\kappa(D_{\mathcal{G}}, v_0)$ is needed.

    As a concrete special case: consider the formulas (2.7) from a previous work [7], for $\det(D_{\mathcal{G}})$ and $\operatorname{cof}(D_{\mathcal{G}})$, where G is a tree. By inspection, these depend only on $\det(D_{\mathcal{G}_e})$ and $\operatorname{cof}(D_{\mathcal{G}_e})$ for all strong blocks – i.e. edges e of G – but previous GHH variants (including [7]) could not explain why, for the *general* additive-multiplicative trees. Now we can.
(3) The identities below specialize and apply to all of the previously studied settings, to yield all previously known GHH identities. See Corollary 2.5.
(4) In a later subsection, we will apply these identities to obtain closed-form expressions for $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}})$ for arbitrary additive-multiplicative hypertrees. This again extends prior formulas in [7, 12] to the additive-multiplicative setting, which could not be handled by the results in previous papers (including *loc. cit.*).

We begin with the main result in this section, which addresses the first two of these points.

Definition 2.1 (From [7])**.** Fix a vertex v_0 of a graph G. Given a subgraph G' induced on a subset of nodes V' containing v_0, with additive-multiplicative block-datum $\mathcal{G}'$, write

$$D_{\mathcal{G}'} := (d(v,w))_{v,w\in V'} = \begin{pmatrix} D|_{V'\setminus\{v_0\}} & \mathbf{u}_1 \\ \mathbf{w}_1^T & 0 \end{pmatrix},$$

by relabelling the nodes, and define the invariant

$$\kappa(D_{\mathcal{G}'}, v_0) := \det\left(D|_{V'\setminus\{v_0\}} - \mathbf{u}_1\,\mathbf{e}(V'\setminus\{v_0\})^T - \mathbf{m}(V'\setminus\{v_0\}, v_0)\mathbf{w}_1^T\right). \tag{2.1}$$

For an alternate formula for $\kappa(D_{\mathcal{G}}, v_0)$, see (3.2) below (and see (3.4) for a generalization). Also, a notational **remark**: in the sequel we may use multiple $\mathbf{e}$-vectors of varying dimensions in our computations, the dimension of each being deducible from context. At other times, we will use the notation $\mathbf{e}(n)$ or $\mathbf{e}(V)$ depending on the context – similarly for the square matrix J or J_n.

Theorem A (Master GHH identities)**.** *Fix a graph G with node-set $[n]$, with strong blocks G_e on p_e nodes for $e \in E$. Let $\mathcal{G} = \{\mathcal{G}_e = (a_e, D^*_{G_e}) : e \in E\}$ be an additive-multiplicative block-datum, and $D_{\mathcal{G}}, D^*_G, D_{\mathcal{G}_e}, D^*_{G_e}$ the corresponding matrices. Also fix a (not necessarily cut) vertex v_0 of G. Then $\kappa(D_{\mathcal{G}}, v_0)$ is independent of v_0, and equals*

$$\kappa(D_{\mathcal{G}}, v_0) = \det(D^*_G)\prod_{e\in E} a_e^{p_e-1}. \tag{2.2}$$

Writing this as $\kappa(D_{\mathcal{G}})$, so that $\kappa(D_{\mathcal{G}_e}) = a_e^{p_e-1}\det(D^*_{G_e})$, the following identities then hold:

$$\kappa(D_{\mathcal{G}}) = \prod_{e\in E}\kappa(D_{\mathcal{G}_e}), \tag{2.3}$$

$$\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} = \sum_{e\in E}\frac{\det(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})}, \tag{2.4}$$

$$\frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} - 1 = \sum_{e\in E}\left(\frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})} - 1\right), \tag{2.5}$$

where the denominators in the second and third formulas are understood to be placeholders, to be canceled by multiplying both sides by $\kappa(D_{\mathcal{G}})$. Finally, we have

$$\operatorname{cof}(D_{\mathcal{G}}) = \operatorname{cof}(D^*_G) \prod_{e \in E} a_e^{p_e - 1}, \qquad \frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} = \frac{\operatorname{cof}(D^*_G)}{\det(D^*_G)}. \tag{2.6}$$

Before proving the theorem, we recall the special case of trees shown in [7], where each G_e is a single edge K_2 (so $p_e = 2$), and $G = T$ has block-datum $\mathcal{T}$:

$$\det(D_{\mathcal{T}} + xJ) = \prod_{e \in E} a_e(1 - m_e m'_e) \left[\sum_{e \in E} \frac{a_e(m_e - 1)(m'_e - 1)}{m_e m'_e - 1} + x \left(1 - \sum_{e \in E} \frac{(m_e - 1)(m'_e - 1)}{m_e m'_e - 1} \right) \right]. \tag{2.7}$$

Observe two points about these, unexplained in [7], which can now be understood via Theorem A:

- The common multiplicative factor $\prod_e (a_e(1 - m_e m'_e))$ is precisely $\kappa(D_{\mathcal{T}})$ (at any vertex v_0), since the set E of strong blocks is precisely the edge-set of the tree, and each $D^*_{G_e} = \begin{pmatrix} 1 & m_e \\ m'_e & 1 \end{pmatrix}$ (up to transposing). Now the formulas for $\frac{\det(D_{\mathcal{T}})}{\kappa(D_{\mathcal{T}})}$ and $1 - \frac{\operatorname{cof}(D_{\mathcal{T}})}{\kappa(D_{\mathcal{T}})}$ are remarkably similar: they involve summing the same terms, with an additional factor of a_e in the former expression.
- $\operatorname{cof}(D_{\mathcal{T}})$ does not depend on the complete datum $\mathcal{T} = \{(a_e, D^*_{G_e}) : e \in E\} = \{(a_e, m_e, m'_e) : e \in E\}$, but rather on the decoupled data $\{a_e\}$ and $\{D^*_{G_e}\} = \{(m_e, m'_e)\}$.

We now explain why both phenomena are special cases of our master GHH identities. In the first case, since $D_{\mathcal{G}_e} = a_e(D^*_{G_e} - J)$, the summands in the right-hand sides of (2.4) and (2.5) are computable by Lemma 0.1 to be, respectively:

$$\frac{\det(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})} = a_e \left(1 - \frac{\operatorname{cof}(D^*_{G_e})}{\det(D^*_{G_e})} \right), \qquad \frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})} - 1 = \frac{\operatorname{cof}(D^*_{G_e})}{\det(D^*_{G_e})} - 1. \tag{2.8}$$

This explains the first phenomenon above, but now for all G – as well as the second phenomenon, since the latter quantity in (2.8) does not depend on a_e. In particular, (2.4) can be restated as:

$$\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} = \sum_{e \in E} a_e \left(1 - \frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})} \right).$$

A pleasing consequence of our master GHH formulas – given the present discussion – is that even for arbitrary graphs, $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}}), \kappa(D_{\mathcal{G}})$ do not depend on the complete datum $\mathcal{G} = \{(a_e, D^*_{G_e}) : e \in E\}$, but once again only on triples and pairs:

Corollary 2.2. *For any additive-multiplicative distance matrix $D_{\mathcal{G}}$ of an arbitrary graph G,*

1. *$\det(D_{\mathcal{G}})$ depends on the datum $\mathcal{G}$ only through the triples $\{(a_e, \det(D^*_{G_e}), \operatorname{cof}(D^*_{G_e})) : e \in E\}$,*
2. *$\operatorname{cof}(D_{\mathcal{G}})$ through the decoupled data $\{a_e : e \in E\} \sqcup \{(\det(D^*_{G_e}), \operatorname{cof}(D^*_{G_e})) : e \in E\}$, and*
3. *$\kappa(D_{\mathcal{G}})$ depends on $\mathcal{G}$ through even less: $\{a_e : e \in E\} \sqcup \{\det(D^*_{G_e}) : e \in E\}$.*

Proof of Theorem A. (We will deduce all previous GHH identities in Corollary 2.5.) We first assume (2.2) and show that it implies the rest. Clearly (2.2) implies (2.3). We now show (2.4) and (2.5) by induction on the number $|E|$ of strong blocks in G. For $|E| = 1$ the identities are tautologies; otherwise the graph G has a "pendant block" (or "end-block") G_e, separated by a cut-vertex v_0 from the rest of the subgraph; call it G'. But then by [7, Theorem D], we have

$$\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}}, v_0)} = \frac{\det(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e}, v_0)} + \frac{\det(D_{\mathcal{G}'})}{\kappa(D_{\mathcal{G}'}, v_0)},$$

$$\frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}}, v_0)} - 1 = \left(\frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e}, v_0)} - 1 \right) + \left(\frac{\operatorname{cof}(D_{\mathcal{G}'})}{\kappa(D_{\mathcal{G}'}, v_0)} - 1 \right).$$

The last term on the right in both equations has denominator $\kappa(D_{\mathcal{G}'})$ by the (assumed) identity (2.2), so we are now done by the induction hypothesis, since G' has fewer blocks than G.

Finally, the identities in (2.6) are shown as follows: by (2.5) and (2.8), we obtain:

$$\frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} - 1 = \sum_{e \in E} \left(\frac{\operatorname{cof}(D^*_{G_e})}{\det(D^*_{G_e})} - 1 \right).$$

In particular, this equation also holds for the special case when $a_e = 1\ \forall e$. Here the right-hand side is unchanged, while as stated in Remark 1.2(2), now $D_{\mathcal{G}} = D^*_G - J$, so the left-hand side is

$$\frac{\operatorname{cof}(D^*_G - J)}{\det(D^*_G)} - 1 = \frac{\operatorname{cof}(D^*_G)}{\det(D^*_G)} - 1$$

by Lemma 0.1. This shows the second of the identities in (2.6), and hence the first.

Remark 2.3. In this paper, the freedom to work with denominators that can vanish at special values of the parameters (in an arbitrary commutative ring R) is permissible via Zariski density arguments – a technique introduced in such settings by our prior work [7]. Concretely: note by specializing that $\kappa(D_{\mathcal{G}_e}, v_0), \kappa(D_{\mathcal{G}'}, v_0), \kappa(D_{\mathcal{G}}, v_0)$ are nonzero polynomials in the a_e and the off-diagonal entries of $D^*_{G_e}$ (over all $e \in E$). Thus we first work over the field of rational functions in all of these variables – with coefficients in $\mathbb{Q}$ – and show that all assertions hold on the nonzero-locus of a finite set of nonzero polynomials over an infinite field. By Zariski density, the assertions hold on the entire affine space. Now the equality of the two sides (in all claimed identities/assertions) holds in the ring of polynomials in the same variables (since there are no denominators) – and with coefficients in $\mathbb{Z}$. Finally, specialize the parameters to take values in R. We refer the reader to [7] for specific examples of this procedure, and omit such demonstrations in this paper for brevity.

We now come to the meat of the proof, which is in showing (2.2), and we again do so by induction on the number $|E|$ of strong blocks in G. For E a singleton, write $D_{\mathcal{G}}$ in the form

$$D_{\mathcal{G}} = a_1(D^*_G - J) = a_1 \begin{pmatrix} D^*_1 - J & \mathbf{u}_1 - \mathbf{e} \\ \mathbf{w}_1^T - \mathbf{e}^T & 0 \end{pmatrix},$$

under a suitable ordering of the nodes (i.e. with v_0 last), and where $\mathbf{e} = \mathbf{e}(V(G) \setminus \{v_0\})$. Then,

$$\kappa(D_{\mathcal{G}}, v_0) = a_1^{p_1 - 1} \det(D^*_1 - J - (\mathbf{u}_1 - \mathbf{e})\mathbf{e}^T - \mathbf{u}_1(\mathbf{w}_1^T - \mathbf{e}^T)) = a_1^{p_1-1} \det \begin{pmatrix} D^*_1 & \mathbf{u}_1 \\ \mathbf{w}_1^T & 1 \end{pmatrix},$$

as desired. For the induction step, the first case is when v_0 a cut-vertex of G. Let $G = G_1 \sqcup_{v_0} G_2$, with $V(G_1) = \{1, \dots, v_0\}$ and $V(G_2) = \{v_0, \dots, n\}$. Set $V'_j := V(G_j) \setminus \{v_0\}$ and write

$$D_{\mathcal{G}} := \begin{pmatrix} a_1(D^*_1 - J) & a_1(\mathbf{u}_1 - \mathbf{e}) & a_1(\mathbf{u}_1 - \mathbf{e})\mathbf{e}^T + a_2\mathbf{u}_1(\mathbf{w}_2^T - \mathbf{e}^T) \\ a_1(\mathbf{w}_1^T - \mathbf{e}^T) & 0 & a_2(\mathbf{w}_2^T - \mathbf{e}^T) \\ a_2(\mathbf{u}_2 - \mathbf{e})\mathbf{e}^T + a_1\mathbf{u}_2(\mathbf{w}_1^T - \mathbf{e}^T) & a_2(\mathbf{u}_2 - \mathbf{e}) & a_2(D^*_2 - J) \end{pmatrix}.$$

Then as asserted in the proof of [7, Theorem D], a straightforward computation shows that

$$\kappa(D_{\mathcal{G}}, v_0) = \kappa(D_{\mathcal{G}_1}, v_0)\kappa(D_{\mathcal{G}_2}, v_0).$$

In particular, if v_0 is a cut-vertex of G then (2.2) follows by the induction hypothesis.

