MULTIPLY POSITIVE FUNCTIONS, CRITICAL EXPONENT PHENOMENA, AND THE JAIN–KARLIN–SCHOENBERG KERNEL

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Abstract. This paper continues the analysis of multiply positive functions, first studied by Schoenberg in [*Ann. of Math.* 1955]. We prove the converse to a result of Karlin [*Trans. Amer. Math. Soc.* 1964], and also strengthen his result and two results of Schoenberg [*Ann. of Math.* 1955]. One of the latter results concerns zeros of Laplace transforms of multiply positive functions. The other results study which powers α of two specific kernels are totally non-negative of order $p \geqslant 2$ (denoted TN_p); both authors showed this happens for $\alpha \geqslant p-2$, and Schoenberg proved that it does not for $\alpha < p-2$. We show more strongly that for every $p \times p$ submatrix of either kernel, up to a 'shift', its αth power is totally positive of order p (TP_p) for every $\alpha > p-2$, and is not TN_p for every $\alpha \in (0, p-2) \setminus \mathbb{Z}$. We also extend Karlin's result to a larger class of non-smooth Pólya frequency functions. In particular, these results reveal 'critical exponent' phenomena in the theory of total positivity. We also prove the converse to a 1968 result of Karlin, revealing yet another critical exponent phenomenon – for Laplace transforms of all Pólya frequency functions. More strongly, these results reveal Berezin/Gindikin/Wallach-type sets (predating these authors) in total positivity.

We further classify the powers preserving all Hankel TN_p kernels on intervals, and isolate individual kernels encoding these powers; the latter strengthens a result in previous joint work in [*J. Eur. Math. Soc.*, in press]. We then transfer results on preservers by Pólya–Szegő (1925), Loewner/Horn [*Trans. Amer. Math. Soc.* 1969], and joint with Tao [*Amer. J. Math.*, in press] from positive semidefinite matrices to Hankel TN_p kernels. An additional application is to construct individual matrices that encode the Loewner convex powers. This complements Jain's results [*Adv. in Oper. Th.* 2020] for Loewner positivity, which we strengthen to total positivity, with self-contained proofs. Remarkably, these (strengthened) results of Jain, those of Schoenberg and Karlin, the latter's converse, and the aforementioned individual Hankel kernels all arise from a single symmetric rank-two kernel and its powers: $\max(1+xy, 0)$.

In addition, we provide a novel characterization of Pólya frequency functions and sequences of order $p \geqslant 3$, following Schoenberg's result for $p = 2$ in [*J. d'Analyse Math.* 1951]. We also correct a small gap in that same paper, in Schoenberg's classification of the discontinuous Pólya frequency functions.

Contents

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Notation:

(1) A *positive semidefinite matrix* is a real symmetric matrix with non-negative eigenvalues. Given $I \subset \mathbb{R}$ and $n \geqslant 1$, denote the space of such $n \times n$ matrices with entries in I by $\mathbb{P}_n(I)$.
(2) The *Loewner ordering* on $\mathbb{R}^{n\times n}$ is the partial order where $M \geqslant N$ if and only if $M - N \in \mathbb{P}_n$.
(3) Following Schur [59], a function $f : I \to \mathbb{R}$ acts *entrywise* on $\mathbb{P}_n(I)$ via: $f[A] := (f(a_{jk}))_{j,k=1}^n$.
(4) We say that a map $f : I \to \mathbb{R}$ preserves *Loewner positivity* on $\mathbb{P}_n(I)$ if $f[A] \geqslant 0$ for all $A \in \mathbb{P}_n(I)$, i.e., for $A \geqslant 0$.
(5) We will adopt the convention $0^0 := 0$, unless otherwise specified.

Definition. Let X, Y be totally ordered sets, and $p \geqslant 1$ an integer.

(1) Define $X^{p,\uparrow}$ to be the set of all p-tuples $\mathbf{x} = (x_1, \dots, x_p) \in X$ with strictly increasing coordinates: $x_1 < \cdots < x_p$. (In his book [39], Karlin denotes this open simplex by $\Delta_p(X)$.)
(2) A kernel $K : X \times Y \to \mathbb{R}$ is *totally non-negative of order* p, denoted TN_p, if for all integers $1 \leqslant r \leqslant p$ and tuples $\mathbf{x} \in X^{r,\uparrow}, \mathbf{y} \in Y^{r,\uparrow}$, the determinant of the matrix
$$K[\mathbf{x}; \mathbf{y}] := (K(x_j, y_k))_{j,k=1}^r$$
is non-negative. We say K is *totally non-negative (*TN*)* if K is TN_p for all $p \geqslant 1$.
(3) Analogously, one defines TP_p and TP kernels. If the domains X, Y are both finite, then this yields TN_p, TN, TP_p, or TP matrices.

1. Introduction and main results

In recent joint works [5, 3], we explored the preservers of various classes of positive semidefinite, TN, and TP kernels on infinite domains – as well as the preservers of TN_p and TP_p kernels on finite domains. The present paper studies preservers of TN_p kernels, albeit on infinite domains – this was initiated by Schoenberg in 1955 [58]. In doing so, we end up bringing under this roof, several old and new results on powers preserving Loewner positivity, monotonicity, and convexity as well.

Positive semidefinite matrices, totally positive (TP) matrices, and operations preserving these structures have been widely studied in the literature. More generally, the same question applies to post-composition operators applied to (structured) kernels with the various notions of positivity. Pólya frequency functions [57] and sequences [16] constitute important classes of totally non-negative (TN) kernels that have been widely studied in analysis [57, 58], interpolation theory [12], differential equations and integrable systems [42, 45], probability and statistics [10, 13, 39], and combinatorics [9] (to name a few areas and a very few sources). More generally, TN and TP matrices occur in multiple areas of mathematics, ranging from the aforementioned fields to representation theory and flag varieties [46, 52], cluster algebras [6, 18], interacting particle systems [20, 21], and Gabor analysis [23, 24]. We refer the reader to the twin surveys [1, 2] and references therein – specifically, to the comprehensive book of Karlin [39] – for more on TN/TP matrices and kernels.

1.1. The critical exponent $n - 2$ in positivity.

A well-studied theme in the matrix positivity literature involves entrywise real powers acting on matrices (say with positive entries), to preserve positive (semi)definiteness or other Loewner properties. This theme owes its origins to Loewner, who was interested in understanding (in connection with the Bieberbach conjecture) which entrywise powers preserve positive semidefiniteness. This was resolved by FitzGerald and Horn:

Theorem 1.1 (FitzGerald and Horn, 1977, [17])**.** *Let $n \geqslant 2$ be an integer and $\alpha \in \mathbb{R}$.*

(1) The entrywise map x^α preserves Loewner positivity on $\mathbb{P}_n((0,\infty))$ if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-2, \infty)$.
(2) The entrywise map x^α preserves Loewner monotonicity on $\mathbb{P}_n((0,\infty))$ if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-1, \infty)$. Here, we say a map $f : I \to \mathbb{R}$ is Loewner monotone on $\mathbb{P}_n(I)$ if $f[A] \geqslant f[B]$ whenever $A \geqslant B$ in $\mathbb{P}_n(I)$.

This phase transition at $\alpha = n-2$ for positivity preservers (resp. $\alpha = n-1$ for monotonicity preservers) is known as a *critical exponent* in the matrix analysis literature. See [37] for a survey of the early history of this phenomenon. More recently, a plethora of papers have studied Loewner positive entrywise powers on the domain $I = (0,\infty)$ or $\mathbb{R}$, and on test sets of positive matrices constrained by rank and sparsity [8, 25, 26, 28, 34, 35]. These have yielded similar critical exponents (including a 'combinatorial' one for every graph [26, 27]).

In fact the earliest occurrence of this critical exponent $(n-2)$ – in the positive semidefiniteness literature – was in Horn's 1969 article [32]. Horn began with an important result of Loewner on continuous maps preserving Loewner positivity on $\mathbb{P}_n((0,\infty))$ (which remains essentially the only known necessary condition to date, for such maps in fixed dimension) – see Theorem 4.6. From this, Horn deduced the 'only if' part of Theorem 1.1(1): for $\alpha \in (0, n-2) \setminus \mathbb{Z}$, there exists a matrix $A_\alpha \in \mathbb{P}_n((0,\infty))$ such that $A_\alpha^{\circ\alpha}$ is not positive semidefinite. Horn's proof was non-constructive; moreover, such a 'counterexample' matrix A_α would *a priori* depend on α, as is also the case in the proof of Theorem 1.1(1),(2). This dependence was recently removed, as we explain presently.

1.2. **The critical exponent $p-2$ in total positivity.** At almost the same time[1] as Horn's aforementioned article containing Loewner's result, Karlin had completed his important and comprehensive monograph [39] on total positivity. One can find in it the same set of powers as above – now acting on a certain Pólya frequency function. In this case, however, Karlin showed (originally in his 1964 paper [38]) the 'reverse' direction to Horn above:

Theorem 1.2 (Karlin, [38] – see also [39, Ch. 4, §4, p. 211])**.** *Let $p \geqslant 2$ be an integer and $\alpha \geqslant 0$. Define the Pólya frequency function*

$$\Omega(x) := \begin{cases} xe^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

If $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$, then the function $\Omega(x)^\alpha$ is TN_p.

In particular, for every integer $\alpha > 0$, the function Ω^α is a Pólya frequency function – this was originally shown by Schoenberg in 1951 [57]. We explain the notation used here and in the sequel:

Definition 1.4. Let $p \geqslant 1$ be an integer, and $\Lambda : \mathbb{R} \to \mathbb{R}$ a Lebesgue measurable function.

(1) We say Λ is a *Pólya frequency function* if Λ is Lebesgue integrable on $\mathbb{R}$, the associated Toeplitz kernel
$$T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x,y) \mapsto \Lambda(x-y)$$
is totally non-negative, and Λ does not vanish at least at two points (whence on an interval).
(2) We say Λ is *totally non-negative of order* $p \geqslant 1$, again denoted TN_p, if T_Λ is TN_p. If Λ is TN_p for all $p \geqslant 1$, then we say Λ is *totally non-negative (*TN*)*.
(3) Analogously, one defines TP_p and TP functions.