It remains to consider the case when v_0 is not a cut-vertex. Let $G_{e_0} = G_0$ be the strong block of G containing v_0, and let $v_1, \dots, v_k$ be the cut-vertices of G lying in G_0. Let $G_1, \dots, G_k$ denote the maximal (pairwise disjoint) induced subgraphs of G such that G_j intersects G_0 precisely at v_j:

$$G = G_0 \sqcup_{v_1} G_1 \sqcup_{v_2} G_2 \cdots \sqcup_{v_k} G_k.$$

For this graph, order the vertices in $k+1$ groups, each with a vertex followed by a set of vertices:

$$v_0,\ V'_0 := V(G_0) \setminus \{v_j\}; \quad v_1,\ V'_1 := V(G_1) \setminus \{v_1\}; \quad \dots; \quad v_k,\ V'_k := V(G_k) \setminus \{v_k\}. \tag{2.9}$$

Corresponding to these, we write $D_{\mathcal{G}}$ as a $(2k+2)\times(2k+2)$ block matrix, consisting of $(k+1)^2$-many 2×2 block matrices $B_{ij} : 0 \leqslant i,j \leqslant k$, where

$$B_{ij} := \begin{cases} a_i \begin{pmatrix} 0 & \mathbf{w}_i^T - \mathbf{e}^T \\ \mathbf{u}_i - \mathbf{e} & D_i - J \end{pmatrix}, & \text{if } i = j \in \{0,\dots,k\}, \\ \begin{pmatrix} a_0(m_{v_0v_j} - 1) & a_0(m_{v_0v_j}-1)\mathbf{e}^T + a_j m_{v_0v_j}(\mathbf{w}_j^T - \mathbf{e}^T) \\ a_0(\mathbf{m}_{V_0',v_j} - \mathbf{e}) & a_0(\mathbf{m}_{V_0',v_j} - \mathbf{e})\mathbf{e}^T + a_j \mathbf{m}_{V_0',v_j}(\mathbf{w}_j^T - \mathbf{e}^T) \end{pmatrix}, & \text{if } 0 = i < j \leqslant k, \\ \begin{pmatrix} a_0(m_{v_iv_0} - 1) & a_0(\mathbf{m}^T_{v_i,V_0'} - \mathbf{e}^T) \\ a_i(\mathbf{u}_i - \mathbf{e}) + a_0(m_{v_iv_0}-1)\mathbf{u}_i & a_i(\mathbf{u}_i - \mathbf{e})\mathbf{e}^T + a_0\mathbf{u}_i(\mathbf{m}^T_{v_i,V_0'} - \mathbf{e}^T) \end{pmatrix}, & \text{if } 0 = j < i \leqslant k, \\ \left(\begin{array}{c|c} a_0(m_{v_iv_j} - 1) & a_0(m_{v_iv_j}-1)\mathbf{e}^T + a_j m_{v_i,v_j}(\mathbf{w}_j^T - \mathbf{e}^T) \\ \hline a_i(\mathbf{u}_i - \mathbf{e}) + & a_i(\mathbf{u}_i - \mathbf{e})\mathbf{e}^T + a_0(m_{v_iv_j}-1)\mathbf{u}_i\mathbf{e}^T + \\ a_0(m_{v_iv_j}-1)\mathbf{u}_i & a_j m_{v_iv_j}\mathbf{u}_i(\mathbf{w}_j^T - \mathbf{e}^T) \end{array}\right), & \text{if } 1 \leqslant i,j \leqslant k,\ i \neq j. \end{cases} \tag{2.10}$$

Recall here that $m_{v_iv_j}$ for $0 \leqslant i,j \leqslant k$ is the corresponding entry in the multiplicative distance matrix $D^*_{G_0}$. Now write $D_{\mathcal{G}} = (B_{ij})_{i,j=0}^k$ in the form $\begin{pmatrix} 0 & \mathbf{w}^T \\ \mathbf{u} & D \end{pmatrix}$; as a result,

$$\kappa(D_{\mathcal{G}}, v_0) = \det(D - \mathbf{u}\mathbf{e}^T - \mathbf{m}(V \setminus \{v_0\}, v_0)\mathbf{w}^T).$$

Next, $\mathbf{u}^T, \mathbf{m}(V\setminus\{v_0\})^T, \mathbf{w}^T$ have $V_0'; v_i, V_i'$ components (for $i \in [k]$) given respectively by:

$$\begin{aligned} \mathbf{u}^T &= (a_0(\mathbf{u}_0^T - \mathbf{e}^T); \quad (a_0(m_{v_iv_0} - 1), \ \ a_i(\mathbf{u}_i^T - \mathbf{e}^T) + a_i(m_{v_iv_0}-1)\mathbf{u}_i^T)_{i=1}^k), \\ \mathbf{m}(V\setminus\{v_0\}, v_0)^T &= (\mathbf{u}_0^T; \quad (m_{v_iv_0}, \ \ m_{v_iv_0}\mathbf{u}_i^T)_{i=1}^k), \\ \mathbf{w}^T &= (a_0(\mathbf{w}_0^T - \mathbf{e}^T); \quad (a_0(m_{v_0v_j}-1), \ \ a_0(m_{v_0v_j}-1)\mathbf{e}^T + a_j m_{v_0v_j}(\mathbf{w}_j^T - \mathbf{e}^T))_{j=1}^k). \end{aligned} \tag{2.11}$$

From this, we obtain the matrix $A_{\mathcal{G}} := \mathbf{u}\mathbf{e}^T + \mathbf{m}(V\setminus\{v_0\}, v_0)\mathbf{w}^T)$ in block form as follows:

$$A_{\mathcal{G}} = \left(\begin{array}{c|c|c} a_0(\mathbf{u}_0\mathbf{w}_0^T - J) & a_0(m_{v_0v_j}\mathbf{u}_0 - \mathbf{e}) & \begin{array}{c} a_0(m_{v_0v_j}\mathbf{u}_0 - \mathbf{e})\mathbf{e}^T + \\ a_j m_{v_0v_j}\mathbf{u}_0(\mathbf{w}_j^T - \mathbf{e}^T) \end{array} \\ \hline a_0(m_{v_iv_0}\mathbf{w}_0^T - \mathbf{e}^T) & a_0(m_{v_iv_0}m_{v_0v_j} - 1) & \begin{array}{c} a_0(m_{v_iv_0}m_{v_0v_j} - 1)\mathbf{e}^T + \\ a_j m_{v_iv_0}m_{v_0v_j}(\mathbf{w}_j^T - \mathbf{e}^T) \end{array} \\ \hline \begin{array}{c} a_i(\mathbf{u}_i - \mathbf{e})\mathbf{e}^T + \\ a_0\mathbf{u}_i(m_{v_iv_0}\mathbf{w}_0^T - \mathbf{e}^T) \end{array} & \begin{array}{c} a_i(\mathbf{u}_i - \mathbf{e}) + \\ a_0(m_{v_iv_0}m_{v_0v_j} - 1)\mathbf{u}_i \end{array} & \begin{array}{c} a_i(\mathbf{u}_i - \mathbf{e})\mathbf{e}^T + \\ a_0(m_{v_iv_0}m_{v_0v_j} - 1)\mathbf{u}_i\mathbf{e}^T + \\ a_j m_{v_iv_0}m_{v_0v_j}\mathbf{u}_i(\mathbf{w}_j^T - \mathbf{e}^T) \end{array} \end{array}\right). \tag{2.12}$$

Here, the rows are labelled by V_0'; v_i, V_i' for $i \in [k]$, and the columns by V_0'; v_j, V_j' for $j \in [k]$. Note that the square matrix $A_{\mathcal{G}}$ has dimension $|V(G)| - 1$. Now the above computations yield:

$$\kappa(D_{\mathcal{G}}, v_0) = \det(D_{\mathcal{G}} - A_{\mathcal{G}}) = \det((B'_{ij})_{i,j=0}^k),$$

where B'_{ij} has the same size as B_{ij} for $i,j > 0$, and B'_{i0}, B'_{0j} have one less row and one less column respectively for all $0 \leqslant i,j \leqslant k$. Moreover, the B'_{ij} are given by:

$$B'_{ij} := \begin{cases} a_0(D_0 - \mathbf{u}_0\mathbf{w}_0^T), & \text{if } i = j = 0, \\ (\mathbf{m}_{V_0',v_j} - m_{v_0v_j}\mathbf{u}_0)\begin{pmatrix} a_0 & a_0\mathbf{e}^T + a_j(\mathbf{w}_j^T - \mathbf{e}^T) \end{pmatrix}, & \text{if } 0 = i < j \leqslant k, \\ a_0 \begin{pmatrix} 1 \\ \mathbf{u}_i \end{pmatrix} (\mathbf{m}_{v_i,V_0'} - m_{v_iv_0}\mathbf{w}_0)^T, & \text{if } 0 = j < i \leqslant k, \\ \gamma_{ij}\begin{pmatrix} 1 \\ \mathbf{u}_i \end{pmatrix}\begin{pmatrix} a_0 & a_0\mathbf{e}^T + a_j(\mathbf{w}_j^T - \mathbf{e}^T) \end{pmatrix} + \delta_{ij}\begin{pmatrix} 0 & 0 \\ 0 & a_i(D_i - \mathbf{u}_i\mathbf{w}_i^T) \end{pmatrix}, & \text{if } 1 \leqslant i,j \leqslant k, \end{cases} \tag{2.13}$$

where δ_{ij} denotes the Kronecker delta, and $\gamma_{ij} := m_{v_i v_j} - m_{v_i v_0} m_{v_0 v_j}$ for all $i, j \in [k]$, with the understanding that $m_{v_i v_i} = 1$ in our notation/convention.

Now to compute $\det((B'_{ij})_{i,j=0}^k)$, we perform block row operations, in which for each $i \in [k]$, from the V'_i-block row we subtract $\mathbf{u}_i$ times the v_i-block row. This kills all entries except the (V'_i, V'_i)-block entry, which is $a_i(D_i - \mathbf{u}_i \mathbf{w}_i^T)$. By e.g. "block upper triangularity", it follows that

$$\kappa(D_{\mathcal{G}}, v_0) = \det(D - A_{\mathcal{G}}) = \prod_{i=1}^k \det(a_i(D_i - \mathbf{u}_i \mathbf{w}_i^T)) \cdot \det A_0, \tag{2.14}$$

for a particular square matrix A_0 (given below) of dimension $|V(G_0)| - 1$. Also note that since the graph G_i (with node set $V'_i \sqcup \{v_i\}$) has a smaller number of blocks, the induction hypothesis yields

$$\det(a_i(D_i - \mathbf{u}_i \mathbf{w}_i^T)) = \kappa(D_{\mathcal{G}_i}, v_i) = \kappa(D_{\mathcal{G}_i}), \qquad \forall i \in [k].$$

It thus remains to show that $\det(A_0) = a_0^{|V(G_0)|-1} \det(D^*_{G_0}) = \kappa(D_{\mathcal{G}_0}, v_0) = \kappa(D_{\mathcal{G}_0})$, where

$$A_0 := a_0 \begin{pmatrix} D_0 - \mathbf{u}_0 \mathbf{w}_0^T & \mathbf{m}_{V'_0, v_1} - m_{v_0 v_1} \mathbf{u}_0 & \mathbf{m}_{V'_0, v_2} - m_{v_0 v_2} \mathbf{u}_0 & \cdots & \mathbf{m}_{V'_0, v_k} - m_{v_0 v_k} \mathbf{u}_0 \\ \mathbf{m}^T_{v_1, V'} - m_{v_1 v_0} \mathbf{w}_0^T & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1k} \\ \mathbf{m}^T_{v_2, V'} - m_{v_2 v_0} \mathbf{w}_0^T & \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{m}^T_{v_k, V'} - m_{v_k v_0} \mathbf{w}_0^T & \gamma_{k1} & \gamma_{k2} & \cdots & \gamma_{kk} \end{pmatrix}.$$

But it is straightforward to verify that if one writes $D^*_{G_0}$ with the same labelling of the nodes v_0, V'_0, and subtracts a suitable multiple of the first row from every other row (to cancel their leading entries) – or does the analogous column operations – then the matrix one obtains has leading column $\mathbf{e}_1$ (or row $\mathbf{e}_1^T$), and the principal submatrix obtained by now removing the first row and column is precisely $a_0^{-1} A_0$. Hence $\det(A_0) = a_0^{|V(G_0)|-1} \det(D^*_{G_0})$, and by (2.14) and the induction hypothesis, the proof is complete. □

2.1. **Application 1: Previous GHH identities.** Theorem A has several applications; some are now listed. First, the result explains all previously known GHH identities, including in our recent work [7]. We begin with a straightforward reformulation of the GHH identities in Theorem A:

Proposition 2.4. *Notation as in Theorem A. Then:*

$$\det(D_{\mathcal{G}}) = \sum_{e \in E} \det(D_{\mathcal{G}_e}) \prod_{f \neq e} a_f^{p_f - 1} \det(D^*_{G_f}), \tag{2.15}$$

$$\operatorname{cof}(D_{\mathcal{G}}) = \det(D^*_G) \prod_{e \in E} a_e^{p_e - 1} + \sum_{e \in E} (\operatorname{cof}(D_{\mathcal{G}_e}) - a_e^{p_e - 1} \det(D^*_{G_e})) \prod_{f \neq e} a_f^{p_f - 1} \det(D^*_{G_f}). \tag{2.16}$$

As a consequence, we obtain the following application of our master GHH identities:

Corollary 2.5. *Proposition 2.4 implies the GHH identities for multiplicative, q, and classical distance matrices, shown in* [7], [7], *and* [8] *respectively (following previously shown special cases):*

$$\begin{aligned} \det(D^*_G) &= \prod_{e \in E} \det(D^*_{G_e}), \\ \operatorname{cof}(D^*_G) &= \prod_{e \in E} \det(D^*_{G_e}) + \sum_{e \in E} (\operatorname{cof}(D^*_{G_e}) - \det(D^*_{G_e})) \prod_{f \neq e} \det(D^*_{G_f}), \end{aligned} \tag{2.17}$$

$$\begin{aligned} \det(D_q(G)) &= \sum_{e \in E} \det(D_q(G_e)) \prod_{f \neq e} d^*_f, \\ \operatorname{cof}(D_q(G)) &= \prod_{e \in E} d^*_e - (q-1) \sum_{e \in E} \det(D_q(G_j)) \prod_{f \neq e} d^*_f, \end{aligned} \tag{2.18}$$

$$\begin{aligned}\det(D_G) &= \sum_{e\in E}\det(D_{G_e})\prod_{f\neq e}\mathrm{cof}(D_{G_f}),\\ \mathrm{cof}(D_G) &= \prod_{e\in E}\mathrm{cof}(D_{G_e}).\end{aligned} \tag{2.19}$$

Here $D_q(G)$ is defined in Remark 1.2(3), $D_G := D_1(G)$, and $d_e^* := (q-1)\det(D_q(G_e)) + \mathrm{cof}(D_q(G_e))$.