Karlin's result is at least the second instance of a critical exponent phenomenon, implicit in the theory of total positivity. Almost a decade earlier, Schoenberg had shown a similar result for powers of a seemingly unrelated kernel, which he termed *Wallis distributions*:

Theorem 1.5 (Schoenberg, 1955, [57, Theorems 4 and 5])**.** *Define the map*

$$W : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} \cos(x), & \text{if } x \in (-\pi/2, \pi/2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.6}$$

Also suppose $\alpha \geqslant 0$ and an integer $p \geqslant 2$. Then $W(x)^\alpha$ is TN_p if and only if $\alpha \geqslant p-2$.

[1] In fact, also at the same place (Stanford University); Karlin, Loewner, Pólya, and Szegő had been colleagues, and FitzGerald and Horn were Loewner's students.

(The 'only if' part implicitly follows from [58, Theorem 4] and was not formulated. For completeness, we write out how this can be achieved, in Remark 6.2.) Thus, Schoenberg's result shows a critical exponent phenomenon from total positivity – with the same point $p-2$ for a TN_p kernel, as for positivity preservers on $p \times p$ matrices.

In parallel: note that Karlin did not address the non-integer powers below $p-2$. We begin by achieving this task, and showing that $\alpha = p-2$ is indeed a 'critical exponent' for total positivity:

Theorem 1.7. *Let $p \geqslant 2$ be an integer and $\alpha \in (0, p-2) \setminus \mathbb{Z}$. Then Ω^α is not TN_p.*

One consequence is that there also exists a sequence of *Pólya frequency sequences*[2] whose αth powers are not TN_p for $\alpha \in (0, p-2) \setminus \mathbb{Z}$. This follows from the continuity of the kernel Ω, via a discretization argument as in our recent joint work [3]. The assertion can be strengthened to show the existence of TN Toeplitz kernels on more general domains $X \times Y$ than $\mathbb{Z} \times \mathbb{Z}$. These subsets X, Y only need to satisfy: for each $p \geqslant 1$, there exist equi-spaced arithmetic progressions $\mathbf{x} \in X^{p,\uparrow}$ and $\mathbf{y} \in Y^{p,\uparrow}$ with $x_2 - x_1 = y_2 - y_1$. A similar argument works for Schoenberg's powers $W(x)^\alpha$.

Thus, Theorems 1.5 and 1.7 say that for each $\alpha \in (0, p-2) \setminus \mathbb{Z}$, one can find tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$ (or in $\mathbb{Q}^{p,\uparrow}$ via discretization), for which the Toeplitz matrices $T_{\Omega^\alpha}[\mathbf{x}; \mathbf{y}]$ and $T_{W^\alpha}[\mathbf{x}; \mathbf{y}]$ each contain a negative minor. Our first main result strengthens both of these conditions, by showing they are satisfied up to a shift at *every* pair $\mathbf{x}, \mathbf{y}$, and simultaneously for *all* powers $\alpha \in (0, p-2) \setminus \mathbb{Z}$:

Theorem A. *Fix an integer $p \geqslant 2$ and subsets $X, Y \subset \mathbb{R}$ of size at least p.*

1. *There exists $a = a(X, Y) \in \mathbb{R}$ such that the restriction of $T_{\Omega_a}(x, y)^\alpha$ to $X \times Y$ (where $\Omega_a(x) = \Omega(x - a)$ as in Theorem 1.2), is not TN_p for all $\alpha \in (0, p-2) \setminus \mathbb{Z}$.*
2. *There exists $m = m(X, Y) \in (0, \infty)$ such that the restriction of $T_{W_m}(x, y)^\alpha$ to $X \times Y$ (where $W_m(x) = W(mx)$ as in Theorem 1.5), is not TN_p for all $\alpha \in (0, p-2) \setminus \mathbb{Z}$.*
3. *Given tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$, there exist $a \in \mathbb{R}$ and $m > 0$ such that the matrices*
$$(\Omega(x_j - y_k - a)^\alpha)_{j,k=1}^p, \qquad (W(m(x_j - y_k))^\alpha)_{j,k=1}^p$$
are TP if $\alpha > p-2$, TN if $\alpha \in \{0, 1, \dots, p-2\}$, and not TN if $\alpha \in (0, p-2) \setminus \mathbb{Z}$.

Note that the additive/multiplicative shifts $a = a(\mathbf{x}, \mathbf{y})$ and $m = m(\mathbf{x}, \mathbf{y})$ are independent of $\alpha \in (0, p-2) \setminus \mathbb{Z}$. Hence so are $a(X, Y), m(X, Y)$.

Remark 1.8. The first two assertions in Theorem A(3) strengthen Karlin's theorem 1.2 and one implication in Schoenberg's theorem 1.5, on a suitable part of their domains. The final assertion in Theorem A(3) is the aforementioned strengthening of the 'converse' Theorem 1.7 (and of the other implication in Theorem 1.5), and follows from parts (1) and (2) by specializing X, Y to the sets of coordinates of $\mathbf{x}, \mathbf{y}$ respectively.

Theorems 1.7 and A lead to a Pólya frequency function whose non-integer powers are not TN:

Corollary 1.9. *If $\alpha \geqslant 0$ and the function $\Omega(x)^\alpha$ is TN, then α is an integer.*

This was observed e.g. in [3], where the 'heavy machinery' of the bilateral Laplace transform was used through deep results of Schoenberg [57].[3] Our proof of Theorem A below is self-contained, shows a stronger result, and avoids these sophisticated tools.

Thus, our first contribution shows that critical exponents for total non-negativity – more strongly, 'total positivity' phenomena – occur in the study of preservers of Pólya frequency functions, Pólya frequency sequences, and Toeplitz kernels on more general domains, in the above (strengthened) results by Schoenberg and Karlin and their converses – and for all submatrices, up to a shift.

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[2]Recall, Pólya frequency sequences are defined to be real sequences $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ such that the Toeplitz kernel $T_{\mathbf{a}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ sending $(m, n) \mapsto a_{m-n}$ is TN.

[3]Briefly, the bilateral Laplace transform of Ω^α is $\Gamma(\alpha+1)/(s+\alpha)^{\alpha+1}$, and if $\alpha \notin \mathbb{Z}^{\geqslant 0}$ then its reciprocal is not analytic in s – not in the Laguerre–Pólya class. Thus Ω^α is not a Pólya frequency function by [57], whence not TN.

Remark 1.10. A direct application of Theorem A yields yet another critical exponent phenomenon in total positivity – this time on on the level of Laplace transforms of Pólya frequency functions. One implication can be found in Karlin's book, and the converse follows from our results. See Corollary 3.3, which reveals the set $\mathbb{Z}^{>0} \cup (p-1,\infty)$ of powers acting on Laplace transforms of arbitrary one-sided Pólya frequency functions to preserve TN_p.

Remark 1.11. Via the Laplace transform, we also show in Section 3.1 that Karlin's result and Theorem A are 'degenerate' cases of a more general phenomenon. Namely, if $\Omega^{(q,r)}(x)$ is the (unique) Pólya frequency function with bilateral Laplace transform $\mathcal{B}\{\Omega^{(q,r)}\}(s) = (1+s/q)^{-1}(1+s/r)^{-1}$ for $q \neq r$ positive scalars, then we show that $\Omega^{(q,r)}(x)^\alpha$ satisfies the same results as does $\Omega(x)^\alpha$ above. The 'degenerate' case of $q = r = 1$ precisely yields $\Omega(x)$ and Theorem A.

The preceding two remarks, and the results alluded to therein (and proved below), may remind the reader of similar ensembles inside the real line, consisting of a finite arithmetic progression followed by an infinite semi-axis, in other contexts. Indeed, such sets have been studied since the 1970s, in representation theory and complex geometry (Rossi–Vergne [53], Wallach [63]), symmetric cones (Gindikin [22], Lassalle [43]), quantization (Berezin [7]), and in subsequent decades in probability theory owing to the non-central Wishart distribution (see e.g. [15, 44, 47, 48]). As mentioned above, the set of powers entrywise preserving positivity in a fixed dimension (computed by FitzGerald–Horn in [17]) is another such example. These occurrences of Berezin/Gindikin/Wallach-type sets were predated by Karlin's work in the preceding decade (with the converses proved below).

1.3. **Single-matrix encoders; Hankel kernels.** As seen above, Schoenberg and Karlin studied individual kernels, for which all powers $\geqslant p-2$ preserve TN_p, and no non-integer power $< p-2$ does so – in close analogy with the FitzGerald–Horn theorem 1.1. In the latter, parallel setting of entrywise powers preserving positivity, such individual matrices were discovered only recently, by Jain [34, 35]. Her results are now stated in parallel to Theorem 1.1, and isolate a smallest possible test set for Loewner positive and monotone powers:

Theorem 1.12 (Jain, 2020, [35])**.** *Let $n \in \mathbb{Z}, n \geqslant 2$ and $\alpha \in \mathbb{R}$. Suppose $x_1, \dots, x_n \in \mathbb{R}$ are pairwise distinct, with $1 + x_j x_k > 0 \ \forall j,k$. Let $A := (1 + x_j x_k)_{j,k=1}^n$ and $B := \mathbf{1}_{n\times n}$, so $A \geqslant B \geqslant 0$.*

1. *The matrix $A^{\circ\alpha}$ is positive semidefinite if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-2,\infty)$.*
2. *Suppose all x_j are non-zero. The matrix $A^{\circ\alpha} \geqslant B = B^{\circ\alpha}$, if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-1,\infty)$.*

In fact Jain does more in [34, 35]: she computes the inertia of the matrices $A^{\circ\alpha}$ as above, for all real $\alpha \geqslant 0$. Our main theorem C below strengthens Theorem 1.12(1), and shows that $A^{\circ\alpha}$ is not just positive definite for $\alpha > n-2$, but totally positive. In particular, as can be shown using Perron's theorem [49] and the folklore theorem of Kronecker on eigenvalues of compound matrices, $A^{\circ\alpha}$ has simple, positive eigenvalues for $\alpha > n-2$, parallel to Jain.

Before proceeding further, we describe two consequences of the first part of Jain's theorem 1.12:

1. Set $x_j := \cot(j\pi/(2n))$; now $A^{\circ\alpha}$ is positive semidefinite if and only if so is the matrix
$$D^{\circ\alpha} A^{\circ\alpha} D^{\circ\alpha} = (DAD)^{\circ\alpha},$$
where D is the diagonal matrix with (j,j) entry $\sin(j\pi/(2n))$. But DAD is the Toeplitz matrix $(\cos((j-k)\pi/(2n)))_{j,k=1}^n$, so Jain's result yields a rank-two positive semidefinite Toeplitz matrix which encodes the Loewner positive powers on $\mathbb{P}_n((0,\infty))$. Notice this is a restriction of Schoenberg's kernel T_W from Theorem 1.5.
2. Setting $x_j := u_0^j$ for $u_0 \in (0,\infty) \setminus \{1\}$, it follows that A is a rank-two positive semidefinite Hankel matrix, which encodes the Loewner positive and monotone powers on $\mathbb{P}_n((0,\infty))$.

This second consequence leads to our next theorem. Recall that Karlin and Schoenberg's results above, together with Theorem A, studied Toeplitz kernels which encoded the (non-integer) powers preserving TN_p. We next produce a Hankel kernel with this property. Unfortunately, the naive guess of $K(x,y) = (x+y)e^{-(x+y)}$ does not work, since this is 'equivalent' to $T_\Omega(x,-y)$, which leads to 'row-reversal' and hence a sign of $(-1)^{p(p-1)/2}$ in $p \times p$ submatrices drawn from K. (As a specific instance, $\det T_\Omega[(3,4);(-2,-1)] < 0$.) However, the 'rank-two' kernel $1 + u_0^{x+y}$ is TN and exhibits the same critical exponent phenomenon. More strongly, this kernel encodes the powers preserving TN_p for *all* Hankel kernels on $\mathbb{R} \times \mathbb{R}$ – in other words, the analogues of Theorems 1.1 and 1.12 hold together, for Hankel kernels on $\mathbb{R} \times \mathbb{R}$. Slightly more strongly, this happens over arbitrary intervals:

Theorem B. *Let $p \geqslant 2$ be an integer, and fix scalars $c_0, u_0 > 0, u_0 \neq 1$ and $\alpha \geqslant 0$. Also fix an interval $X_0 \subset \mathbb{R}$ with positive measure. The following are equivalent:*

1. *If $X \subset \mathbb{R}$ is an interval with positive measure, and $H : X \times X \to \mathbb{R}$ is a continuous TN_p Hankel kernel, then H^α is TN_p. Here, by a Hankel kernel we mean $K : X \times X \to \mathbb{R}$ such that there exists a function $f : X + X \to \mathbb{R}$ satisfying: $K(x,y) = f(x+y)$ for $x, y \in X$.*
2. *Define the Hankel kernel*
$$H_{u_0} : X_0 \times X_0 \to \mathbb{R}, \qquad (x,y) \mapsto 1 + c_0 u_0^{x+y}.$$
Then $H_{u_0}^\alpha$ is TN_p on $X_0 \times X_0$.
3. *$\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$.*

In particular, every $\alpha \in \mathbb{Z}^{\geqslant 0}$ preserves TN Hankel kernels. Moreover, for every $\mathbf{x}, \mathbf{y} \in {X_0}^{p,\uparrow}$, the kernel $H_{u_0}^\alpha$ is TP_p if $\alpha > p-2$, and not TN_p if $\alpha \in (0, p-2) \setminus \mathbb{Z}$.

This strengthens results in recent work [5, 3], which study powers preserving TN Hankel kernels. Theorem B studies power preservers of TN_p Hankel kernels, for each $p \geqslant 2$.

1.4. **The Jain–Karlin–Schoenberg kernel.** Our next main result again concerns power-preservers of TN_p kernels. We show that remarkably, the multitude of kernels studied above are all related. More precisely, Karlin's theorem 1.2 and our converse, Schoenberg's theorem 1.5, the FitzGerald–Horn theorem 1.1, Jain's theorem 1.12(1), the aforementioned strengthenings of these, the Hankel kernels H_{u_0}, and the related critical exponent phenomena *all* arise from studying a particular symmetric kernel having 'rank two' (on part of its domain) – restricted to various sub-domains. In particular, this will explain why the same critical exponent of $p-2$ (plus, all powers above $p-2$, and no non-integer power below it) shows up in each of these settings.

We begin by introducing this simple kernel:

Definition 1.13. Define the *Jain–Karlin–Schoenberg kernel* $K_{\mathcal{JKS}}$ as follows:
$$K_{\mathcal{JKS}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad (x,y) \mapsto \max(1+xy, 0). \tag{1.14}$$

The choice of name is because – as we explain in Remark 5.2 – the restrictions of this kernel to $(-\infty, 0] \times (0, \infty)$, to $(0,\infty) \times (0,\infty)$, and on the full domain $\mathbb{R}^2$, are intimately related to Karlin's kernel Ω, to Jain's matrices $(1 + x_j x_k)$, and to Schoenberg's cosine-kernel W, respectively.

Our next result studies the powers of $K_{\mathcal{JKS}}$ that are TN_p on the plane or on the X or Y half-planes. Remark 5.2 will then explain how this connects to all of the results stated above.

Theorem C. *Fix an integer $p \geqslant 2$, an interval $I \subset \mathbb{R}$, and let a scalar $\alpha \geqslant 0$.*

1. *$K_{\mathcal{JKS}}^\alpha$ is TN_p on $\mathbb{R} \times \mathbb{R}$ for $\alpha \geqslant p-2$.*
2. *If the power $K_{\mathcal{JKS}}^\alpha$ is TN_p, then $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$. More strongly, given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$ such that $1 + x_j y_k > 0 \ \forall j,k$, the matrix $K_{\mathcal{JKS}}[\mathbf{x};\mathbf{y}]^{\circ\alpha}$ is:*
   - (a) TP *if $\alpha > p-2$;*
   - (b) TN *if $\alpha \in \{0, 1, \dots, p-2\}$; and*
   - (c) *not* TN *if $\alpha \in (0, p-2) \setminus \mathbb{Z}$.*
3. *Suppose $I \subset [0, \infty)$ or $I \subset (-\infty, 0]$. The kernel $K_{\mathcal{JKS}}^\alpha$ is TN_p on $I \times \mathbb{R}$ (or $\mathbb{R} \times I$) if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$. (In particular, $K_{\mathcal{JKS}}^\alpha$ is* TN *on $I \times \mathbb{R}$ or $\mathbb{R} \times I$ for $\alpha \in \mathbb{Z}^{\geqslant 0}$.)*

As an aside, integer powers of the kernel $K_{\mathcal{JKS}}$ (more precisely, of $1 + xy$) have featured in the statistics and machine learning literature, as non-homogeneous polynomial kernels of dot-product type. See e.g. [31, 33, 41, 61, 62].

1.5. **General** TN_p **functions.** Our final two results deal with general TN_p kernels. A closely related result to Theorem C is a 1955 theorem by Schoenberg [58], which implies that no power $\alpha < p - 2$ of the kernel W is TN_p. This is a result on arbitrary compactly supported, multiply positive functions Λ, and we strengthen it by restricting the domain of Λ:

Theorem D. *Suppose* $0 < \rho \leqslant \widetilde{\rho} \leqslant +\infty$ *and* $0 < \epsilon \leqslant \widetilde{\rho} - \rho/2$ *are scalars, with* $\rho < \infty$. *Suppose* $p \geqslant 2$ *is an integer, and the integrable function* $\Lambda : (-\widetilde{\rho}, \widetilde{\rho}) \to \mathbb{R}$ *is positive on* $(-\rho/2, \rho/2)$, *vanishes outside* $[-\rho/2, \rho/2]$, *and induces the* TN_p *kernel*

$$T_\Lambda : [0, \epsilon) \times (-\rho/2, (\rho/2) + \epsilon) \to \mathbb{R}, \qquad (x, y) \mapsto \Lambda(x - y).$$

Then the Fourier–Laplace transform

$$\mathcal{B}\{\Lambda\}(s) := \int_{-\rho/2}^{\rho/2} e^{-sx} \Lambda(x) \, dx, \qquad s \in \mathbb{C}$$

has no zeros in the strip $|\Im(s)| < p\pi/\rho$.

Schoenberg proved this result in [58], assuming $\widetilde{\rho} = +\infty$ and that $T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is TN_p. (He also 'changed variables' so that $\rho = \pi$.) This means that all minors of order $\leqslant p$ drawn from Λ are required to be non-negative. We arrive at the same conclusions as Schoenberg, using far fewer minors – indeed, the aforementioned domain of T_Λ means that we only need to work with the restriction of Λ to $(-(\rho/2) - \epsilon, (\rho/2) + \epsilon)$ for arbitrarily small $\epsilon > 0$.

Our final result provides a characterization of TN_p functions (or Pólya frequency functions of order p). Recall that such a result was shown for $p = 2$ by Schoenberg in 1951 [57], and Weinberger mentioned in 1983 a variant for $p = 3$ in [64] (which turns out to have a small gap). To our knowledge, no such characterization is known for $p \geqslant 4$. This is provided by the next result, by considering only the largest-sized minors:

Theorem E. *Let* $p \geqslant 3$ *be an integer, and a function* $\Lambda : \mathbb{R} \to [0, \infty)$. *The following are equivalent.*

(1) Either $\Lambda(x) = e^{ax+b}$ *for* $a, b \in \mathbb{R}$, *or: (a)* Λ *is Lebesgue measurable; (b) for all scalars* x_0, y_0, *the function* $\Lambda(x_0 - y)\Lambda(y - y_0) \to 0$ *as* $y \to \infty$; *and (c)* $\det T_\Lambda[\mathbf{x}; \mathbf{y}] \geqslant 0$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$.

(2) The function $\Lambda : \mathbb{R} \to \mathbb{R}$ *is* TN_p.

The result also holds for $p = 2$, in which case it is a tautology. The proof-technique also yields similar results for 'Pólya frequency sequences of order p' – or more generally, for (not necessarily Toeplitz) TN_p kernels on $X \times Y$ for general subsets $X, Y \subset \mathbb{R}$ – under similar decay assumptions. See the final section of the paper.

Organization of the paper. The next section develops a few preliminaries – specifically, novel homotopy arguments that are used in our proofs. The subsequent five sections of the paper prove our main theorems, one per section. Four of these sections contain four other features:

(a) After proving Theorem A, we show akin to Jain's theorem 1.12 that the same 'individual' matrices $(1 + x_j x_k)_{j,k=1}^n$ and $\mathbf{1}_{n \times n}$ also encode the powers preserving *Loewner convexity*. See Section 3.2 for the definition of Loewner convexity as well as the precise result.