As mentioned above, there are also GHH identities involving the invariant $\kappa(D_{\mathcal{G}}, v_0)$; these are subsumed by our main result in this section – see Theorem A.

Proof. Setting all $a_e = 1$, we observe that $D^*_{\mathcal{G}} = D_{\mathcal{G}} - J$ (see e.g. Remark 1.2(2)). Now the first two identities (2.17) are easy consequences of Proposition 2.4, via Lemma 0.1. That these identities (2.17) for $D^*_{\mathcal{G}}$ imply the identities (2.18) for $D_q(G)$ was shown in [7, Section 2]; and specializing (2.18) to $q \to 1$ yields the classical GHH identities (2.19) for $D_G = D_1(G)$. □

2.2. **Application 2: Hypertrees.** Our next application of Theorem A is to *hypertrees* – graphs whose strong blocks are cliques. Here we provide a novel additive-multiplicative model, which generalizes and subsumes the known variants. In our model, every block of the hypertree G is a bidirected clique K_p, with vertex set $V = [p] = \{1, \dots, p\}$. We begin with the multiplicative component, in which every edge $i \to j$ is equipped with both a "head" and "tail" contribution:

$$(D^*_{K_p})_{i\to j} := \begin{cases} m_i m'_j, & \text{if } i \neq j,\\ 1 & \text{otherwise.}\end{cases} \tag{2.20}$$

For example, one can take all $m_i = q, m'_i = 1$, leading to the q-exponential distance matrix $D^*_{K_p}$ (hence to the q-distance matrix $D_q(K_p)$ if we choose $a = 1/(q-1)$). We now have:

Lemma 2.6. *If $\mathbf{d}$ denotes the vector with vth component $\frac{m_v m'_v}{1-m_v m'_v}$, then*

$$\begin{aligned}\det D^*_{K_p} &= \prod_{v\in V}(1 - m_v m'_v)\cdot(1 + \mathbf{e}(p)^T\mathbf{d}),\\ \mathrm{cof}\, D^*_{K_p} &= \prod_{v\in V}(1 - m_v m'_v)\cdot\left(p + \mathbf{e}(p)^T\mathbf{d} + \sum_{v<w}\frac{(m_v - m_w)(m'_v - m'_w)}{(1-m_v m'_v)(1-m_w m'_w)}\right).\end{aligned} \tag{2.21}$$

The denominators in both right-hand sides (including in $\mathbf{d}$) are understood to be placeholders, which cancel with factors of the products on the right.

Proof. We sketch a proof. Let $\mathbf{m}' := (m'_v)_{v\in V}$. Then $D^*_{K_p} = \mathrm{diag}(1 - m_v m'_v)_{v\in V} + \mathbf{m}(\mathbf{m}')^T$. This easily yields $\det D^*_{K_p}$. Next, the Sherman–Morrison formula for a rank-one update implies:

$$(D^*_{K_p})^{-1} = \mathrm{diag}(\mathbf{m}')^{-1}\cdot C\cdot \mathrm{diag}(\mathbf{m})^{-1}, \qquad \text{where } C := \mathrm{diag}(\mathbf{d}) - \frac{1}{1+\mathbf{e}(p)^T\mathbf{d}}\mathbf{d}\mathbf{d}^T,$$

and from this one obtains $\mathrm{cof}(D^*_{K_p}) = \det(D^*_{K_p})\cdot \mathbf{e}(p)^T(D^*_{K_p})^{-1}\mathbf{e}(p)$, if $D^*_{K_p}$ is invertible. The general case now follows by Zariski density, see e.g. Remark 2.3 or proofs of special cases in [7]. □

From these computations, one derives the desired invariants for arbitrary hypertrees:

Proposition 2.7. *Suppose G is a hypertree with strong blocks $G_e = K_{p_e}$ for $e \in E$. Suppose the nodes of each clique are labelled by $[p_e]$, with block-data given by $a_e, m_{e,v}, m'_{e,v}$ as above. Also define the vector $\mathbf{d}_e := \left(\frac{m_{e,v}m'_{e,v}}{1-m_{e,v}m'_{e,v}}\right)_{v\in[p_e]}$ for all blocks $e \in E$. Then,*

$$\kappa(D_{\mathcal{G}}) = \prod_{e\in E}\left(a_e^{p_e-1}(1+\mathbf{e}(p_e)^T\mathbf{d}_e)\prod_{v\in[p_e]}(1 - m_{e,v}m'_{e,v})\right),$$

$$\det(D_{\mathcal{G}}) = \kappa(D_{\mathcal{G}}) \left[\sum_{e \in E} \frac{-a_e}{1 + \mathbf{e}(p_e)^T \mathbf{d}_e} \left(p_e - 1 + \sum_{v < w \in [p_e]} \frac{(m_{e,v} - m_{e,w})(m'_{e,v} - m'_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})} \right) \right],$$

$$\operatorname{cof}(D_{\mathcal{G}}) = \kappa(D_{\mathcal{G}}) \left[1 + \sum_{e \in E} \frac{1}{1 + \mathbf{e}(p_e)^T \mathbf{d}_e} \left(p_e - 1 + \sum_{v < w \in [p_e]} \frac{(m_{e,v} - m_{e,w})(m'_{e,v} - m'_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})} \right) \right].$$

Proof. With the present notation, the additive-multiplicative matrix of the block G_e is precisely $D_{\mathcal{G}_e} := a_e(D^*_{G_e} - J)$ as above, for each $e \in E$. For this matrix, Lemmas 2.6 and 0.1 imply:

$$\det(D_{\mathcal{G}_e} + xJ) = a_e^{p_e - 1} \prod_{v \in [p_e]} (1 - m_{e,v} m'_{e,v})$$
$$\times \left[x \cdot \mathbf{e}(p_e)^T \mathbf{d}_e + a_e + (x - a_e) \left(p_e + \sum_{v < w} \frac{(m_{e,v} - m_{e,w})(m'_{e,v} - m'_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})} \right) \right].$$

Now apply Lemma 2.6 and Theorem A to complete the calculations. ☐

As a sample corollary, specialize $m_{e,v} = m'_{e,v} = \sqrt{q}, a_e = w_e/(q-1)$ for all $e \in E$ and $v \in [p_e]$. This yields the invariants $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}}), \kappa(D_{\mathcal{G}})$ for $D_{\mathcal{G}} = D_q^w(G)$, the weighted q-distance matrix of a hypertree – both for general q and for the "classical" case $q = 1$:

Corollary 2.8. *Let G be a hypertree with strong blocks K_{p_e}, additive weights $w_e/(q-1)$ for $e \in E$, and all multiplicative weights $m_{e,v} = m'_{e,v} = \sqrt{q}$. Then,*

$$\kappa(D_{\mathcal{G}}) = \prod_{e \in E} (-w_e)^{p_e - 1}(1 + (p_e - 1)q),$$

$$\det(D_{\mathcal{G}} + xJ) = \kappa(D_{\mathcal{G}}) \left(x + \sum_{e \in E} \frac{(p_e - 1)(w_e + x(1-q))}{1 + (p_e - 1)q} \right).$$

Recall [12] that a hypertree is *d-regular* if every strong block is a clique on d nodes. The $d = 2$ case (i.e., trees) was worked out in the above generality in our prior work [7], while Sivasubramanian [12] computed $\det(D_{\mathcal{G}})$, in the special case $m_v = m'_v = \sqrt{q}$ for all blocks. Both results – and hence all prior variants for (hyper)trees – are subsumed by Corollary 2.8, so by Proposition 2.7.

2.3. **Application 3: Adding pendant hypertrees.**

Our final application in this section extends a prior result of Bapat–Kirkland–Neumann (for trees) and its subsequent generalizations (e.g. in [7] for additive-multiplicative trees) to arbitrary hypertrees:

Proposition 2.9. *Let $k, p'_1, \dots, p'_k \geqslant 1$ be integers. For each $j \in [k]$ let G_j be a weighted bi-directed graph with node set $[p'_j]$, no cut-vertices, and with additive-multiplicative distance matrix $D_{\mathcal{G}_j} = a_j(D^*_{G_j} - J_{p'_j})$ that satisfies:*

$$D_{\mathcal{G}_j} \mathbf{e}(p'_j) = d_j \mathbf{e}(p'_j), \qquad \forall j \in [k], \tag{2.22}$$

where $d_j \in R$ are scalars. Now let G' be any graph with strong blocks $G_1, \dots, G_k$, and let G be obtained from G' by further attaching finitely many cliques inductively – i.e. to a vertex of the graph constructed at each step, so that these cliques are also strong blocks. Let these cliques have block-data as in Proposition 2.7 (so they do not necessarily satisfy (2.22)).

Then $\det(D_{\mathcal{G}}), \operatorname{cof}(D_{\mathcal{G}})$ (and $\kappa(D_{\mathcal{G}})$) depend not on the structure and locations of the attached cliques G_e, $e \in E$ or the blocks G_j, $j \in [k]$ but only on their block-data, and as follows:

$$\det(D_{\mathcal{G}} + xJ) \tag{2.23}$$

$$= \prod_{j=1}^{k} \Big(\det(D_{\mathcal{G}_j})(a_j^{-1} + (p'_j/d_j))\Big) \prod_{e\in E} \left(a_e^{p_e-1}(1+\mathbf{e}(p_e)^T\mathbf{d}_e) \prod_{v\in[p_e]} (1-m_{e,v}m'_{e,v}) \right)$$
$$\times \left[x + \sum_{j=1}^{k} \frac{a_j - x}{1+\frac{a_j p'_j}{d_j}} + \sum_{e\in E} \frac{x-a_e}{1+\mathbf{e}(p_e)^T\mathbf{d}_e} \left(p_e - 1 + \sum_{v<w\in[p_e]} \frac{(m_{e,v}-m_{e,w})(m'_{e,v}-m'_{e,w})}{(1-m_{e,v}m'_{e,v})(1-m_{e,w}m'_{e,w})} \right) \right].$$

We make three concluding remarks: first, the proof of Proposition 2.9 involves straightforward computations using Theorem A and Proposition 2.7, hence is omitted. Second, the a_j^{-1} in the first product on the right is a placeholder (when working over an arbitrary commutative unital ring). Indeed, the entire factor in the first product is precisely $\kappa(D_{\mathcal{G}_j}) = a_j^{p_j-1}\det(D^*_{G_j})$.

Third, Proposition 2.9 subsumes all of the known results in the literature along these lines, including for trees (see the results and references in [7]) as well as Proposition 2.7 – but also covering the special case of cycle-clique graphs. These are graphs whose strong blocks are cliques or cycles, the latter of which fit the description of the G_j – in that they have $\mathbf{e}$ as an eigenvector for $D_{\mathcal{G}_j}$. Cycle-clique graphs include unicyclic, bicyclic, polycyclic (including cactus) graphs as well as (hyper)trees – see [7, Section 2] and the references therein for the numerous special cases studied in the literature. A final comment is that the literature studied only the special case of $D_q(G)$ (the q-distance matrix); but our result holds more generally for *all* additive-multiplicative $D_{\mathcal{G}}$.

3. Theorem B: The three invariants for minors of additive-multiplicative distance matrices

In recent work [7], in addition to computing $\det(D_{\mathcal{G}})$ and $\operatorname{cof}(D_{\mathcal{G}})$ in terms of the strong blocks of G, we also focused on trees, and computed these invariants for certain "admissible" minors of the additive-multiplicative distance matrix.

The goal in this section is to show that these formulas extend to similarly admissible minors for an arbitrary graph G – where we remove rows and columns corresponding to nodes such that what remains is a connected induced subgraph. Moreover, we also compute the third invariant above – namely, κ – at these minors of $D_{\mathcal{G}}$ and at arbitrary nodes v_0, and show the same dependence purely on the block-data. We begin with $\det(\cdot)$ and $\operatorname{cof}(\cdot)$, via the second main result of this paper:

Theorem B. *Suppose G has strong blocks $\{G_e : e \in E\}$, vertex set V, and block-datum $\mathcal{G} = \{\mathcal{G}_e = (a_e, D^*_{G_e}) : e \in E\}$. Let I, J' denote subsets of nodes of G satisfying: (a) $|I| = |J'| \leqslant |V| - 3$; (b) every vertex in $I \setminus J'$ is connected to $V \setminus I$ through a unique cut-vertex in $V \setminus (I \cup J')$ – and similarly upon exchanging I and J'; (c) the induced subgraphs on $V \setminus I$, $V \setminus J'$, $V \setminus (I \cap J')$ are strongly connected. Now let $E_\circ := E_{(I\cap J')^c}$ index the strong blocks G_e in the induced directed subgraph on the vertices $V \setminus (I \cap J')$.*

As an additional notation, given a $V \times V$ matrix D, let $D_{I|J'}$ the submatrix formed by removing the rows and columns labelled by I, J' respectively. Then $\det(D_{\mathcal{G}} + xJ)_{I|J'}$ depends on the block-data but not on the block-structure:

$$\det(D_{\mathcal{G}} + xJ)_{I|J'} = \begin{cases} \prod_{e\in E_\circ} \kappa(D_{\mathcal{G}_e}) \left[x + \sum_{e\in E_\circ} (a_e - x) \left(1 - \frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})}\right)\right], & \text{if } |I\Delta J'| = 0, \\ \displaystyle\prod_{e\in E_\circ\setminus\{\{p(i_0),i_0\},\{j_0,p(j_0)\}\}} \kappa(D_{\mathcal{G}_e}) \cdot (-1)^{|V(G)|-1} a_{\{p(i_0),i_0\}} & \\ \qquad \times (a_{\{j_0,p(j_0)\}} - x)(m_{(p(i_0),i_0)} - 1)(m_{(j_0,p(j_0))} - 1), & \text{if } |I\Delta J'| = 2, \\ 0, & \text{if } |I\Delta J'| > 2. \end{cases} \tag{3.1}$$

Here, the denominators (for $I = J'$) are simply placeholders to cancel with a factor in the numerators. We also assume that if $|I\Delta J'| = 2$, then the nodes i_0, j_0 are given by $I\setminus J' = \{i_0\}$, $J'\setminus I = \{j_0\}$.