(b) After proving Theorem B, we present results – now for Hankel TN_p kernel preservers – parallel to Loewner's aforementioned necessary condition in [32], to an old observation of Pólya–Szegő [51], and to our recent work with Tao [40] on polynomial preservers of positivity on $p \times p$ matrices.

(c) After proving Theorem C, we explain in Remark 5.2 how this result for $K_{\mathcal{JKS}}$ subsumes our results above, as well as results of Karlin, Jain, and Schoenberg.

(d) Prior to proving Theorem E, we correct via Theorem 7.4 a small gap in Schoenberg's landmark 1951 paper [57], thus completing the classification of continuous Pólya frequency / TN functions.

The Appendix, which may safely be skipped during a first reading, contains proofs of several results pertaining to power-preservers of TN_p, and is included for the convenience of the reader.

2. A variant of Descartes' rule of signs, and homotopy arguments

The proofs of the above results rely on new tools and old. We begin with a variant from [35] of Descartes' rule of signs, in which exponentials are replaced by powers $(1 + ux_j)^r$. To state this result requires the following notation.

Definition 2.1. Given an integer $n \geqslant 1$ and a tuple $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$A_{\mathbf{x}} := \begin{cases} -\infty, & \text{if } \max_j x_j \leqslant 0, \\ -1/\max_j x_j, & \text{otherwise}, \end{cases} \qquad B_{\mathbf{x}} := \begin{cases} \infty, & \text{if } \min_j x_j \geqslant 0, \\ -1/\min_j x_j, & \text{otherwise}. \end{cases}$$

Proposition 2.2 (Jain, [35])**.** *Fix an integer $n \geqslant 1$ and real tuples $\mathbf{c} = (c_1, \dots, c_n) \neq 0$ and $\mathbf{x} = (x_1, \dots, x_n)$, where the x_j are pairwise distinct. For a real number r, define the function*

$$\varphi_{\mathbf{x},\mathbf{c},r} : (A_{\mathbf{x}}, B_{\mathbf{x}}) \to \mathbb{R}, \qquad u \mapsto \sum_{j=1}^n c_j (1 + ux_j)^r.$$

Then either $\varphi_{\mathbf{x},\mathbf{c},r} \equiv 0$, or it has at most $n-1$ zeros, counting multiplicities.

In the interest of keeping this paper self-contained, we sketch this proof in a somewhat vestigial Appendix, together with proofs of the results from other works that are used in this paper.

The next step is a (novel) homotopy argument for symmetric matrices; a non-symmetric variant will also be proved and used below.

Proposition 2.3. *Fix an integer $n \geqslant 2$ and real scalars*

$$x_1 < \cdots < x_n \quad \textit{and} \quad 0 < y_1 < \cdots < y_n, \quad \textit{with} \quad 1 + x_j x_k > 0 \ \forall j, k.$$

There exists $\delta > 0$ such that for all $0 < \epsilon \leqslant \delta$, the 'linear homotopies' between x_j and ϵy_j, given by

$$x_j^{(\epsilon)}(t) := x_j + t(\epsilon y_j - x_j), \qquad t \in [0,1]$$

satisfy

$$1 + x_j^{(\epsilon)}(t) x_k^{(\epsilon)}(t) > 0, \qquad \forall 1 \leqslant j, k \leqslant n, \ t \in [0,1].$$

Remark 2.4. The above result is (implicitly stated, and explicitly) used in [35] with all $y_j = j$, and without the factor of ϵ. The use of this result is key if one wishes to avoid using Jain's prior work [34] in proving Theorem 1.12. Unfortunately, the factor of ϵ here is crucial, otherwise the result fails to hold. Here are two explicit examples; in both of them, $n = 2$, $\epsilon = 1$, and $(y_1, y_2) = (1, 2)$. Suppose first that $(x_1, x_2) = (-199, 0)$; then 'completing the square' shows that the above assertion fails to hold at 'most' times in the homotopy:

$$1 + x_1^{(1)}(t) x_2^{(1)}(t) \leqslant 0, \quad \forall t \in \left[\frac{398}{800} - \frac{1}{20}\sqrt{\frac{398^2}{40^2} - 1}, \ \frac{398}{800} + \frac{1}{20}\sqrt{\frac{398^2}{40^2} - 1} \right],$$

and this interval contains $[0.0026, 0.9924]$. As another example, if $(x_1, x_2) = (-8.5, 0.1)$, then

$$1 + x_1^{(1)}(t) x_2^{(1)}(t) \leqslant 0, \quad \forall t \in \left[\frac{8 - \sqrt{61}}{19}, \ \frac{8 + \sqrt{61}}{19} \right] \supset [0.01, 0.8321].$$

Remark 2.5. Jain has communicated to us [36] a short workaround to the above gap in [35], as follows: if all $x_j \leqslant 0$ then to prove Theorem 1.12(1) one can replace all x_j with $-x_j$. If $x_1 < 0 < x_n$ then one lets $0 < y_1 < \cdots < y_n < x_n$, and for these *specific* y_j, the homotopy argument works. However, we then need to show Theorem 1.12(1) in the special case when all $x_j > 0$ – which is a result in Jain's prior work; see [34] and the references and results cited therein. These prior results involve strictly sign regular (SSR) matrices and earlier papers. In this paper we avoid SSR matrices, and hence our approach additionally serves to provide a shorter, direct proof of Theorem 1.12.

We now show the above homotopy result.

Proof of Proposition 2.3. We make three clarifying observations to start the proof, with $x_j(t)$ denoting $x_j^{(\epsilon)}(t)$ throughout for a fixed $\epsilon > 0$. First, the assumptions imply $x_1(t) < \cdots < x_n(t)$ for all $t \in [0,1]$.

Second, if $x_1 = x_1(0) \geqslant 0$, then clearly $x_j(t) \geqslant 0$ for all $t \in [0,1]$ and all $1 \leqslant j \leqslant n$, and in this case the result follows at once. We will thus assume in the sequel that $x_1 < 0$.

Third, suppose there exist integers $1 \leqslant j < k \leqslant n$ and a time $t \in [0,1]$ such that $1 + x_j(t)x_k(t) \leqslant 0$, then we have $x_j(t) < 0 < x_k(t)$, and so $x_1(t) < 0 < x_n(t)$. A straightforward computation shows

$$1 + x_1(t)x_n(t) \leqslant 1 + x_j(t)x_k(t) \leqslant 0.$$

Given these observations, suppose we have initial data x_j, y_j, with $x_1 < 0$ from above. It suffices to find $\delta > 0$ such that

$$1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) > 0, \qquad \forall \epsilon \in (0, \delta],\ t \in (0,1).$$

Depending on the sign of x_n, we consider two cases:

Case 1: $x_n \geqslant 0$, in which case $x_n < 1/|x_1|$. We claim that $\delta := 1/(|x_1|y_n)$ works. Indeed, given $0 < \epsilon \leqslant \delta$, and $t \in (0,1)$, compute:

$$\begin{aligned} 1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) &= 1 + (t\epsilon y_1 + (1-t)x_1)(t\epsilon y_n + (1-t)x_n) \\ &> 1 + (1-t)x_1(t\epsilon y_n + (1-t)x_n) \\ &> 1 + (1-t)x_1(t\epsilon y_n + (1-t)/|x_1|), \end{aligned}$$

with both inequalities strict because $t \in (0,1)$. Now the final expression equals

$$= 1 - (1-t)^2 + t(1-t)\epsilon y_n x_1 \geqslant t\,(2 - t - (1-t)\delta y_n |x_1|) = t > 0.$$

Case 2: $x_n < 0$. Define the continuous function

$$g(\epsilon) := 1 - \frac{\epsilon^2 (x_n y_1 - x_1 y_n)^2}{4(\epsilon y_1 - x_1)(\epsilon y_n - x_n)}, \qquad \epsilon \geqslant 0.$$

Since $g(0) > 0$, there exists $\delta > 0$ such that g is positive on $[0, \delta]$. We claim this choice of δ works. Fix $0 < \epsilon \leqslant \delta$, and define

$$t_j^{(\epsilon)} := -x_j/(\epsilon y_j - x_j), \qquad \forall j \in [1,n].$$

It is easy to check that $x_j^{(\epsilon)}(t)$ is positive, zero, or negative when $t > t_j^{(\epsilon)}, t = t_j^{(\epsilon)}, t < t_j^{(\epsilon)}$ respectively; moreover, since all $x_j < 0$, the above observations imply

$$0 < t_n^{(\epsilon)} < t_{n-1}^{(\epsilon)} < \cdots < t_1^{(\epsilon)} < 1.$$

In particular, if $0 \leqslant t \leqslant t_n^{(\epsilon)}$ or $t_1^{(\epsilon)} \leqslant t \leqslant 1$, then $x_1^{(\epsilon)}(t)$ and $x_n^{(\epsilon)}(t)$ both have the same sign, whence $1 + x_1^{(\epsilon)}(t)x_n^{(\epsilon)}(t) \geqslant 1$, so is positive. Otherwise $t_n^{(\epsilon)} < t < t_1^{(\epsilon)}$, in which case we first note that

$$x_j^{(\epsilon)}(t) = t\epsilon y_j + (1-t)x_j = (t - t_j^{(\epsilon)})(\epsilon y_j - x_j), \qquad \forall j \in [1,n],\ t \in [0,1].$$

But now we compute, using the AM–GM inequality and choice of δ:

$$\begin{aligned} 1 + x_1^{(\epsilon)}(t) x_n^{(\epsilon)}(t) &= 1 + (t - t_1^{(\epsilon)})(t - t_n^{(\epsilon)})(\epsilon y_1 - x_1)(\epsilon y_n - x_n) \\ &\geqslant 1 - \frac{1}{4}(t_1^{(\epsilon)} - t_n^{(\epsilon)})^2 (\epsilon y_1 - x_1)(\epsilon y_n - x_n) = g(\epsilon) > 0. \end{aligned}$$

□

Our next result is more widely applicable, at the cost of making the homotopy 'piecewise linear':

Proposition 2.6. *Fix an integer $n \geqslant 2$ and tuples of real scalars*

$$\mathbf{x}, \mathbf{y}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^{n,\uparrow}$$

such that $1 + x_j y_k > 0 \ \forall j, k$ and $p_1, q_1 > 0$. Then there exists piecewise linear homotopies

$$x_j(t), y_j(t) : [0,1] \to \mathbb{R}, \qquad 1 \leqslant j \leqslant n$$

such that $\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^{n,\uparrow}$ for all times $t \in [0,1]$, with

$$x_j(0) = x_j, \quad x_j(1) = p_j, \quad y_j(0) = y_j, \quad x_j(1) = q_j,$$

and such that $1 + x_j(t) y_k(t) > 0$ for all $t \in [0,1]$.