In this case, i_0, j_0 are pendant in the induced subgraph on $V \setminus (I \cap J')$, and $p(i_0), p(j_0)$ are their unique respective neighbors in $V \setminus (I \cap J')$.

We also compute the invariant $\kappa((D_{\mathcal{G}})_{I|J'}, v_0)$, in Proposition 3.4. This is new even for trees.

Notice that in addition to the original case of $\det(D_{\mathcal{G}} + xJ)$ when $I = J' = \emptyset$, Theorem B subsumes [7, Theorem A]. The latter is because now G is a tree, so every $v \in I \setminus J'$ is indeed connected to $V \setminus I$ through a unique cut-vertex $v_0 \notin I$. If $v_0 \in J'$, consider the node $x \sim v_0$ that lies along the path from v_0 to v. Disconnecting the edge $x - v_0$ separates V into I and J', which contradicts the hypotheses of [7, Theorem A]. (Similarly if I and J' are exchanged.)

Remark 3.1. Our recent work [7, Theorem A] showed the special case for trees. The $|I \Delta J'| = 2$ case in it was proved using multiple ingredients: (i) the case of $|I \Delta J'| > 2$, (ii) the case of $|I \Delta J'| = 0$ (which is precisely (2.7)), (iii) Dodgson condensation (i.e. the Desnanot–Jacobi identity), and (iv) Zariski density. Our alternate proof below for $|I \Delta J'| = 2$ and all graphs, avoids using *all* of these ingredients, and works using just the definitions and the multiplicativity of κ. Thus it uses simpler ingredients (modulo the longwinded computations proving (2.2) in Theorem A); it is conceptually more transparent even for trees; and it applies more generally – in fact to all graphs.

Before proving the theorem, we mention another special case, where $x = a_e = 1\ \forall e$:

Corollary 3.2. *Let $I \neq J'$ be as in Theorem B. Then* $\det(D^*_G)_{I|J'} = 0$.

Proof of Theorem B. Note that we may replace G by the induced subgraph on the nodes in $E_\circ$, i.e. in $\bigcup_{e \in E_\circ} V(G_e)$. In other words, we may assume $E = E_\circ$, or equivalently, that I, J' are disjoint. Now if $I = J'$ then the result reduces to Theorem A; see also the discussion following its statement.

Henceforth assume $I \Delta J'$ is nonempty. The first claim is that there exist $w \in J' \setminus I$ and $v \in V \setminus (I \cup J')$ such that $v \sim w$. For if not, consider any such w, v for which $d(v, w)$ is minimal – here, $d(v, w)$ denotes the shortest path distance using unweighted, bidirected edges. Then $v \sim w$, for if not then all paths from v to w contain a node in I, contradicting the connectedness of $V \setminus I$.

There are two other observations here: first, if $d(v, w)$ is minimal then v is a cut-vertex by assumption (b). Thus v disconnects the graph; moreover, in the graph block corresponding to w, there cannot be a node in $V \setminus J'$ by assumption (b).

We next work out the (shorter) case of $|I \Delta J'| > 2$. There are two sub-cases:

(1) Suppose there are two such vertices $w_1 \neq w_2 \in J' \setminus I$ that are adjacent to (cut) vertices $v_1, v_2 \in V \setminus (I \cup J')$. Note here that v_1 may equal v_2. In this case, the w_1, w_2, v_1, v_2 rows (with the columns for J' removed) occur in $(D_{\mathcal{G}} + xJ)_{I|J'}$.

   Given a cut-vertex v and a node $w \neq v$, let $G_{v \to w}$ be the unique maximum induced subgraph of G that contains v, w and for which v is not a cut-vertex. Then (i) the nodes in $G_{v_l \to w_l} \setminus \{v_l\}$ for $l = 1, 2$ are necessarily contained in J' by the hypotheses, hence (ii) the corresponding columns are deleted from $(D_{\mathcal{G}} + xJ)_{I|J'}$.

   Now let the truncated row of $(D_{\mathcal{G}} + xJ)_{I|J'}$ (with columns indexed by $V \setminus J'$) corresponding to $v \in V$ be denoted by $\mathbf{d}_v^T = (d(v, w) + x)_{w \in V}$. Since $\mathbf{d}_{w_l}^T, \mathbf{d}_{v_l}^T$ indeed occur for $l = 1, 2$, the determinant of $(D_{\mathcal{G}} + xJ)_{I|J'}$ remains unchanged if one first performs the row operations

$$\mathbf{d}_{w_l}^T \mapsto \mathbf{d}_{w_l}^T - m_{w_l v_l} \mathbf{d}_{v_l}^T, \qquad l = 1, 2.$$

   But this gives a matrix with two proportional rows, since

$$\mathbf{d}_{w_l}^T - m_{w_l v_l} \mathbf{d}_{v_l}^T = (a_{e_l} - x)(m_{w_l v_l} - 1)\mathbf{e}^T, \qquad l = 1, 2,$$

   where e_l is the unique block of G containing the edge $\{w_l, v_l\}$. Hence $\det(D_{\mathcal{G}} + xJ)_{I|J'} = 0$.

(2) If the previous scenario does not occur, then there is a unique cut-vertex $v_0 \in V \setminus (I \cup J')$ and a unique node $w_0 \in J' \setminus I$ such that $w_0 \sim v_0$ and all other nodes in J' are connected to v_0 through w_0. Choose $w_1 \in J' \setminus (I \sqcup \{w_0\})$ that is closest in the (unweighted,

undirected) shortest path distance to w_0. If w_1 is not adjacent to w_0, then (i) all nodes on all (unweighted, undirected) paths between $w_0 \in J'$ and $w_1 \in J'$ lie in J', since $V \setminus J'$ is connected; and (ii) every path from w_1 to w_0 contains a node in I, and hence in $I \cap J'$ by (i). But then $w_0, w_1 \in J' \setminus I \subseteq V \setminus I$ are separated by I, contradicting the hypotheses.

It follows that $w_1 \longleftrightarrow w_0 \longleftrightarrow v_0$. Moreover, the columns corresponding to the nodes in $G_{v_0 \to w_0} \setminus \{v_0\}$ are removed in $(D_{\mathcal{G}} + xJ)_{I|J'}$, while the rows corresponding to w_1, w_0, v_0 are all present. Hence as in the previous case, the determinant of the minor $(D_{\mathcal{G}} + xJ)_{I|J'}$ remains unchanged upon carrying out – in order – the following row operations:

$$\mathbf{d}_{w_1}^T \mapsto \mathbf{d}_{w_1}^T - m_{w_1 w_0} \mathbf{d}_{w_0}^T, \qquad \mathbf{d}_{w_0}^T \mapsto \mathbf{d}_{w_0}^T - m_{w_0 v_0} \mathbf{d}_{v_0}^T.$$

As above, this gives a matrix with two proportional rows, since

$$\mathbf{d}_{w_1}^T - m_{w_1 w_0} \mathbf{d}_{w_0}^T = (a_{e_1} - x)(m_{w_1 w_0} - 1)\mathbf{e}^T, \qquad \mathbf{d}_{w_0}^T - m_{w_0 v_0} \mathbf{d}_{v_0}^T = (a_{e_0} - x)(m_{w_0 v_0} - 1)\mathbf{e}^T,$$

where e_0, e_1 are the blocks of G containing the edges $\{v_0, w_0\}$ and $\{w_0, w_1\}$ respectively. It again follows that $\det(D_{\mathcal{G}} + xJ)_{I|J'} = 0$.

The final case is when $|I \Delta J'| = 2$, so that we can assume

$$I = I \setminus J' = \{i_0\}, \qquad J' = J' \setminus I = \{j_0\}.$$

Let i_0, j_0 be connected to $V \setminus I, V \setminus J'$ through the cut-vertices $p(i_0), p(j_0)$ respectively. By the same reasoning as above,

$$\{i_0, p(i_0)\} \subseteq G_{p(i_0) \to i_0} \subseteq \{p(i_0)\} \sqcup (I \setminus J') = \{p(i_0)\} \sqcup I = \{i_0, p(i_0)\},$$

so $G_{p(i_0) \to i_0} = \{i_0, p(i_0)\}$. The analogous results also hold for j_0, so i_0, j_0 are pendant as claimed.

It remains to show (3.1). Without loss of generality, let

$$i_0 = 1, \quad j_0 = n, \quad V' := \{2, \dots, n-1\} = V(G) \setminus \{i_0, j_0\}.$$

Also note from above that the edges $\{i_0, p(i_0)\}, \{j_0, p(j_0)\}$ are strong blocks of G; we denote the corresponding additive block-data by a_1 and a_n respectively.

Using the definitions, it follows that

$$D_{\mathcal{G}} + xJ = \begin{pmatrix} x & \mathbf{w}^T + x\mathbf{e}^T & d_{1,n} + x \\ \mathbf{d}(V', p(1)) + a_1(m_{p(1),1} - 1)\mathbf{m}(V', p(1)) + x\mathbf{e} & D_{\mathcal{G}}|_{V' \times V'} + xJ & \mathbf{u} + x\mathbf{e} \\ d_{n,p(1)} + a_1(m_{p(1),1} - 1)m_{n,p(1)} + x & \mathbf{d}(n, V')^T + x\mathbf{e}^T & x \end{pmatrix}$$

for a suitable choice of vectors $\mathbf{u}, \mathbf{w} \in R^{n-2}$. First remove the first row and last column of this matrix. Next, the determinant of the truncated matrix is unchanged upon subtracting the $p(1)$th column from the first, so it suffices to compute the determinant of

$$D' := \begin{pmatrix} a_1(m_{p(1),1} - 1)\mathbf{m}(V', p(1)) & D_{\mathcal{G}}|_{V' \times V'} + xJ \\ a_1(m_{p(1),1} - 1)m_{n,p(n)} m_{p(n),p(1)} & \mathbf{d}(n, V')^T + x\mathbf{e}^T \end{pmatrix}.$$

To compute $\det(D')$, the factor $a_1(m_{p(1),1} - 1)$ can be taken out of the first column. Next, since

$$\mathbf{d}(n, V') = a_n(m_{n,p(n)} - 1)\mathbf{e} + m_{n,p(n)} \mathbf{d}(p(n), V'),$$

we subtract $m_{n,p(n)}$ times the $p(n)$th row of D' from its final row, to obtain:

$$\det(D_{\mathcal{G}} + xJ)_{1|n} = a_1(m_{p(1),1} - 1) \det \begin{pmatrix} \mathbf{m}(V', p(1)) & D_{\mathcal{G}}|_{V' \times V'} + xJ \\ 0 & (a_n - x)(m_{n,p(n)} - 1)\mathbf{e}^T \end{pmatrix}$$

We think of this 2×2 block matrix as relating to the graph G' with two vertices and two edges removed, i.e., with nodes V' and strong blocks precisely the strong blocks of G with two fewer edge-blocks. Now take the factor $(a_n - x)(m_{n,p(n)} - 1)$ out of the last row, and use the following identity (3.2), which we isolate into a standalone lemma of possible independent interest, to obtain:

$$\det(D_{\mathcal{G}} + xJ)_{1|n} = a_1(m_{p(1),1} - 1) \cdot (a_n - x)(m_{n,p(n)} - 1) \cdot \kappa(D_{\mathcal{G}}|_{V' \times V'}).$$

But G' has the same strong blocks as G (with two fewer edges), so by (2.3) we obtain the desired result (3.1) for $|I \Delta J'| = 2$. $\square$

The following identity was used in the closing arguments above.

Lemma 3.3. *Suppose G is a graph with strong blocks $\{G_e : e \in E\}$ and block-datum $\mathcal{G} = \{(a_e, D^*_{G_e}) \in R^{1+|V|^2} : e \in E\}$ as above. Fix $v \in V$, and say x commutes with R. Then,*

$$M(x) := \begin{pmatrix} \mathbf{m}(V, v) & D_{\mathcal{G}} + xJ \\ 0 & \mathbf{e}^T \end{pmatrix} \quad \Longrightarrow \quad \det(M(x)) = (-1)^{|V|-1} \kappa(D_{\mathcal{G}}). \tag{3.2}$$

Proof. Label the rows and columns of the matrix $M(x)$ by $(V; \infty)$ and $(\infty; V)$ respectively. First pre-multiply the final row by $x\mathbf{e}$ and subtract this from the upper 1×2 block-submatrix of $M(x)$. This shows that $\det(M(x)) = \det(M(0))$, and so we work with $x = 0$ henceforth.

Subtract the vth column of $M(0)$ from all columns indexed by nodes in $V \setminus \{v\}$. Note that since $D_{\mathcal{G}}$ has (v, v)-entry 0, this leaves unchanged the vth row of $M(0)$. Call the new matrix M', and subtract $m(w, v)$-times the vth row of M' from the wth row, for each $w \in V \setminus \{v\}$. Now the first column and the last row (both indexed by ∞) are standard basis vectors. Expanding along the first column and the last row, and using (2.1), we obtain $\det(M(x)) = (-1)^{|V|-1} \kappa(D_{\mathcal{G}})$, as claimed. $\square$

3.1. **Vanishing of κ for minors.** Having computed $\det(D_{\mathcal{G}})_{I|J'}$ and $\operatorname{cof}(D_{\mathcal{G}})_{I|J'}$ in Theorem B, we complete the set by proving:

Proposition 3.4. *Suppose $\mathcal{G}$ and $I, J' \subseteq V(G)$ are as in Theorem B. Let $v_0 \in V \setminus (I \cup J')$. Then:*

$$\kappa((D_{\mathcal{G}})_{I|J'}, v_0) = \begin{cases} \prod_{e \in E_\circ} \kappa(D_{\mathcal{G}_e}), & \text{if } I = J', \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

where $E_\circ = E_{(I \cap J')^c}$ is as in Theorem B. In particular, this too is independent of $v_0 \in V \setminus (I \cup J')$.