Proof. Let $\delta_1 := \frac{1}{2|y_1|p_n}$ if $y_1 \neq 0$, and 1 otherwise. Define

$$x'_j(t) := x_j + t(\delta_1 p_j - x_j), \qquad 1 \leqslant j \leqslant n, \ t \in [0,1].$$

We claim that $1 + x'_j(t) y_k > 0$ for all $1 \leqslant j, k \leqslant n$ and $t \in [0,1]$. This is true at $t = 0$ for all j, k; now suppose it fails for some $t_0 \in (0,1]$ and $j, k \in [1,n]$. If $y_k \geqslant 0$ then $0 > x'_j(t_0) \geqslant x_j$, so

$$0 \geqslant 1 + x'_j(t_0) y_k \geqslant 1 + x_j y_k > 0,$$

which is impossible. Thus we must have

$$y_k < 0 < x'_j(t_0) \leqslant x'_n(t_0) \leqslant \max(\delta_1 p_n, x_n).$$

Using this,

$$0 \geqslant 1 + x'_j(t_0) y_k \geqslant 1 + x'_j(t_0) y_1 \geqslant 1 + y_1 \max(\delta_1 p_n, x_n) = \min(1 + y_1 x_n, 1 + \delta_1 y_1 p_n) > 0,$$

which is similarly impossible.

This reasoning shows that one can define a linear homotopy $\mathbf{x}(t), \ t \in [0, 1/3]$ going from $\mathbf{x}$ to $\delta_1 \mathbf{p}$ for some $\delta_1 > 0$, such that $1 + x_j(t) y_k > 0$ for all t. Throughout, we define $\mathbf{y}(t) \equiv \mathbf{y}$ for $t \in [0, 1/3]$.

In a similar fashion, we let $\mathbf{x}(t) \equiv \delta_1 \mathbf{p}$ for $t \in [1/3, 2/3]$, and write down a linear homotopy $\mathbf{y}(t)$ from $\mathbf{y}$ to $\delta_2 \mathbf{q}$ for some $\delta_2 > 0$, such that $1 + x_j(t) y_k(t) > 0$ for $t \in [1/3, 2/3]$.

Finally, let $\mathbf{x}(t)$ (respectively $\mathbf{y}(t)$) for $t \in [2/3, 1]$ be the linear homotopy from $\delta_1 \mathbf{p}$ to $\mathbf{p}$ (respectively from $\delta_2 \mathbf{q}$ to $\mathbf{q}$). Since $p_1, q_1 > 0$, it is trivially true that $1 + x_j(t) y_k(t) > 0$ for $t \in [2/3, 1]$. □

3. Proof of Theorem A and its strengthening: Critical exponent for PF functions

We now show the main results above. The next step is a direct application of Proposition 2.2:

Proposition 3.1 (Jain, [35])**.** *Suppose $x_1, \dots, x_n \in \mathbb{R}$ are pairwise distinct, as are $y_1, \dots, y_n \in \mathbb{R}$. If $1 + x_j y_k > 0$ for all j, k, and $\alpha \in \mathbb{R} \setminus \{0, 1, \dots, n-2\}$, then $S^{\circ\alpha}$ is non-singular, where $S := (1 + x_j y_k)_{j,k=1}^n$. If $\alpha \in \{0, 1, \dots, n-2\}$, then $S^{\circ\alpha}$ has rank $\alpha + 1$.*

Once again, the short proof is outlined in the Appendix.

In this and later sections, we provide applications of Proposition 3.1: to our main theorems, as well as to Jain's theorem 1.12. All of these applications also rely on the (novel) homotopy argument in Proposition 2.3; this keeps the proofs in this paper self-contained. We begin with Theorem 1.12, as it is used in the subsequent proofs.

Proof of Theorem 1.12.

(1) If $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-2,\infty)$ then $A^{\circ\alpha} \in \mathbb{P}_n$ by Theorem 1.1(1) (the proof of which is outlined in the Appendix). We will need a refinement of the converse result, so we sketch this argument, taken from [17]. The result is easily shown for $\alpha < 0$, so we suppose $\alpha \in (0, n-2) \setminus \mathbb{Z}$ – in particular, $n \geqslant 3$ now. Let $\mathbf{x} = \mathbf{x}(\epsilon) := \epsilon(1,2,\dots,n)^T$ with $\epsilon > 0$, and choose any vector $v \in \mathbb{R}^n$ that is orthogonal to $\mathbf{1}, \mathbf{x}, \mathbf{x}^{\circ 2}, \dots, \mathbf{x}^{\circ(\lfloor\alpha\rfloor+1)}$ but not to $\mathbf{x}^{\circ(\lfloor\alpha\rfloor+2)}$. (Here, $\mathbf{x}^{\circ m} = \epsilon^m (1,\dots,n^m)^T$ for an integer m.) Now using binomial series, one computes:

$$v^T(\mathbf{1}_{n\times n} + \mathbf{x}\mathbf{x}^T)^{\circ\alpha} v = \epsilon^{2(\lfloor\alpha\rfloor+2)} \binom{\alpha}{\lfloor\alpha\rfloor+2} (v^T \mathbf{x}^{\circ(\lfloor\alpha\rfloor+2)})^2 + o(\epsilon^{2(\lfloor\alpha\rfloor+2)}).$$

Divide by $\epsilon^{2(\lfloor\alpha\rfloor+2)}$ and let $\epsilon \to 0^+$; as the right-hand side has a negative limit, the matrix-power on the left cannot be positive semidefinite.

With this special case at hand, the general case follows, via a more direct argument than in [34, 35]. Given pairwise distinct x_j such that $1 + x_j x_k > 0 \ \forall j,k$, let $y_j := \epsilon j$, where $\epsilon > 0$ is small enough to satisfy both the argument in the preceding paragraph, as well as the conclusions of Proposition 2.3. Now let $x_j(t) := x_j + t(\epsilon j - x_j)$ and let $C(t) := (1 + x_j(t) x_k(t))^{\circ\alpha}$. Then the smallest eigenvalue $\lambda_{\min}(C(1)) < 0$ from above, and $C(t)$ is always non-singular by Proposition 3.1. It follows by the continuity of eigenvalues (or a simpler, direct argument) that $\lambda_{\min}(C(0)) < 0$, as desired.

(2) We show the 'if' part of Theorem 1.1(2) from [17] for self-completeness (and also because it is used presently). If $\alpha \in \mathbb{Z}^{\geqslant 0}$ and $C \geqslant D \geqslant 0$ in $\mathbb{P}_n((0,\infty))$, then

$$C^{\circ\alpha} \geqslant C^{\circ(\alpha-1)} \circ D \geqslant \cdots \geqslant D^{\circ\alpha},$$

by the Schur product theorem.[4] If $\alpha \geqslant n-1$, then by the fundamental theorem of calculus,

$$C^{\circ\alpha} - D^{\circ\alpha} = \alpha \int_0^1 (C - D) \circ (\lambda C + (1-\lambda) D)^{\circ(\alpha-1)} \, d\lambda.$$

By Theorem 1.1(1) and the Schur product theorem, the integrand is positive semidefinite, whence we are done.

The 'only if' part of Theorem 1.1(2) follows from Theorem 1.12(2), which is immediate from the preceding part: Suppose $A^{\circ\alpha} \geqslant B = B^{\circ\alpha}$, with $A = (1 + x_j x_k)_{j,k=1}^n$ and $B = \mathbf{1}$ as given. If $\mathbf{x}' := (\mathbf{x}^T, 0)^T \in \mathbb{R}^{n+1}$, then the matrix

$$\widetilde{A} := \mathbf{1}_{(n+1)\times(n+1)} + \mathbf{x}'(\mathbf{x}')^T = \begin{pmatrix} A & \mathbf{1} \\ \mathbf{1}^T & 1 \end{pmatrix}$$

satisfies the hypotheses of part (1). Using Schur complements and part (1), we thus have:

$$A^{\circ\alpha} \geqslant \mathbf{1}_{n\times n} \iff \widetilde{A}^{\circ\alpha} \in \mathbb{P}_{n+1} \iff \alpha \in \mathbb{Z}^{\geqslant 0} \cup [n-1,\infty). \qquad \square$$

This concludes a self-contained (modulo the Appendix) proof of Theorem 1.12, avoiding the use of SSR matrices as in [34, 35] (see Remark 2.4). A key corollary, used repeatedly below, now strengthens Theorem 1.12 from positive (semi)definiteness to total positivity, as promised above:

Corollary 3.2. *Let $p \geqslant 2$ be an integer, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$ be tuples such that $1 + x_j y_k > 0$ for all j,k. Let the matrix $C := (1 + x_j y_k)_{j,k=1}^p$.*

(1) If $\alpha > p-2$ then $C^{\circ\alpha}$ is TP.
(2) If $\alpha \in \{0,1,\dots,p-2\}$, then $C^{\circ\alpha}$ has rank $\alpha+1$.
(3) If $\alpha \in (0,p-2) \setminus \mathbb{Z}$, then $C^{\circ\alpha}$ is not TN – in fact, it has a principal minor that is negative.

See also Corollary 5.4 below, for a stronger version with more detailed information. (This corollary is not required in the present work, but we include it below for completeness.)

[4] The Schur product theorem [59] says that if $A, B \in \mathbb{P}_n(\mathbb{R})$, then so is their entrywise product $A \circ B := (a_{jk} b_{jk})_{j,k=1}^n$. (For self-completeness: This is easily checked using the spectral eigen-decompositions of A, B.)

Proof. The second part follows from Proposition 3.1. For the third, fix any tuple $\mathbf{q} \in (0,\infty)^{p,\uparrow}$, and use Proposition 2.6 to construct piecewise linear homotopies $\mathbf{x}(t), \mathbf{y}(t),\ t \in [0,1]$ from $\mathbf{x}, \mathbf{y}$ to $\mathbf{q}$ respectively, such that $1 + x_j(t)y_k(t) > 0$ for all $1 \leqslant j,k \leqslant p$ and $t \in [0,1]$. Let $C(t) := \mathbf{1}_{p\times p} + \mathbf{x}(t)\mathbf{y}(t)^T$. Then $C(1)^{\circ\alpha} = (\mathbf{1}_{p\times p} + \mathbf{q}\mathbf{q}^T)^{\circ\alpha}$ is not positive semidefinite by Theorem 1.12(1), hence has a negative principal minor. Now use Proposition 2.6 to show that the same principal minor of $C(0)^{\circ\alpha} = C^{\circ\alpha}$ is negative, again by Proposition 3.1.

For the first part, let B be any square submatrix of C of order $p' \in [2,p]$; then one can repeat the preceding argument with $C = B$ for this part. Thus, let $\mathbf{q}$ and $C(t)_{p'\times p'} = B(t)$ be as in the previous paragraph. Since now $\alpha > p' - 2$, so $\det B(t)^{\circ\alpha}$ does not change sign, by Proposition 3.1. But $\det B(1)^{\circ\alpha} > 0$ by Theorem 1.12(1). This shows that every minor of the original matrix $C^{\circ\alpha}_{p\times p}$ is positive, whence $C^{\circ\alpha}$ is TP. □

With this and the preceding ingredients at hand, we show our first main result.

Proof of Theorem A.

(1) We first show the result for $X = Y = \mathbb{R}$. Notice in this case that the result for any $a \in \mathbb{R}$ shows the result for any other, so we work with $a = 0$. Suppose $\alpha \in (0, p-2) \setminus \mathbb{Z}$, and

$$0 < v_p < \cdots < v_1 < u_p < \cdots < u_1$$

are fixed scalars. Set $x_j := u_j^{-1}$ and $y_k := -v_k$; thus $x_j > 0$ and $1 + x_j y_k > 0$ for all j,k. By (the proof of) Corollary 3.2(3), the matrix $C := ((1 + x_j y_k)^\alpha)_{j,k=1}^p$ has a negative minor, hence is not TN. Pre- and post-multiply by diagonal matrices with (j,j) entry $u_j^\alpha e^{-\alpha u_j}$ and $e^{\alpha v_j}$ respectively. This shows, via applying the order-reversing permutation to the rows and to the columns, that given

$$\mathbf{u}' := (u'_1, \dots, u'_p),\ \mathbf{v}' := (v'_1, \dots, v'_p) \in \mathbb{R}^{p,\uparrow}, \quad \text{with} \quad v'_p < u'_1,$$

the matrix $T_{\Omega^\alpha}[\mathbf{u}'; \mathbf{v}'] = T_{\Omega^\alpha}[\mathbf{u}' - (v'_1 - 1)\mathbf{1}; \mathbf{v}' - (v'_1 - 1)\mathbf{1}]$ has a submatrix with negative determinant.

This shows the result when $X = Y = \mathbb{R}$, e.g. with $a = 0$. For arbitrary $X, Y \subset \mathbb{R}$ of sizes at least p, first choose and fix increasing p-tuples $\mathbf{u}' \in X^{p,\uparrow}, \mathbf{v}' \in Y^{p,\uparrow}$; now choose any $a < u'_1 - v'_p$. By the above proof, the matrix $T_{\Omega_a(x)^\alpha}[\mathbf{u}'; \mathbf{v}'] = T_{\Omega^\alpha}[\mathbf{u}' - a\mathbf{1}; \mathbf{v}']$ is not TN_p. This shows the result for all X, Y.

(2) Choose $m > 0$ and tuples $\mathbf{x} \in X^{p,\uparrow}$, $\mathbf{y} \in Y^{p,\uparrow}$ such that $|mx_j|, |my_j| < \pi/4$ for all j. Now,

$$T_{W_m}[\mathbf{x};\mathbf{y}]^{\circ\alpha} = (\cos(mx_j - my_k)^\alpha)_{j,k=1}^p = D_{\mathbf{x}}(1 + \tan(mx_j)\tan(my_k))^{\circ\alpha} D_{\mathbf{y}},$$

where $D_{\mathbf{x}}$ for a vector $\mathbf{x}$ equals $\operatorname{diag}(\cos(mx_j)^\alpha)_j$. Now since $m\mathbf{x}, m\mathbf{y}$ have increasing coordinates, all in $(-\pi/4, \pi/4)$, Corollary 3.2(3) applies to show that $T_{W_m}^\alpha$ is not TN_p.

(3) The previous two parts in fact show the case of $\alpha \in (0, p-2) \setminus \mathbb{Z}$. The other two cases follow by using similar arguments, via Corollary 3.2(1),(2). □

With Theorem A at hand, we show that the same set of powers (shifted by 1) works to 'preserve TN_p' on Laplace transforms of Pólya frequency functions. Notably, the following result accommodates *all* PF functions – in the spirit of Theorem B – and not just individual ones as in Theorem A.

Corollary 3.3. *Given scalars $\alpha \geqslant 1$ and $a_0 > 0$ and an integer $p \geqslant 2$, the following are equivalent.*

(1) If Λ is a one-sided Pólya frequency function (i.e. one that vanishes on a semi-axis), then $\mathcal{B}\{\Lambda\}^\alpha$ is the Laplace transform of a TN_p function.

(2) The exponent $\alpha \in \mathbb{Z}^{>0} \cup (p-1, \infty)$.

The implication (2) $\implies$ (1) was proved by Karlin in his book – see [39, Chapter 7, Theorem 12.2], and his (short) proof is included in the Appendix for completeness.

Proof. In the spirit of this paper, we first reduce the test set in (1) to a single Pólya frequency function, and show (2). For the kernel $\lambda_0(x) = e^{-x}\mathbf{1}_{x>0}$ – which is shown to be a Pólya frequency function in Lemma 7.1 below – standard computations reveal:

$$\mathcal{B}\{\lambda_0\}(s)^\alpha = \frac{1}{(s+1)^\alpha} = \mathcal{B}\{\Lambda_\alpha\}(s), \qquad \text{where } \Lambda_\alpha(x) = \frac{e^{(\alpha-2)x}\Omega(x)^{\alpha-1}}{\Gamma(\alpha)}.$$

Since $\alpha \geqslant 1$, Laplace inversion yields: if $\mathcal{B}\{\Lambda\}(s) = (s+1)^{-\alpha}$, then $\Lambda = \Lambda_\alpha$. Thus Λ_α is TN_p, whence so is $\Omega(x)^{\alpha-1}$ (e.g. see the argument that concludes the proof of Lemma 7.1). Now Theorem A shows $\alpha - 1 \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$, which implies (2). ☐

3.1. **Non-degenerate extension of Karlin's result.** We next formulate (and prove) the strengthening of Karlin's theorem 1.2 promised above. Karlin was studying the function $\Omega(x) := xe^{-x}\mathbf{1}_{x>0}$. This function has Laplace transform $1/(1+s)^2$, and is an example of a non-smooth, one-sided Pólya frequency function, as shown by Schoenberg in [57]. More generally, given scalars $q, r > 0$, define

$$\Omega^{(q,r)}(x) := \begin{cases} \dfrac{qr(e^{-qx} - e^{-rx})}{r-q}, & \text{if } x > 0 \text{ and } q \neq r, \\ r^2 x e^{-rx}, & \text{if } x > 0 \text{ and } q = r, \\ 0, & \text{if } x \leqslant 0. \end{cases}$$

By inspection, $\Omega^{(q,r)}(x) = \Omega^{(r,q)}(x) \to \Omega^{(r,r)}(x)$ as $q \to r$. Moreover, for all $q, r > 0$ the map $\Omega^{(q,r)}$ is a probability density function for all $q, r > 0$; and it has bilateral Laplace transform

$$\mathcal{B}\{\Omega^{(q,r)}\}(s) = \frac{qr}{(s+q)(s+r)}.$$

Thus by Schoenberg's representation theorems [57], $\Omega^{(q,r)}$ is a Pólya frequency function for all $q, r > 0$. These functions were studied by Hirschman and Widder [29, 30] in greater generality; Schoenberg [57] showed that they are not smooth at the origin; and their powers are also studied in recent joint work [4]. In this paper we restrict ourselves to working with $\Omega^{(q,r)}(x)$.

Notice that $\Omega^{(q,r)}(x) = \Omega(x)$ is Karlin's kernel when the parameters specialize to $q = r = 1$. The following result extends Theorems A and 1.2 to all other $q, r > 0$:

Theorem 3.4. *Fix an integer $p \geqslant 2$ and subsets $X, Y \subset \mathbb{R}$ of size at least p. Also fix real numbers $q, r > 0$.*

1. *If $\alpha \geqslant p - 2$, then $\Omega^{(q,r)}(x)^\alpha$ is TN_p on $X \times Y$.*
2. *There exists $a = a(X,Y) \in \mathbb{R}$ such that the restriction of $T_{\Omega_a^{(q,r)}}(x,y)^\alpha$ to $X \times Y$ (where $\Omega_a^{(q,r)}(x) = \Omega^{(q,r)}(x-a)$), is not TN_p for all $\alpha \in (0, p-2) \setminus \mathbb{Z}$.*
3. *Given tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$, there exist $a \in \mathbb{R}$ such that the matrix*
$$(\Omega^{(q,r)}(x_j - y_k - a)^\alpha)_{j,k=1}^p$$
is TP if $\alpha > p - 2$, TN if $\alpha \in \{0, 1, \dots, p-2\}$, and not TN if $\alpha \in (0, p-2) \setminus \mathbb{Z}$.