Notice from Definition 2.1 that κ is not yet defined for non-principal submatrices of $D_{\mathcal{G}}$. We begin by doing so; now the final part of Proposition 3.4 is in the spirit of Theorem A. (In fact, Theorem A mostly involved showing that $\kappa((D_{\mathcal{G}})_{I|J'}, v_0)$ is independent of v_0, for $I = J' = \emptyset$.)

Definition 3.5. Suppose G is a graph with additive-multiplicative block-datum $\mathcal{G}$, and $I, J' \subseteq V(G)$ be equal-sized subsets. Let $v_0 \in V \setminus (I \cup J')$, and write

$$(D_{\mathcal{G}})_{I|J'} = \begin{pmatrix} D_1 & \mathbf{u}_1 \\ \mathbf{w}_1^T & 0 \end{pmatrix},$$

where $D_1 = (D_{\mathcal{G}})_{(I \cup \{v_0\})^c \times (J' \cup \{v_0\})^c}$. Now define:

$$\kappa((D_{\mathcal{G}})_{I|J'}, v_0) := \det\left(D_1 - \mathbf{u}_1\, \mathbf{e}((J' \cup \{v_0\})^c)^T - \mathbf{m}((I \cup \{v_0\})^c, v_0) \mathbf{w}_1^T\right). \tag{3.4}$$

Notice this generalizes the special case of $I = J'$ (i.e., $\kappa(D_{\mathcal{G}}, v_0)$) in Definition 2.1; however, we did not mention this more general case earlier because it is only applied (in this paper) in proving Proposition 3.4. Indeed, the more restrictive Definition 2.1 is sufficient in order to show our three main results: Theorems A, B, and C.

Proof of Proposition 3.4. The $I = J'$ case follows by Theorem A. Now suppose $I \neq J'$; as in the proof of Theorem B, we may assume without loss of generality that I, J' are disjoint. The first step is to generalize Lemma 3.3:

Lemma 3.6. *With notation as above, and for all x commuting with R,*

$$M(x) := \begin{pmatrix} \mathbf{m}(V \setminus I, v_0) & (D_{\mathcal{G}})_{I|J'} + xJ \\ 0 & \mathbf{e}(V \setminus J')^T \end{pmatrix} \quad \Longrightarrow \quad \det(M(x)) = (-1)^{|V \setminus I|-1} \kappa((D_{\mathcal{G}})_{I|J'}, v_0). \tag{3.5}$$

The proof is the same as that of Lemma 3.3, and is hence omitted.

We now show the result for $I \neq J'$. By (3.5), it suffices to show that $M(0)$ is singular. As in the proof of Theorem B, choose $w \in J' \setminus I$ and a cut-vertex $v \in V \setminus (I \cup J')$ which is adjacent to w. Now the columns corresponding to the nodes of $G_{v \to w} \setminus \{v\}$ are removed from $(D_{\mathcal{G}})_{I|J'}$. Label every row of $M(0)$ – except the last row – by the corresponding node in $V \setminus I$, and subtract $m_{w,v}$ times the vth row from the wth row. Then the new wth row is $d(w,v) \cdot (0, \mathbf{e}(V \setminus J')^T)$. As this is proportional to the final row, the proof is complete. □

4. Theorem C: The inverse matrix for additive-multiplicative graphs

In this section, we provide a closed-form expression for the inverse of the distance matrix $D_{\mathcal{G}}$ for an arbitrary graph G, in terms of the additive-multiplicative block-datum $\mathcal{G}$ and the geometry of G. In particular, our result subsumes [7, Theorem B] for "additive-multiplicative trees", which in turn implies all known formulas for trees – including by Graham–Lovász, Bapat and his coauthors, and Zhou–Ding (see the discussion and references around *loc. cit.*). More precisely, that result for trees involved explicit formulas, which we now explain more conceptually (for general graphs).

A striking feature of our formula is that the matrix $D_{\mathcal{G}}^{-1}$ is a rank-one update not of the graph Laplacian $L_{\mathcal{G}}$ (as in previous works [3, 9]) but of a different matrix $C_{\mathcal{G}}(D_G^*)^{-1}$, where $C_{\mathcal{G}}$ was defined for trees in [7]. That said, our formula specializes to all previous versions in the literature.

Definition 4.1. Given the block-datum $\mathcal{G} = \{\mathcal{G}_e = (a_e, D_{G_e}^*) : e \in E\}$ for a graph G, with strong blocks G_e and multiplicative distance matrix D_G^* that is invertible, define

$$\boldsymbol{\tau}_{\text{in}}^T := \mathbf{e}^T (D_G^*)^{-1}, \qquad \boldsymbol{\tau}_{\text{out}} := (D_G^*)^{-1}\mathbf{e}. \tag{4.1}$$

Given a node i lying in a strong block $e \in E$, define $G_{i \to e}$ to be the unique maximum induced subgraph of G that contains e and for which i is not a cut-vertex. Now define for $i \in V(G)$:

$$\beta_i := \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \sum_{e \in E: i \in e} \frac{1}{a_e} \sum_{f \in E: f \subseteq G_{i \to e}} \frac{\det(D_{\mathcal{G}_f})}{\kappa(D_{\mathcal{G}_f})}, \tag{4.2}$$

and the matrix $C_{\mathcal{G}} \in R^{V(G) \times V(G)}$ via:

$$(C_{\mathcal{G}})_{ij} := \begin{cases} \beta_i, & \text{if } j = i, \\ \beta_i - \frac{1}{a_e}, & \text{if } j \neq i, \ j \in G_{i \to e}. \end{cases} \tag{4.3}$$

Remark 4.2. For every node $i \in V(G)$, the subgraphs $G_{i \to e}$ partition the edges and the nodes:

$$E(G) = \bigsqcup_{e \in E: i \in e} E(G_{i \to e}), \qquad V(G) = \{i\} \sqcup \bigsqcup_{e \in E: i \in e} (V(G_{i \to e}) \setminus \{i\}).$$

Also note that if i is not a cut-vertex, then $G_{i \to e} = G$, and hence $\beta_i = \frac{1}{a_e}$ where $e \in E$ is the unique strong block containing i. In particular, $C_{\mathcal{G}}$ has ith row $\frac{1}{a_e}\mathbf{e}_i^T$.

When G is a tree with edge-datum $\mathcal{T}$, Definition 4.1 specializes to the corresponding one in [7] – see (4.24) below. That said, the following formula for $D_{\mathcal{G}}^{-1}$ is simpler than that for $D_{\mathcal{T}}^{-1}$ in [7]:

Theorem C. *With the above notation – in particular assuming that $D_{\mathcal{G}}, D_G^*$ are invertible,*

$$D_{\mathcal{G}}^{-1} = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \boldsymbol{\tau}_{\text{out}} \boldsymbol{\tau}_{\text{in}}^T + C_{\mathcal{G}}(D_G^*)^{-1}. \tag{4.4}$$

Theorem C applies to all additive-multiplicative matrices, of all graphs. As we show in Section 4.1, the formula (4.4) indeed specializes to the formula (4.24) for $D_{\mathcal{T}}^{-1}$ for trees, shown in [7, Theorem B]. This is in spite of the absence of a Laplacian matrix as in the literature for trees [3, 9]. On a related note, in Section 4.1 we define such a Laplacian $L_{\mathcal{G}}$ for general graphs, and it is closely related to the matrix $C_{\mathcal{G}}(D_G^*)^{-1}$, which by Theorem C is the "correct" matrix to use for $D_{\mathcal{G}}^{-1}$.

Remark 4.3. Akin to the invariants $\kappa(D_{\mathcal{G}}), \det(D_{\mathcal{G}})$, $\operatorname{cof}(D_{\mathcal{G}})$, the inverse $D_{\mathcal{G}}^{-1}$ can also be recovered from the block-datum $\mathcal{G} = \{(a_e, D^*_{G_e}) : e \in E\}$. This is because by [7, Theorem C],

$$(D^*_G)^{-1} = \sum_{e\in E}[(D^*_{G_e})^{-1}]_{V(G_e)} + \mathrm{Id}_{V(G)} - \sum_{e\in E}[\mathrm{Id}_{G_e}]_{V(G_e)}, \tag{4.5}$$

where given $S \subseteq V(G)$ and a matrix $A \in R^{|S|\times|S|}$, we define $[A]_S$ to be the $|V(G)| \times |V(G)|$ matrix with principal submatrix A over the rows and columns indexed by S, and all other entries zero.

Remark 4.4. In the sequel, we will assume that $D_{\mathcal{G}}$ and $\kappa(D_{\mathcal{G}})$ are invertible, and hence so are all a_e (if $|V(G_e)| > 1$) and $D^*_{G_e}$ – by Zariski density. For more on this, we refer the reader to Remark 2.3 and to similar applications of Zariski density in the special case in [7].

Remark 4.5. Suppose $a_e \equiv a\ \forall e \in E$, for some fixed scalar a. By Remark 1.2(2) or [7], in this case $D_{\mathcal{G}} = a(D^*_G - J)$. Moreover, $C_{\mathcal{G}} = \frac{1}{a}\mathrm{Id}_{V(G)}$ by Theorem A. This yields:

$$\begin{aligned}\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\boldsymbol{\tau}_{\text{out}}\boldsymbol{\tau}_{\text{in}}^T + C_{\mathcal{G}}(D^*_G)^{-1} &= a^{-1}\left(\frac{\det(D^*_G) - \operatorname{cof}(D^*_G)}{\det(D^*_G)}\right)^{-1} \cdot (D^*_G)^{-1}\mathbf{e}\cdot\mathbf{e}^T(D^*_G)^{-1} + a^{-1}(D^*_G)^{-1}\\ &= a^{-1}\left[(D^*_G)^{-1} + \frac{(D^*_G)^{-1}\mathbf{e}\cdot\mathbf{e}^T(D^*_G)^{-1}}{1-\mathbf{e}^T(D^*_G)^{-1}\mathbf{e}}\right],\end{aligned}$$

which by the Sherman–Morrison formula equals $a^{-1}(D^*_G - \mathbf{e}\mathbf{e}^T)^{-1} = D_{\mathcal{G}}^{-1}$. This applies e.g. to all q-distance matrices $D_q(G)$ where $a_e = 1/(q-1)$, and to blocks (graphs with no cut-vertices).

We now turn to the proof of Theorem C, and begin by isolating some preliminary identities that will be used below.

Proposition 4.6. *Notation as above. Then:*

$$\boldsymbol{\tau}_{\text{in}}^T\mathbf{e} = \mathbf{e}^T\boldsymbol{\tau}_{\text{out}} = \frac{\operatorname{cof}(D^*_G)}{\det(D^*_G)} = \frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}; \tag{4.6}$$

$$\mathbf{e}^T C_{\mathcal{G}} = \frac{\kappa(D_{\mathcal{G}}) - \operatorname{cof}(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\cdot\mathbf{e}^T. \tag{4.7}$$

Proof. The identity (4.6) is immediate from the definitions and Theorem A. We now show the equality in (4.7) of the jth coordinate on both sides, for each fixed $j \in V(G)$. The left-side equals

$$\sum_{i\in V(G)}\beta_i - \sum_{i\in V(G)}\frac{1}{a_{e_{i\to j}}},$$

where $e_{i\to j}$ denotes the unique strong block $e \in E$ containing i such that $j \in G_{i\to e}$. We now convert the second sum, which is over vertices, to one over blocks. It is not hard to see that every block e contains a unique node i_0 such that $e \neq e_{i_0\to j}$; hence the latter sum equals

$$\sum_{e\in E}\frac{|V(G_e)|-1}{a_e}.$$

Next, we consider the former sum, again converting it into a sum over blocks:

$$\sum_{i\in V(G)}\beta_i = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\sum_{e\in E}\frac{1}{a_e}\sum_{i\in V(G_e)}\ \sum_{f\in E: f\subseteq G_{i\to e}}\frac{\det(D_{\mathcal{G}_f})}{\kappa(D_{\mathcal{G}_f})}$$

For each fixed block $e \in E$, the inner double sum on the right is an integer-linear combination of the ratios $\det(D_{\mathcal{G}_f})/\kappa(D_{\mathcal{G}_f})$ over $f \in E$. Count the coefficient of this term, say $n_{e,f} \in \mathbb{Z}$ for fixed e and arbitrary $f \in E$. If $f = e$, then the ratio $\det(D_{\mathcal{G}_e})/\kappa(D_{\mathcal{G}_e})$ occurs for every summand $i \in V(G_e)$ in the inner double sum above, so $n_{e,e} = |V(G_e)|$. If instead $f \neq e$, then the ratio $\det(D_{\mathcal{G}_f})/\kappa(D_{\mathcal{G}_f})$

occurs for every summand $i \in V(G_e)$, except for the unique cut-vertex $i_0 \in V(G_e)$ separating f from $G_{i_0 \to e}$ (which includes e). Hence, $n_{e,f} = |V(G_e)| - 1$ if $f \neq e$. Therefore:

$$\begin{aligned}\sum_{i \in V(G)} \beta_i &= \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \sum_{e \in E} \frac{1}{a_e} \left(\frac{\det(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})} + (|V(G_e)| - 1) \sum_{f \in E} \frac{\det(D_{\mathcal{G}_f})}{\kappa(D_{\mathcal{G}_f})} \right) \\ &= \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \sum_{e \in E} \frac{\det(D_{\mathcal{G}_e})}{a_e \kappa(D_{\mathcal{G}_e})} + \sum_{e \in E} \frac{|V(G_e)| - 1}{a_e},\end{aligned}$$

where the final equality follows from (2.4).

Putting together these computations, for each $j \in V(G)$ we have by (2.8) and Theorem A:

$$(\mathbf{e}^T C_{\mathcal{G}})_j = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \sum_{e \in E} \frac{\det(D_{\mathcal{G}_e})}{a_e \kappa(D_{\mathcal{G}_e})} = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \sum_{e \in E} \left(1 - \frac{\operatorname{cof}(D_{\mathcal{G}_e})}{\kappa(D_{\mathcal{G}_e})}\right) = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \left(1 - \frac{\operatorname{cof}(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}\right).$$

Since this holds for every vertex $j \in V(G)$, the proof of (4.7) is complete. □

With Proposition 4.6 in hand, we show our final main result.