Clearly, setting $q = r = 1$ yields the corresponding assertions for $\Omega(x)$ in Theorem A. Similarly, we deduce from this result the following extension of Corollary 1.9:

Corollary 3.5. *Fix real scalars $q, r > 0$ and $\alpha \geqslant 0$. The function $\Omega^{(q,r)}(x)^\alpha$ is* TN *if and only if α is an integer.*

Proof. If $\Omega^{(q,r)}(x)^\alpha$ is TN_p for all $p \geqslant 2$, then the preceding theorem yields $\alpha \in \mathbb{Z}^{\geqslant 0}$. Conversely, using $0^0 := 0$ we have that $\Omega^{(q,r)}(x)^\alpha$ is indeed TN for $\alpha = 0, 1$. Suppose $\alpha > 1$ is an integer. Clearly $\Omega^{(q,r)}(x)^\alpha$ is integrable and non-vanishing on $(0, \infty)$, so it suffices to show it is a TN function. This holds for $q = r$ by the same proof as for Theorem 1.2 for integer powers (see the Appendix). If instead $q \neq r$, then we directly compute its Laplace transform:

$$\mathcal{B}\{(\Omega^{(q,r)})^\alpha\}(s) = \frac{(qr)^\alpha}{(r-q)^\alpha} \sum_{j=0}^\alpha \binom{\alpha}{j} \frac{(-1)^j}{s + jr + (\alpha - j)q} = \frac{f(s)}{g(s)},$$

say. Here the polynomial $g(s) := \prod_{j=0}^\alpha (s + jr + (\alpha - j)q)$, and $f(s)$ is written by taking common denominators:

$$f(s) := \frac{(qr)^\alpha}{(r-q)^\alpha} \sum_{j=0}^\alpha \binom{\alpha}{j} (-1)^j \prod_{k \neq j} (s + kr + (\alpha - k)q).$$

Clearly, $\deg(f) \leqslant \alpha$; but a straightforward computation shows that at the $\alpha + 1$ points $s_0 = -(kr + (\alpha - k)q)$, $k = 0, 1, \dots, \alpha$, we have $f(s_0) = \alpha!(qr)^\alpha$. Hence $f(s)$ is a constant and $g(s)/f(s)$ is a polynomial, whence in the Laguerre–Pólya class. It follows that $\Omega^{(q,r)}(x)^\alpha$ is indeed a Pólya frequency function via Schoenberg's characterization of such functions [57]. □

It remains to prove the above extension of Theorems A and 1.2.

Proof of Theorem 3.4. When $q = r$, the result follows from Theorems A and 1.2 by a change of scale. Thus we assume henceforth – without loss of generality – that $0 < q < r$.

(1) We show the assertion for $X = Y = \mathbb{R}$. Suppose $\alpha \geqslant p - 2$, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$ for $p \geqslant 1$. Set $C := (\Omega^{(q,r)}(x_j - y_k)^\alpha)_{j,k=1}^p$, and define

$$u_j := -e^{(q-r)x_j}, \ v_j := e^{(r-q)y_j}, \qquad 1 \leqslant j \leqslant p.$$

Then $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{p,\uparrow}$, and a straightforward computation shows that the matrix C can be described using the Jain–Karlin–Schoenberg kernel:

$$C = T_{\Omega^{(q,r)}}[\mathbf{x}; \mathbf{y}]^{\circ\alpha} = D^{\circ\alpha} K_{\mathcal{JKS}}[\mathbf{u}; \mathbf{v}]^{\circ\alpha} D_1^{\circ\alpha},$$

where D, D_1 are diagonal matrices

$$D = \frac{qr}{r-q} \operatorname{diag}(e^{-qx_1}, \dots, e^{-qx_p}), \qquad D_1 = \operatorname{diag}(e^{qy_1}, \dots, e^{qy_p}).$$

It follows by Theorem C (proved below) that C is TN, as desired.

(2) As in the proof of Theorem A, this part follows from the next, by fixing p elements $x_j \in X$ and $y_j \in Y$.

(3) Choose $a \leqslant x_1 - y_p$. Using $u_j := -e^{(q-r)x_j}$ and $D_{p \times p}$ as in the first part, and setting

$$v'_k := e^{(r-q)(y_k + a)}, \qquad D'_1 = \operatorname{diag}(e^{q(y_1+a)}, \dots, e^{q(y_p+a)}),$$

it follows that $\mathbf{u}, \mathbf{v}' \in \mathbb{R}^{p,\uparrow}$, that $1 + u_j v'_k > 0$ for all j, k, and furthermore that

$$C' = T_{\Omega_a^{(q,r)}}[\mathbf{x}; \mathbf{y}]^{\circ\alpha} = D^{\circ\alpha}((1 + u_j v'_k)^\alpha)_{j,k=1}^p (D'_1)^{\circ\alpha}.$$

Now the result follows from Corollary 3.2, akin to the proof of Theorem A(3). □

3.2. **Single-matrix encoders of Loewner convexity.** As an application of the methods used above, we provide single-matrix encoders of the entrywise powers preserving Loewner convexity. Recall for $I \subset \mathbb{R}$ that a function $f : I \to \mathbb{R}$ preserves *Loewner convexity* on a set $V \subset \mathbb{P}_n(I)$ if $f[\lambda A + (1-\lambda)B] \leqslant \lambda f[A] + (1-\lambda) f[B]$ whenever $\lambda \in [0,1]$ and $A \geqslant B \geqslant 0$ in V.

The powers preserving Loewner convexity were classified by Hiai in 2009:

Theorem 3.6 (Hiai, [28]). *Let $n \geqslant 2$ be an integer and $\alpha \in \mathbb{R}$. The entrywise map x^α preserves Loewner convexity on $\mathbb{P}_n([0,\infty))$ if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n, \infty)$.*

In the spirit of Theorem 1.12, we provide single-matrix encoders of these powers:

Theorem 3.7. *Let $n \geqslant 2$ be an integer and $\alpha \in \mathbb{R}$. Suppose $x_1, \dots, x_n \in \mathbb{R}$ are pairwise distinct, non-zero scalars such that $1 + x_j x_k > 0$ for all j, k. Let $A := (1 + x_j x_k)_{j,k=1}^n$ and $B := \mathbf{1}_{n\times n}$. Then x^α preserves Loewner convexity on $A \geqslant B \geqslant 0$ if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [n, \infty)$.*

The proof relies on the following preliminary lemma, which can be shown by an argument of Hiai – see the Appendix.

Lemma 3.8. *Let $n \geqslant 2$ and $A \geqslant B \geqslant 0$ in $\mathbb{P}_n(\mathbb{R})$ be such that $A - B = uu^T$ has rank one and no non-zero entries. Choose an open interval $I \subset \mathbb{R}$ containing the entries of A, B, and suppose $f : I \to \mathbb{R}$ is differentiable. If the entrywise map $f[-]$ preserves Loewner convexity on the interval*

$$[B, A] := \{\lambda A + (1 - \lambda)B : \lambda \in [0, 1]\}$$

then $f'[-]$ preserves Loewner monotonicity on $[B, A]$. The converse holds for arbitrary matrices $0 \leqslant B \leqslant A$.

We now prove Theorem 3.7 – in the process also proving Hiai's result:

Proof of Theorems 3.7 and 3.6. By Lemma 3.8 and Theorem 1.1(2), x^α preserves Loewner convexity on $\mathbb{P}_n((0,\infty))$ for $\alpha \in \mathbb{Z}^{>0} \cup [n, \infty)$, and obviously so for $\alpha = 0$. The result for $\mathbb{P}_n([0,\infty))$ follows by continuity. Next, if x^α preserves Loewner convexity on $\mathbb{P}_n([0,\infty))$, then it does so on the given matrices $A \geqslant B \geqslant 0$. Finally, if the latter condition holds and $\alpha \notin \mathbb{Z}^{\geqslant 0}$, then Lemma 3.8 applies, so $\alpha \geqslant n$ via Theorem 1.12(2). □

4. Hankel TN_p kernels: preservers, critical exponent, and Theorem B

In this section we first prove Theorem B. The key tool is a result of Fekete from 1912 [16], subsequently strengthened by Schoenberg in 1955 [58]:

Lemma 4.1. *Suppose $1 \leqslant p \leqslant m, n$ are integers, and $A \in \mathbb{R}^{m\times n}$. Then A is TP_p if and only if all contiguous minors of orders $\leqslant p$ are positive. (Here, 'contiguous' means that the rows and columns for the minor are both consecutive.)*

The proof is not too long, relying on computational lemmas by Gantmacher and Krein. See [20].

Corollary 4.2. *Suppose $1 \leqslant p \leqslant n$ are integers and $A \in \mathbb{R}^{n\times n}$ is Hankel. Let $A^{(1)}$ denote the truncation of A, i.e. the submatrix with the first row and last column of A removed. Then A is TN_p (respectively TP_p) if and only if every contiguous principal minor of A and of $A^{(1)}$ of size $\leqslant p$ is non-negative (respectively positive).*

This result can be found in [50, Chapter 4] for the TP case, and in [14] for the TP_p, TN, TN_p cases. These sources do not use the word 'contiguous' – the advantage of using contiguous (principal) minors is that they are all Hankel. We provide a quick proof of Corollary 4.2 in the Appendix, for self-completeness. For now, we apply this result to prove Theorem B and other results. The relevant part of this argument is isolated into the following standalone result.

Proposition 4.3. *Suppose $p \geqslant 2$ is an integer, $X \subset \mathbb{R}$ is an interval with positive measure, and $H : X \times X \to \mathbb{R}$ is a continuous Hankel TN_p kernel. If $f : [0,\infty) \to [0,\infty)$ is continuous at 0^+, and preserves positive semidefiniteness when acting entrywise on $r \times r$ Hankel matrices for $1 \leqslant r \leqslant p$, then $f \circ H : X \times X \to \mathbb{R}$ is continuous, Hankel, and TN_p.*

Proof. The first step is to show that f is continuous on $(0,\infty)$; this quickly follows e.g. from work of Hiai [28], and is sketched in the Appendix for completeness. (A longer proof is via using a 1929 result of Ostrowski; see e.g. [5].) Thus $f \circ H$ is continuous and Hankel on $X \times X$.

Now let $2 \leqslant r \leqslant p$ and choose $\mathbf{x}, \mathbf{y} \in X^{r,\uparrow}$. We need to show $\det(f \circ H)[\mathbf{x}; \mathbf{y}] \geqslant 0$. Let $\mathbf{u} = (u_1, \dots, u_m)$ denote the ordered tuple whose coordinates are the union of the x_j, y_k (without repetitions). We claim that $(f \circ H)[\mathbf{u}; \mathbf{u}]$ is TN_p; this would suffice to complete the proof.