Proof of Theorem C. The proof is by induction on the number of strong blocks $|E|$ of G, with the case $|E| = 1$ worked out in Remark 4.5. For the induction step, assume that we know the result for G; now add to G the pendant block f, separated from G by the cut-vertex v_0. We first set notation. Let $\overline{G}$ be the new graph, and write $\overline{D}_{\overline{\mathcal{G}}}$ for the additive-multiplicative matrix for $\overline{G}$. Thus:

$$\begin{aligned}\overline{D}_{\overline{\mathcal{G}}} &:= \begin{pmatrix} D_{\mathcal{G}} & H \\ K & a_f(D_2^* - J) \end{pmatrix}, \\ \text{where} \quad D_f^* &= \begin{pmatrix} 1 & \mathbf{w}^T \\ \mathbf{u} & D_2^* \end{pmatrix}, \qquad H := D_{\mathcal{G}} \mathbf{e}_{v_0} \mathbf{e}^T + a_f D_G^* \mathbf{e}_{v_0} (\mathbf{w} - \mathbf{e})^T, \\ K &:= a_f(\mathbf{u} - \mathbf{e})\mathbf{e}^T + \mathbf{u}\mathbf{e}_{v_0}^T D_{\mathcal{G}}.\end{aligned} \tag{4.8}$$

In the sequel, we also use the formula for the inverse of a 2×2 square block matrix:

$$M = \begin{pmatrix} D_1 & H \\ K & D_2 \end{pmatrix} \quad \Longrightarrow \quad M^{-1} = \begin{pmatrix} D_1^{-1} + D_1^{-1} H \Psi^{-1} K D_1^{-1} & -D_1^{-1} H \Psi^{-1} \\ -\Psi^{-1} K D_1^{-1} & \Psi^{-1} \end{pmatrix}, \tag{4.9}$$

where the $(1,1)$-block is assumed to be invertible, and Ψ denotes the Schur complement

$$\Psi = D_2 - K D_1^{-1} H.$$

The following special case is of interest:

$$(D_f^*)^{-1} = \begin{pmatrix} 1 + \mathbf{w}^T X^{-1} \mathbf{u} & -\mathbf{w}^T X^{-1} \\ -X^{-1} \mathbf{u} & X^{-1} \end{pmatrix}, \qquad \text{where } X := D_2^* - \mathbf{u}\mathbf{w}^T. \tag{4.10}$$

From this and (4.5), one has the following formula for $(D_{\overline{G}}^*)^{-1}$:

$$\begin{aligned}(D_{\overline{G}}^*)^{-1} &= [(D_G^*)^{-1}]_{V(G)} + [(D_f^*)^{-1}]_{V_f} - \mathbf{e}_{v_0} \mathbf{e}_{v_0}^T \\ &= \begin{pmatrix} (D_G^*)^{-1} + (\mathbf{w}^T X^{-1} \mathbf{u}) \mathbf{e}_{v_0} \mathbf{e}_{v_0}^T & \mathbf{e}_{v_0}(-\mathbf{w}^T X^{-1}) \\ (-X^{-1}\mathbf{u})\mathbf{e}_{v_0}^T & X^{-1} \end{pmatrix}.\end{aligned} \tag{4.11}$$

Step 1: We break up the proof into steps for ease of exposition. Let D_f denote the additive-multiplicative distance matrix of f:

$$D_f := a_f(D_f^* - J) = a_f \begin{pmatrix} 0 & (\mathbf{w} - \mathbf{e})^T \\ \mathbf{u} - \mathbf{e} & D_2^* - J \end{pmatrix}. \tag{4.12}$$

Then we claim:

$$-(\mathbf{w} - \mathbf{e})^T X^{-1} (\mathbf{u} - \mathbf{e}) = \frac{\det(D_f)}{a_f \kappa(D_f)}. \tag{4.13}$$

Indeed, we compute using a row and a column operation and the above formulas in this proof:

$$\begin{aligned}\det(D_f) = &\det a_f \begin{pmatrix} 0 & (\mathbf{w}-\mathbf{e})^T \\ \mathbf{u}-\mathbf{e} & X \end{pmatrix} = a_f^{|V_f|} \det(X) \left(-(\mathbf{w}-\mathbf{e})^T X^{-1} (\mathbf{u}-\mathbf{e})\right) \\ &= a_f^{|V_f|} \det(D_f^*) \left(-(\mathbf{w}-\mathbf{e})^T X^{-1} (\mathbf{u}-\mathbf{e})\right),\end{aligned}$$

from which the claim follows. This identity will be used repeatedly in our computations below.

Step 2: We now explain our strategy. The left-hand side of (4.4), namely $(\overline{D}_{\overline{\mathcal{G}}})^{-1}$ where $\overline{D}_{\overline{\mathcal{G}}}$ is given by (4.8), can be computed using the formula (4.9). On the other hand, the right-hand side of (4.4) can be explicitly written out in 2×2 block form as well. We will carry out both of these steps and show the equality of the two sides of (4.4), block by block.

We begin by writing out analogues for $\overline{D}_{\overline{\mathcal{G}}}$ of the vectors $\boldsymbol{\tau}_{\rm in}, \boldsymbol{\tau}_{\rm out}$ for $D_{\mathcal{G}}$; we denote these analogues by $\overline{\boldsymbol{\tau}_{\rm in}}, \overline{\boldsymbol{\tau}_{\rm out}}$ respectively:

$$\overline{\boldsymbol{\tau}_{\rm in}} = \begin{pmatrix} \boldsymbol{\tau}_{\rm in}^1 \\ \boldsymbol{\tau}_{\rm in}^2 \end{pmatrix}, \qquad \overline{\boldsymbol{\tau}_{\rm out}} = \begin{pmatrix} \boldsymbol{\tau}_{\rm out}^1 \\ \boldsymbol{\tau}_{\rm out}^2 \end{pmatrix}, \tag{4.14}$$

where we have by (4.11):

$$\begin{aligned} (\boldsymbol{\tau}_{\rm in}^1)^T &= \boldsymbol{\tau}_{\rm in}^T + (\mathbf{w}-\mathbf{e})^T X^{-1} \mathbf{u} \cdot \mathbf{e}_{v_0}^T, \qquad & (\boldsymbol{\tau}_{\rm in}^2)^T &= -(\mathbf{w}-\mathbf{e})^T X^{-1}, \\ \boldsymbol{\tau}_{\rm out}^1 &= \boldsymbol{\tau}_{\rm out} + \mathbf{w}^T X^{-1} (\mathbf{u}-\mathbf{e}) \cdot \mathbf{e}_{v_0}, \qquad & \boldsymbol{\tau}_{\rm out}^2 &= -X^{-1}(\mathbf{u}-\mathbf{e}). \end{aligned} \tag{4.15}$$

Also note that $C_{\overline{\mathcal{G}}}$ is block upper-triangular, with $(2,1)$-block zero, and $(2,2)$-block $a_f^{-1}\,\mathrm{Id}$, since $V_f \setminus \{v_0\}$ contains no cut-vertices. Writing $C_{\overline{\mathcal{G}}} = \begin{pmatrix} \overline{C}_{11} & \overline{C}_{12} \\ 0 & a_f^{-1}\,\mathrm{Id} \end{pmatrix}$, the theorem reduces by (4.9) to showing:

$$\begin{aligned} &\begin{pmatrix} D_{\mathcal{G}}^{-1} + D_{\mathcal{G}}^{-1} H \Psi^{-1} K D_{\mathcal{G}}^{-1} & -D_{\mathcal{G}}^{-1} H \Psi^{-1} \\ -\Psi^{-1} K D_{\mathcal{G}}^{-1} & \Psi^{-1} \end{pmatrix} \\ = &\ \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})} \begin{pmatrix} \boldsymbol{\tau}_{\rm out}^1 (\boldsymbol{\tau}_{\rm in}^1)^T & \boldsymbol{\tau}_{\rm out}^1 (\boldsymbol{\tau}_{\rm in}^2)^T \\ \boldsymbol{\tau}_{\rm out}^2 (\boldsymbol{\tau}_{\rm in}^1)^T & \boldsymbol{\tau}_{\rm out}^2 (\boldsymbol{\tau}_{\rm in}^2)^T \end{pmatrix} \\ + &\begin{pmatrix} \overline{C}_{11} & \overline{C}_{12} \\ 0 & a_f^{-1}\,\mathrm{Id} \end{pmatrix} \begin{pmatrix} (D_{\mathcal{G}}^*)^{-1} + (\mathbf{w}^T X^{-1} \mathbf{u}) \mathbf{e}_{v_0} \mathbf{e}_{v_0}^T & \mathbf{e}_{v_0} (-\mathbf{w}^T X^{-1}) \\ (-X^{-1}\mathbf{u}) \mathbf{e}_{v_0}^T & X^{-1} \end{pmatrix}. \end{aligned} \tag{4.16}$$

Finally, we state a useful consequence of Proposition 4.6 and (4.4) (via the induction hypothesis):

$$\mathbf{e}^T D_{\mathcal{G}}^{-1} = \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} \boldsymbol{\tau}_{\rm in}^T. \tag{4.17}$$

Step 3: We now compute Ψ using (4.17), and equate Ψ^{-1} to the $(2,2)$-block on the right in (4.16):

$$\begin{aligned} \Psi &= a_f (D_2^* - J) - K D_{\mathcal{G}}^{-1} H \\ &= a_f (D_2^* - J) - \left(a_f (\mathbf{u}-\mathbf{e}) \mathbf{e}^T + \mathbf{u} \mathbf{e}_{v_0}^T D_{\mathcal{G}}\right) D_{\mathcal{G}}^{-1} \left(D_{\mathcal{G}} \mathbf{e}_{v_0} \mathbf{e}^T + a_f D_{\mathcal{G}}^* \mathbf{e}_{v_0} (\mathbf{w}-\mathbf{e})^T\right) \\ &= a_f (D_2^* - J) - \left(a_f (\mathbf{u}-\mathbf{e}) \mathbf{e}^T + a_f \mathbf{u} (\mathbf{w}-\mathbf{e})^T + a_f^2 \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} (\mathbf{u}-\mathbf{e})(\mathbf{w}-\mathbf{e})^T\right) \\ &= a_f \left(X - a_f \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} (\mathbf{u}-\mathbf{e})(\mathbf{w}-\mathbf{e})^T\right). \end{aligned}$$

By the Sherman–Morrison formula and (4.13),

$$\begin{aligned}\Psi^{-1} &= a_f^{-1}\left[X^{-1} + \frac{a_f \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})} X^{-1}(\mathbf{u}-\mathbf{e})(\mathbf{w}-\mathbf{e})^T X^{-1}}{1 - a_f \frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}(\mathbf{w}-\mathbf{e})^T X^{-1}(\mathbf{u}-\mathbf{e})}\right]\\
&= a_f^{-1}X^{-1} + a_f^{-1}\frac{X^{-1}(\mathbf{u}-\mathbf{e})(\mathbf{w}-\mathbf{e})^T X^{-1}}{a_f^{-1}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} + a_f^{-1}\frac{\det(D_f)}{\kappa(D_f)}}\\
&= a_f^{-1}X^{-1} + \frac{\boldsymbol{\tau}^2_{\text{out}}(\boldsymbol{\tau}^2_{\text{in}})^T}{\det(\overline{D}_{\overline{\mathcal{G}}})/\kappa(\overline{D}_{\overline{\mathcal{G}}})},\end{aligned} \tag{4.18}$$

where the final equality uses (4.15) and the Master GHH-formula (2.4). Note, this is the computation of the $(2,2)$-block in the left-hand side of (4.16). But it also clearly equals the $(2,2)$-block in that right-hand side, as desired.

Step 4: We next check the equality of the $(2,1)$-blocks in (4.16). Using (4.9), (4.13), and (4.17):

$$\begin{aligned}& -\Psi^{-1}KD_{\mathcal{G}}^{-1}\\
=& -\Psi^{-1}\left(a_f\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}(\mathbf{u}-\mathbf{e})\boldsymbol{\tau}^T_{\text{in}} + \mathbf{u}\mathbf{e}^T_{v_0}\right)\\
=& \frac{-\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\left[-\boldsymbol{\tau}^2_{\text{out}} + a_f\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}^2_{\text{out}}\cdot\frac{\det(D_f)}{a_f\kappa(D_f)}\right]\boldsymbol{\tau}^T_{\text{in}} + \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\cdot(\mathbf{w}-\mathbf{e})^T X^{-1}\mathbf{u}\cdot\boldsymbol{\tau}^2_{\text{out}}\mathbf{e}^T_{v_0}\\
& - a_f^{-1}X^{-1}\mathbf{u}\mathbf{e}^T_{v_0}\\
=& \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\left[\boldsymbol{\tau}^2_{\text{out}}\boldsymbol{\tau}^T_{\text{in}} + (\mathbf{w}-\mathbf{e})^T X^{-1}\mathbf{u}\cdot\boldsymbol{\tau}^2_{\text{out}}\mathbf{e}^T_{v_0}\right] - a_f^{-1}X^{-1}\mathbf{u}\mathbf{e}^T_{v_0}\\
=& \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}^2_{\text{out}}(\boldsymbol{\tau}^1_{\text{in}})^T - a_f^{-1}X^{-1}\mathbf{u}\mathbf{e}^T_{v_0},\end{aligned}$$

where the penultimate equality uses the Master GHH-formula (2.4), and the final equality uses (4.15). But this is, once again, easily seen to equal the $(2,1)$-block of the right-hand side of (4.16).

Step 5: We now examine the $(1,2)$-blocks in (4.16). Using the induction hypothesis for (4.4), the left-hand side yields:

$$-D_{\mathcal{G}}^{-1}H\Psi^{-1} = -\left[\mathbf{e}_{v_0}\mathbf{e}^T + a_f\left(\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\boldsymbol{\tau}_{\text{out}} + C_{\mathcal{G}}\mathbf{e}_{v_0}\right)(\mathbf{w}-\mathbf{e})^T\right]\Psi^{-1}.$$

By (4.13), the first term of this product equals

$$-a_f^{-1}\mathbf{e}_{v_0}\mathbf{e}^T X^{-1} + \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\left[\mathbf{w}^T X^{-1}(\mathbf{u}-\mathbf{e}) + \frac{\det(D_f)}{a_f\kappa(D_f)}\right]\mathbf{e}_{v_0}(\boldsymbol{\tau}^2_{\text{in}})^T.$$

Moreover, a parallel computation to the previous step yields:

$$a_f(\mathbf{w}-\mathbf{e})^T\Psi^{-1} = -\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}(\boldsymbol{\tau}^2_{\text{in}})^T.$$

Hence the $(1,2)$-block $-D_{\mathcal{G}}^{-1}H\Psi^{-1}$ on the left of (4.16) equals:

$$= -a_f^{-1}\mathbf{e}_{v_0}\mathbf{e}^T X^{-1} + \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\cdot\mathbf{w}^T X^{-1}(\mathbf{u}-\mathbf{e})\cdot\mathbf{e}_{v_0}(\boldsymbol{\tau}^2_{\text{in}})^T + \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\cdot\frac{\det(D_f)}{a_f\kappa(D_f)}\cdot\mathbf{e}_{v_0}(\boldsymbol{\tau}^2_{\text{in}})^T$$

$$+\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}_{\text{out}}(\boldsymbol{\tau}_{\text{in}}^2)^T+\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}C_{\mathcal{G}}\mathbf{e}_{v_0}(\boldsymbol{\tau}_{\text{in}}^2)^T. \tag{4.19}$$

The first and third terms on the right-hand side of (4.19) add up to give

$$-a_f^{-1}\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}\mathbf{e}_{v_0}\mathbf{e}^TX^{-1}-a_f^{-1}\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{\kappa(D_f)}\mathbf{e}_{v_0}\mathbf{w}^TX^{-1}, \tag{4.20}$$

while combining the second and fourth terms of (4.19) yields $\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}_{\text{out}}^1(\boldsymbol{\tau}_{\text{in}}^2)^T$ via (4.15). Now split the final term in (4.19) using

$$(\boldsymbol{\tau}_{\text{in}}^2)^T=\mathbf{e}^TX^{-1}-\mathbf{w}^TX^{-1}$$

and pair these two terms with the two terms in (4.20) to obtain:

$$\begin{aligned}-D_{\mathcal{G}}^{-1}H\Psi^{-1}=&\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}_{\text{out}}^1(\boldsymbol{\tau}_{\text{in}}^2)^T+\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}\left(-a_f^{-1}\mathbf{e}_{v_0}+C_{\mathcal{G}}\mathbf{e}_{v_0}\right)\mathbf{e}^TX^{-1}\\&+\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\left(a_f^{-1}\frac{\det(D_f)}{\kappa(D_f)}\mathbf{e}_{v_0}+\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}C_{\mathcal{G}}\mathbf{e}_{v_0}\right)(-\mathbf{w}^TX^{-1}).\end{aligned}$$

This is the $(1,2)$-block on the left-hand side of (4.16). The first term exactly matches the first term on the right-hand side of (4.16). Moreover, a careful computation reveals that

$$\begin{aligned}\overline{C}_{11}\mathbf{e}_{v_0}&=\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\left(a_f^{-1}\frac{\det(D_f)}{\kappa(D_f)}\mathbf{e}_{v_0}+\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}C_{\mathcal{G}}\mathbf{e}_{v_0}\right),\\\overline{C}_{12}&=\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}\left(-a_f^{-1}\mathbf{e}_{v_0}+C_{\mathcal{G}}\mathbf{e}_{v_0}\right)\mathbf{e}^T.\end{aligned} \tag{4.21}$$

From this and the above computations, the $(1,2)$-blocks in (4.16) agree.

Step 6: Finally, we reconcile the $(1,1)$-blocks in (4.16). On the left-hand side is $D_{\mathcal{G}}^{-1}+D_{\mathcal{G}}^{-1}H\cdot\Psi^{-1}KD_{\mathcal{G}}^{-1}$. By the induction hypothesis, Step 4, and the first line in Step 5, we have

$$\begin{aligned}&D_{\mathcal{G}}^{-1}+D_{\mathcal{G}}^{-1}H\cdot\Psi^{-1}KD_{\mathcal{G}}^{-1}\\&=\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\boldsymbol{\tau}_{\text{out}}\boldsymbol{\tau}_{\text{in}}^T+C_{\mathcal{G}}(D_G^*)^{-1}+\left[a_f\left(\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\boldsymbol{\tau}_{\text{out}}+C_{\mathcal{G}}\mathbf{e}_{v_0}\right)(\mathbf{w}-\mathbf{e})^T+\mathbf{e}_{v_0}\mathbf{e}^T\right]\times\\&\qquad\times\left(\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\left[X^{-1}(\mathbf{u}-\mathbf{e})\boldsymbol{\tau}_{\text{in}}^T+(\mathbf{w}-\mathbf{e})^TX^{-1}\mathbf{u}\cdot X^{-1}(\mathbf{u}-\mathbf{e})\mathbf{e}_{v_0}^T\right]+a_f^{-1}X^{-1}\mathbf{u}\mathbf{e}_{v_0}^T\right)\\&=\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\boldsymbol{\tau}_{\text{out}}\boldsymbol{\tau}_{\text{in}}^T+C_{\mathcal{G}}(D_G^*)^{-1}+S_1+\cdots+S_7,\end{aligned}$$

where

$$\begin{aligned}
S_1 &:= -\frac{\kappa(D_{\mathcal{G}})}{\det(D_{\mathcal{G}})}\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{\kappa(D_f)}\boldsymbol{\tau}_{\rm out}\boldsymbol{\tau}_{\rm in}^T,\\
S_2 &:= \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}(\mathbf{w}-\mathbf{e})^T X^{-1}\mathbf{u}\cdot\boldsymbol{\tau}_{\rm out}\mathbf{e}_{v_0}^T,\\
S_3 &:= -\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{\kappa(D_f)}C_{\mathcal{G}}\mathbf{e}_{v_0}\boldsymbol{\tau}_{\rm in}^T,\\
S_4 &:= \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})}(\mathbf{w}^T X^{-1}\mathbf{u}-\mathbf{e}^T X^{-1}\mathbf{u})\cdot C_{\mathcal{G}}\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T,\\
S_5 &:= \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\mathbf{e}^T X^{-1}(\mathbf{u}-\mathbf{e})\cdot\mathbf{e}_{v_0}\boldsymbol{\tau}_{\rm in}^T,\\
S_6 &:= \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\mathbf{e}^T X^{-1}(\mathbf{u}-\mathbf{e})\cdot(\mathbf{w}-\mathbf{e})^T X^{-1}\mathbf{u}\cdot\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T,\\
S_7 &:= a_f^{-1}\mathbf{e}^T X^{-1}\mathbf{u}\cdot\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T.
\end{aligned} \tag{4.22}$$

We now start to combine these nine terms. The first term added to S_1 and S_2 yields via (4.15):

$$\begin{aligned}
S_8 := {} & \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}_{\rm out}^1(\boldsymbol{\tau}_{\rm in}^1)^T - \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\mathbf{w}^T X^{-1}(\mathbf{u}-\mathbf{e})\cdot\mathbf{e}_{v_0}\boldsymbol{\tau}_{\rm in}^T\\
& - \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\mathbf{w}^T X^{-1}(\mathbf{u}-\mathbf{e})\cdot(\mathbf{w}-\mathbf{e})^T X^{-1}\mathbf{u}\cdot\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T.
\end{aligned}$$

The second term above, combined with S_3, S_5, and the second sub-term in S_8, yields:

$$S_9 := \left(C_{\mathcal{G}} - \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{\kappa(D_f)}C_{\mathcal{G}}\mathbf{e}_{v_0}\mathbf{e}^T + a_f^{-1}\frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{\kappa(D_f)}\mathbf{e}_{v_0}\mathbf{e}^T\right)(D_G^*)^{-1}.$$

But the factor on the right preceding $(D_G^*)^{-1}$ is precisely $\overline{C}_{11}$ from (4.16), as a careful verification reveals. Finally, S_6 and the third sub-term of S_8 add up to yield via (4.13):

$$S_{10} := \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\frac{\det(D_f)}{a_f\kappa(D_f)}(\mathbf{w}^T X^{-1}\mathbf{u}-\mathbf{e}^T X^{-1}\mathbf{u})\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T.$$

Combining all of these, the $(1,1)$-block on the left-hand side of (4.16) equals

$$\begin{aligned}
& D_{\mathcal{G}}^{-1} + D_{\mathcal{G}}^{-1}H\cdot\Psi^{-1}KD_{\mathcal{G}}^{-1}\\
&= \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})}\boldsymbol{\tau}_{\rm out}^1(\boldsymbol{\tau}_{\rm in}^1)^T + \overline{C}_{11}(D_G^*)^{-1} + S_4 + S_7 + S_{10}.
\end{aligned}$$

Examining the $(1,1)$-blocks of (4.16), it remains to show that

$$S_4 + S_7 + S_{10} = \overline{C}_{11}\cdot(\mathbf{w}^T X^{-1}\mathbf{u})\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T + \overline{C}_{12}(-X^{-1}\mathbf{u})\mathbf{e}_{v_0}^T.$$

Notice from above that S_4, S_7, S_{10} are all combinations of

$$\mathbf{w}^T X^{-1}\mathbf{u}\cdot\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T, \qquad \mathbf{e}^T X^{-1}\mathbf{u}\cdot\mathbf{e}_{v_0}\mathbf{e}_{v_0}^T.$$

Regroup $S_4 + S_7 + S_{10}$ in terms of these, and remove the $\mathbf{e}_{v_0}^T$ from both sides of the preceding equation. Thus, it suffices to show:

$$\begin{aligned} & \mathbf{w}^T X^{-1}\mathbf{u} \cdot \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})} \left(a_f^{-1} \frac{\det(D_f)}{\kappa(D_f)} \mathbf{e}_{v_0} + \frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} C_{\mathcal{G}} \mathbf{e}_{v_0} \right) \\ + & \mathbf{e}^T X^{-1}\mathbf{u} \cdot \frac{\kappa(\overline{D}_{\overline{\mathcal{G}}})}{\det(\overline{D}_{\overline{\mathcal{G}}})} \left(a_f^{-1} \frac{\det(\overline{D}_{\overline{\mathcal{G}}})}{\kappa(\overline{D}_{\overline{\mathcal{G}}})} \mathbf{e}_{v_0} - a_f^{-1} \frac{\det(D_f)}{\kappa(D_f)} \mathbf{e}_{v_0} - \frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})} C_{\mathcal{G}} \mathbf{e}_{v_0} \right) \\ = & \mathbf{w}^T X^{-1}\mathbf{u} \cdot \overline{C}_{11} \mathbf{e}_{v_0} + \overline{C}_{12}(-X^{-1}\mathbf{u}). \end{aligned} \tag{4.23}$$

But this follows by applying (2.4) and (4.21). ☐

4.1. **Special case 1: additive-multiplicative Laplacian, and additive-multiplicative (hyper)trees.** We begin this final subsection by recalling the formula for $D_{\mathcal{T}}^{-1}$ for trees equipped with a general additive-multiplicative datum $\{(a_e, m_e, m'_e) : e \in E\}$. Specifically, in [7] we defined certain vectors $\boldsymbol{\tau}_{\rm in}, \boldsymbol{\tau}_{\rm out}$, a scalar $\alpha_{\mathcal{T}}$, the graph Laplacian matrix $L_{\mathcal{T}}$, and a matrix $C_{\mathcal{T}}$, such that

$$D_{\mathcal{T}}^{-1} = \frac{1}{\alpha_{\mathcal{T}}} \boldsymbol{\tau}_{\rm out} \boldsymbol{\tau}_{\rm in}^T - L_{\mathcal{T}} + C_{\mathcal{T}} \operatorname{diag}(\boldsymbol{\tau}_{\rm in}). \tag{4.24}$$

We now show that Theorem C can be recast into a similar formula for $D_{\mathcal{G}}^{-1}$, for any graph G. We begin by defining $\alpha_{\mathcal{G}}$ and $L_{\mathcal{G}}$ in general – the symbols $\boldsymbol{\tau}_{\rm in}, \boldsymbol{\tau}_{\rm out}, C_{\mathcal{G}}$ were already defined above.

Definition 4.7. Given the block-datum $\mathcal{G} = \{\mathcal{G}_e = (a_e, D^*_{G_e}) : e \in E\}$ for a graph G, with strong blocks G_e and invertible $a_e, D^*_{G_e}$, define

$$\alpha_{\mathcal{G}} := \frac{\det(D_{\mathcal{G}})}{\kappa(D_{\mathcal{G}})},$$

and define the *additive-multiplicative Laplacian* to be the $|V(G)| \times |V(G)|$ matrix $L_{\mathcal{G}} = (l_{ij})$, with:

$$l_{ij} := \begin{cases} \frac{-1}{a_e}((D^*_G)^{-1})_{ij}, & \text{if } i \neq j \text{ in a block } e \in E, \\ 0 & \text{if } i, j \text{ lie in different blocks}, \\ -\sum_{k \neq j} l_{kj}, & \text{if } i = j. \end{cases}$$

Note that $L_{\mathcal{G}}$ has column sums zero; in the special case of trees, for q- and classical distance matrices which are symmetric, $L_{\mathcal{G}}$ was symmetric and had zero row sums, e.g. in [9]. Moreover, it follows by (4.5) that D^*_G is a sum of block diagonal matrices (overlapping at the cut-vertex diagonal entries), hence so is the additive-multiplicative Laplacian matrix $L_{\mathcal{G}}$.

Proposition 4.8. *Notation as above. Then:*

$$C_{\mathcal{G}} = (-L_{\mathcal{G}} + C_{\mathcal{G}} \operatorname{diag}(\boldsymbol{\tau}_{\rm in})) D^*_G. \tag{4.25}$$

In particular, and parallel to (4.24)*, one has:*

$$D_{\mathcal{G}}^{-1} = \frac{1}{\alpha_{\mathcal{G}}} \boldsymbol{\tau}_{\rm out} \boldsymbol{\tau}_{\rm in}^T - L_{\mathcal{G}} + C_{\mathcal{G}} \operatorname{diag}(\boldsymbol{\tau}_{\rm in}).$$

In particular, this implies (4.24). A related observation is that in [7], we showed for trees via explicit computations:

$$\boldsymbol{\tau}_{\rm in}^T \mathbf{m}_{\bullet \to l} = 1, \qquad \forall l \in V(G),$$

where $\mathbf{m}_{\bullet \to l} := D^*_G \mathbf{e}_l$ denotes the vector $(m_{vl})_{v \in V(G)}$. With our newfound understanding of $\boldsymbol{\tau}_{\rm in}$, this is now obvious, and for any graph G.

Proof. Define the "block-degree" of a vertex $v \in V(G)$ as follows:

$$d_E(v) := \#\{e \in E : v \in e\}, \qquad V^{cut} := d_E^{-1}([2,\infty)). \tag{4.26}$$

Notice that V^{cut} is precisely the set of cut-vertices; in the case of a tree, these are precisely the non-pendant nodes and $d_E(v)$ is the degree of v.[4]

Once again assume (by Zariski density) that $\kappa(D_{\mathcal{G}})$ is invertible, and hence so are all a_e and $D^*_{G_e}$. Also define for convenience $V_e := V(G_e)$ for each block $e \in E$. Then it follows from (4.5), the lines preceding Remark 4.5, and the definitions that

$$\begin{aligned} C_{\mathcal{G}} =&\ \sum_{e\in E}\left[\frac{1}{a_e}\operatorname{Id}_{V_e}\right]_{V_e} + \sum_{e\in E}\sum_{v\in e\cap V^{cut}}\left[\left(\beta_v - \frac{1}{a_e}\right)\mathbf{e}_v\mathbf{e}(V(G_{v\to e}))^T\right]_{V(G_{v\to e})} \\ &+ \sum_{v\in V^{cut}}(1-d_E(v))\beta_v\mathbf{e}_v\mathbf{e}_v^T, \\ (D^*_G)^{-1} =&\ \sum_{e\in E}[(D^*_{G_e})^{-1}]_{V_e} + \sum_{v\in V^{cut}}(1-d_E(v))\mathbf{e}_v\mathbf{e}_v^T \\ -L_{\mathcal{G}} =&\ \sum_{e\in E}\frac{1}{a_e}\left[(D^*_{G_e})^{-1} - \operatorname{diag}(\boldsymbol{\tau}^e_{\rm in})\right]_{V_e}, \\ \operatorname{diag}(\boldsymbol{\tau}_{\rm in}) =&\ \operatorname{diag}(\mathbf{e}^T(D^*_G)^{-1}) = \sum_{e\in E}[\operatorname{diag}(\boldsymbol{\tau}^e_{\rm in})]_{V_e} + \sum_{v\in V^{cut}}(1-d_E(v))\mathbf{e}_v\mathbf{e}_v^T, \end{aligned} \tag{4.27}$$

where $(\boldsymbol{\tau}^e_{\rm in})^T := \mathbf{e}(V_e)^T(D^*_{G_e})^{-1}$. (The fourth of these formulas follows from the second.)

Notice that the final assertion in the proposition follows from (4.25) via Theorem C. Now, showing (4.25) is equivalent – by Zariski density – to showing:

$$C_{\mathcal{G}}((D^*_G)^{-1} - \operatorname{diag}(\boldsymbol{\tau}_{\rm in})) + L_{\mathcal{G}} = 0.$$

Using the formulas (4.27), we see that the terms in $L_{\mathcal{G}}$ cancel out some of the terms in the first summand of $C_{\mathcal{G}}$ times $((D^*_G)^{-1} - \operatorname{diag}(\boldsymbol{\tau}_{\rm in}))$. Now writing

$$M_f := \left[(D^*_{G_f})^{-1} - \operatorname{diag}(\boldsymbol{\tau}^f_{\rm in})\right]_{V_f}, \qquad f \in E$$

it follows that

$$\begin{aligned} C_{\mathcal{G}}((D^*_G)^{-1} - \operatorname{diag}(\boldsymbol{\tau}_{\rm in})) + L_{\mathcal{G}} =&\ \sum_{e\in E}\sum_{f\in E, f\neq e}\frac{1}{a_e}[\operatorname{Id}_{V_e}]_{V_e}M_f + \sum_{v\in V^{cut}, f\in E}(1-d_E(v))\beta_v\mathbf{e}_v\mathbf{e}_v^T M_f \\ &+ \sum_{e,f\in E}\sum_{v\in e\cap V^{cut}}\left[\left(\beta_v - \frac{1}{a_e}\right)\mathbf{e}_v\mathbf{e}(V(G_{v\to e}))^T\right]_{V(G_{v\to e})} M_f. \end{aligned} \tag{4.28}$$

Denote the three sums on the right-side of (4.28) by S_1, S_2, S_3 respectively. To show $S_1+S_2+S_3$ vanishes, we first **claim** that each S_j can be combinatorially reindexed to a sum over the set

$$E_\cap := \{(e,f) \in E^2 : e \neq f,\ V_e \cap V_f \text{ is nonempty}\}. \tag{4.29}$$

Notice that $V_e \cap V_f$ is a unique cut-vertex if and only if $(e,f) \in E_\cap$; denote this vertex by v_{ef}.

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[4]As an aside, this suggests a different definition/interpretation of pendant nodes in a graph: those which lie in a unique strong block. This differs from the usual notion of a pendant node – i.e. a node with "usual" degree one – and we do not use this alternate interpretation further.

We now show the claim. The first sum S_1 clearly vanishes if $V_e \cap V_f$ is empty; otherwise since $e \neq f$, it follows that $(e,f) \in E_\cap$. Moreover, $\mathrm{Id}_{V_e} = \sum_{v \in V_e} \mathbf{e}_v \mathbf{e}_v^T$, so

$$S_1 = \sum_{(e,f) \in E_\cap} \frac{1}{a_e} \mathbf{e}_{v_{ef}} \mathbf{e}_{v_{ef}}^T M_f.$$

Similarly, the second sum S_2 vanishes unless $v \in f$. Changing the multiplicative factor $(1 - d_E(v))$ to summing over $\{e \in E : e \neq f,\ e \ni v\}$, we have:

$$S_2 = -\sum_{(e,f) \in E_\cap} \beta_{v_{ef}} \mathbf{e}_{v_{ef}} \mathbf{e}_{v_{ef}}^T M_f.$$

Finally, the third sum S_3 in the right-hand side of (4.28) involves row vectors of the form $\mathbf{e}(V(G_{v \to e}))^T M_f$. Notice that if $V_f \cap V(G_{v \to e})$ is empty then this product row vector is trivially zero; the same holds if $f \subseteq G_{v \to e}$, since in that case $\mathbf{e}(V(G_{v \to e}))^T M_f = \mathbf{e}(V_f)^T M_f$, which vanishes by definition. By the geometry of the graph (i.e., the definition of $G_{v \to e}$), it follows that if $\mathbf{e}(V(G_{v \to e}))^T M_f$ is nonzero, then $(e,f) \in E_\cap$ and $v = v_{ef}$. But then,

$$[\mathbf{e}_v \mathbf{e}(V(G_{v \to e}))^T]_{V(G_{v \to e})} M_f = \mathbf{e}_v \mathbf{e}_v^T M_f = \mathbf{e}_{v_{ef}} \mathbf{e}_{v_{ef}}^T M_f,$$

from which it follows that

$$S_3 = \sum_{(e,f) \in E_\cap} \left(\beta_{v_{ef}} - \frac{1}{a_e} \right) \mathbf{e}_{v_{ef}} \mathbf{e}_{v_{ef}}^T M_f.$$

This shows the above claim. Finally, adding up the previous computations,

$$C_{\mathcal{G}}((D_G^*)^{-1} - \mathrm{diag}(\boldsymbol{\tau}_{\mathrm{in}})) + L_{\mathcal{G}} = S_1 + S_2 + S_3 = 0.$$

This shows (4.25) by Zariski density (see the remarks after (4.27)), and concludes the proof. □

To conclude, we remark that the explicit formula for $D_{\mathcal{T}}^{-1}$ for trees, provided in [7, Theorem B], is indeed a special case of Theorem C, by explicitly writing down $(D_e^*)^{-1}$ for every edge (i.e. strong block) e of the tree. More generally, one can do the same for hypertrees:

Proposition 4.9 (Inverse formula for additive-multiplicative hypertrees)**.** *Let G be a hypertree, with block-datum $\mathcal{G}$ as in Proposition 2.7. Then $D_{\mathcal{G}}^{-1}$ is as in Theorem C or Proposition 4.8, with*

$$\alpha_{\mathcal{G}} = \sum_{e \in E} \frac{-a_e}{1 + \mathbf{e}(p_e)^T \mathbf{d}_e} \left(p_e - 1 + \sum_{v < w \in [p_e]} \frac{(m_{e,v} - m_{e,w})(m'_{e,v} - m'_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})} \right),$$

$$(\boldsymbol{\tau}_{\mathrm{in}})_v = 1 - \sum_{e : v \in e} \frac{1}{1 + \mathbf{e}(p_e)^T \mathbf{d}_e} \sum_{w \in e, w \neq v} \frac{m_{e,w} m'_{e,v}(1 - m_{e,v} m'_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})},$$

$$(\boldsymbol{\tau}_{\mathrm{out}})_v = 1 - \sum_{e : v \in e} \frac{1}{1 + \mathbf{e}(p_e)^T \mathbf{d}_e} \sum_{w \in e, w \neq v} \frac{m'_{e,w} m_{e,v}(1 - m'_{e,v} m_{e,w})}{(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})}.$$

Moreover, the additive-multiplicative Laplacian matrix is given by

$$(L_{\mathcal{G}})_{v,w} = \begin{cases} 0 & \text{if } v \neq w, v \not\sim w, \\ \dfrac{m_{e,v} m'_{e,w}}{a_e(1 + \mathbf{e}(p_e)^T \mathbf{d}_e)(1 - m_{e,v} m'_{e,v})(1 - m_{e,w} m'_{e,w})} & \text{if } v \sim w \in e, \\ \displaystyle\sum_{e \in E : w \in e} \frac{-1}{a_e(1 + \mathbf{e}(p_e)^T \mathbf{d}_e)} \sum_{u \in e, u \neq w} \frac{m_{e,u} m'_{e,w}}{(1 - m_{e,u} m'_{e,u})(1 - m_{e,w} m'_{e,w})} & \text{if } v = w, \end{cases}$$

and we also have $C_{\mathcal{G}}$ *given by* (4.3), *with*

$$\beta_i = \frac{1}{\alpha_{\mathcal{G}}} \sum_{e \in E: i \in e} \frac{1}{a_e} \sum_{f \in E: f \subseteq G_{i \to e}} \frac{-a_f}{1 + \mathbf{e}(p_f)^T \mathbf{d}_f} \left(p_f - 1 + \sum_{v < w \in [p_f]} \frac{(m_{f,v} - m_{f,w})(m'_{f,v} - m'_{f,w})}{(1 - m_{f,v} m'_{f,v})(1 - m_{f,w} m'_{f,w})} \right).$$

For instance, for the weighted q-distance matrix whose determinant was computed in [12], one sets $m_{e,v} = m'_{e,v} = \sqrt{q}$ and $a_e = w_e/(q-1)$ for all $e \in E$ and $v \in e$ in the closed-form expressions above, to obtain $D_q(G)^{-1}$.

Proposition 4.9 follows from Lemma 2.6 and Proposition 2.7 via explicit computations. In particular, it is possible to suitably modify the arguments used in proving [7, Theorem B] for trees, and obtain $D_{\mathcal{G}}^{-1}$ for hypertrees via a different proof. In this case we do not use the Master GHH identities (i.e. Theorem A) since all formulas in the preceding proposition are explicit. A consequence of this explicit proof is that as for trees [7], one obtains an alternate, "computational" derivation of the closed-form expressions for $\det(D_{\mathcal{G}})$ and $\operatorname{cof}(D_{\mathcal{G}})$ given in Proposition 2.7.

4.2. **Special case 2: q- and additive distance matrices.** Another special case involves the q-distance matrix of an arbitrary graph, in which case $a_e = 1/(q-1)$ for all e. In particular,

$$D_{\mathcal{G}} = D_q(G) = \frac{1}{q-1}(D_G^* - J), \qquad C_{\mathcal{G}} = (q-1)\,\mathrm{Id}$$

from the definitions and Theorem A (see Remark 4.5). Hence by the Sherman–Morrison formula or via Theorem C,

$$D_q(G)^{-1} = (q-1)\left[(D_G^*)^{-1} + \frac{(D_G^*)^{-1}\mathbf{e} \cdot \mathbf{e}^T (D_G^*)^{-1}}{1 - \mathbf{e}^T (D_G^*)^{-1}\mathbf{e}}\right],$$

and $(D_G^*)^{-1}$ is obtainable from the block submatrices $D_{G_e}^*$ via (4.5). In turn, another use of the Sherman–Morrison formula shows how to convert all occurrences of $(D_{G_e}^*)^{-1}$ into $D_q(G_e)^{-1}, e \in E$. This shows how to obtain a formula for $D_{\mathcal{G}}^{-1} = D_q(G)^{-1}$ in terms of the $D_q(G_e)^{-1}$; specializing to $q \to 1$, we also recover an alternate formulation (and proof) of a recent result of Zhou–Ding–Jia [17] – namely, how to obtain D_G^{-1} in terms of the $D_{G_e}^{-1}$ for *additive* matrices. Note that our method has the added advantage of also obtaining the corresponding result for the q-matrices (for general q). We leave further details to the interested reader.

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