To show the claim, approximate the increasing tuple $\mathbf{u}$ by tuples $\mathbf{u}^{(k)} \in (X \cap \mathbb{Q})^{m,\uparrow}$ of rational numbers in X, with $\mathbf{u}^{(k)} \to \mathbf{u}$ as $k \to \infty$. Choose integers $N_k > 0$ such that $N_k \mathbf{u}^{(k)}$ has integer coordinates. Now if the matrices

$$(f \circ H)[\mathbf{v}_k; \mathbf{v}_k], \quad \text{where} \quad \mathbf{v}_k := (u_1^{(k)}, u_1^{(k)} + \frac{1}{N_k}, \dots, u_m^{(k)})$$

can be shown to be TN_p, then by taking submatrices and the limit as $k \to \infty$, it follows that $(f \circ H)[\mathbf{u}; \mathbf{u}]$ is TN_p, as claimed. We use here that f is continuous.

Since each $\mathbf{v}_k$ is an arithmetic progression, it is easy to see that the matrices $A_k := H[\mathbf{v}_k; \mathbf{v}_k]$ are Hankel, and TN_p because H is so. Now observe that all contiguous principal submatrices C of A_k or of $A_k^{(1)}$ of size $2 \leqslant r \leqslant p$ are symmetric Hankel positive semidefinite matrices. Thus $f[C]$ is positive semidefinite by assumption, hence has determinant $\geqslant 0$. It follows by Corollary 4.2 that $(f \circ H)[\mathbf{v}_k; \mathbf{v}_k]$ is TN_p for all k, and this completes the proof. ☐

With Proposition 4.3 and the previous results at hand, our next main result follows.

Proof of Theorem B. The first step is to verify that H_{u_0} is TN; this is easy because H_{u_0} has 'rank two', being the moment sequence/kernel of the two-point measure $\delta_1 + c_0 \delta_{u_0}$, so all $r \times r$ minors vanish for $r \geqslant 3$. We next prove a chain of cyclic implications. Clearly (1) $\implies$ (2), and (3) $\implies$ (1) by Proposition 4.3 and Theorem 1.1(1). Finally, suppose $\alpha \notin \{0, 1, \dots, p-2\}$. Choose tuples $\mathbf{x}, \mathbf{y} \in {X_0}^{p,\uparrow}$ and apply Corollary 3.2 with x_j, y_j replaced by $\sqrt{c_0} u_0^{x_j}, \sqrt{c_0} u_0^{y_j}$ respectively; we also reverse the rows and columns if $u_0 \in (0,1)$. This yields a TN matrix $H_{u_0}[\mathbf{x}; \mathbf{y}]$, whose αth entrywise power is not TN_p if $\alpha \in (0, p-2) \setminus \mathbb{Z}$, and is TP if $\alpha > p - 2$. This shows both (2) $\implies$ (3) as well as the remaining assertions. ☐

For the curious reader, Theorem B leads to a question about Toeplitz analogues that may be of theoretical interest. One can ask if this 'clean' phenomenon holds for the parallel class of Toeplitz kernels – namely, if for all integers $p \geqslant 2$, the TN_p-preserving powers x^α are precisely $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$. This is easily verified to hold for $p = 2$; see e.g. [57] (or Lemma 6.1 below), where TN_2 functions are characterized as exponentials of concave functions. However, such a clean result fails to hold in general. Specifically, considering the question from the 'dual' viewpoint of the powers α: while x^α for $\alpha = 0, 1$ obviously preserves TN_p for all p, this fails to hold for every other integer $\alpha \geqslant 2$. Namely, one can find a TN_p kernel (for some $p \geqslant 0$), whose αth power is not TN_p. This can be refined further, to work with a single kernel – which is moreover TN – that provides a counterexample for all integer powers:

Lemma 4.4. *There exists a Pólya frequency function $M : \mathbb{R} \to \mathbb{R}$, such that for every integer power $\alpha \geqslant 2$, there exists an integer $p(\alpha) \geqslant 1$ satisfying: M^α is not* $\text{TN}_{p(\alpha)}$.

Proof. Let $M(x) := 2e^{-|x|} - e^{-2|x|}$ for $x \in \mathbb{R}$. It was shown in [3] that M^α is not TN for any $\alpha \geqslant 2$, while M is. (See the Appendix for details.) This proves the result. ☐

In light of Lemma 4.4, one can ask more refined questions, e.g. if all non-integer powers $\alpha > p-2$ preserve TN_p Toeplitz functions/kernels, with $p \geqslant 4$. A challenge in tackling such questions comes from the lack of a well-developed theory for Pólya frequency functions of finite order, i.e., integrable TN_p functions. For instance, to our knowledge there was no known characterization to date of Pólya frequency functions of order $p = 4, 5, \dots$. (While Theorem E now fulfils this need, it does not provide enough information to help here.)

Remark 4.5. In light of Lemma 4.4 and the above results, one can also ask about the classification of powers – or more generally, arbitrary functions – that preserve the class of TN kernels, whether Hankel or Toeplitz, upon composing. These characterizations were recently achieved in joint work [3]: for continuous Hankel kernels, the preservers are precisely the convergent power series with non-negative Maclaurin coefficients (see also Lemma 4.8), while for Toeplitz kernels, the preservers are precisely constants c or homotheties cx or Heaviside functions $c\mathbf{1}_{x>0}$, with $c \geqslant 0$.

4.1. **Connection to fixed-dimension results on positivity preservers.** Given an integer $p \geqslant 1$ and a subset $I \subset \mathbb{R}$, let $\mathbb{P}_p(I)$ denote the set of real symmetric $p \times p$ matrices, which are positive semidefinite and have all entries in I. The critical exponent phenomena studied above suggest that TN_p-preservers are closely related to entrywise functions preserving positive semidefiniteness on $\mathbb{P}_p((0,\infty))$ – especially for Hankel kernels, in light of Proposition 4.3. Although our focus in this paper is on powers, we briefly digress to point out a few such connections. The first is Loewner's necessary condition for preserving positivity on such matrices:

Theorem 4.6 (Loewner / Horn, 1969, [32])**.** *Suppose $I = (0,\infty)$, $f : I \to \mathbb{R}$ is continuous, and $p \geqslant 3$ is an integer such that $f[-]$ applied entrywise to matrices in $\mathbb{P}_p(I)$ preserves positivity. Then $f \in C^{p-3}(I)$, $f^{(p-3)}$ is convex on I, and $f, f', \dots, f^{(p-3)} \geqslant 0$ on I. If in particular $f \in C^{p-1}(I)$, then $f^{(p-2)}, f^{(p-1)} \geqslant 0$ on I as well.*

We claim that the same conclusions hold if f preserves the TN_p Hankel kernels – in fact on a far smaller test set, and without the continuity assumption from [32]:

Theorem 4.7. *Suppose $I = (0,\infty)$, $f : I \to \mathbb{R}$, and $X_0 \subset \mathbb{R}$ is any interval with positive measure. Suppose $p \geqslant 3$ is an integer such that the post-composition transform $f \circ -$ preserves* TN_p *on Hankel* TN *kernels corresponding to non-negative measures supported on at most two points. Then the conclusions of Theorem 4.6 hold.*

That this result is sharp – in the number of non-negative derivatives $f, \dots, f^{(p-1)}$ on I – follows from Theorem B, by considering a suitable power function f.

Proof. We appeal to results in [5], which assert that if $f[-]$ preserves positivity on the matrices

$$(a_0 + c_0 u_0^{j+k})_{j,k=0}^{p-1}, \qquad a_0, c_0 \geqslant 0,\ a_0 + c_0 > 0,$$
$$\begin{pmatrix} a & b \\ b & b \end{pmatrix}, \begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix}, \qquad a,b,c,d > 0,\ a > b > 0,\ c \geqslant d > 0$$

for some fixed $u_0 \in (0,1)$, then f satisfies the conclusions of Theorem 4.6. It thus suffices to embed these test matrices in TN Hankel kernels. We do so on $\mathbb{R} \times \mathbb{R}$; the restriction to $X_0 \times X_0$ follows by a linear change of variables that contains an appropriate compact sub-interval of $\mathbb{R}$. The first class of test matrices above embeds in the Hankel kernels

$$H_{a_0,c_0}(x,y) := a_0 + c_0 u_0^{x+y}, \qquad x, y \in \mathbb{R},$$

for $a_0, c_0 \geqslant 0$, while the 'rank-one' matrices above embed in the kernel $H_{c,0}$ if $c = d$, and in H_{0,c^2} with $u_0 = d/c$, if $c > d > 0$. Recently in [3], the remaining class of matrices $\begin{pmatrix} a & b \\ b & b \end{pmatrix}$ above was shown to embed in the following 'rank-two' TN Hankel kernel, which completes the proof:

$$\frac{(2a-b)^2}{4a-3b} \left(\frac{b}{2a-b} \right)^{x+y} + \frac{b(a-b)}{4a-3b} 2^{x+y}, \qquad x, y \in \mathbb{R}. \qquad \square$$

The next connection is to an – even older – observation of Pólya and Szegő [51] from 1925:

Lemma 4.8. *Suppose f_0 is the restriction to $[0,\infty)$ of an entire function with non-negative Maclaurin coefficients. Then $f_0 \circ -$ preserves the class of continuous* TN_p *Hankel kernels on $X \times X$, for all integers $p \geqslant 1$ and intervals $X \subset \mathbb{R}$.*

Proof. By the Schur product theorem, x^k entrywise preserves positivity on $\mathbb{P}_p([0,\infty))$ for all integers $k \geqslant 0$; here we set $0^0 := 1$. Since $\mathbb{P}_p([0,\infty))$ is a closed convex cone, it follows that all functions f_0 as in the lemma share the same property. We are now done by Proposition 4.3. $\square$

Our third connection is to entrywise polynomials that preserve TN_p. By the preceding lemma, all power series with non-negative coefficients preserve TN_p on continuous Hankel TN_p kernels. It is natural to ask is if a wider class of polynomials shares this property.[5] We conclude this section by providing a positive answer, essentially coming from recent joint work with Tao [40]:

Theorem 4.9. *Let $p > 0$ and $0 \leqslant n_0 < \cdots < n_{p-1} < M < n_p < \cdots < n_{2p-1}$ be integers, and let $c_{n_0}, \ldots, c_{n_{2p-1}} > 0$ be reals. There exists a negative number c_M such that the polynomial*

$$x \mapsto c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{p-1}} x^{n_{p-1}} + c_M x^M + c_{n_p} x^{n_p} + \cdots + c_{n_{2p-1}} x^{n_{2p-1}},$$

preserves the continuous Hankel TN_p kernels on $X \times X$, for intervals $X \subset \mathbb{R}$ with positive measure.

Via Proposition 4.3, Theorem 4.9 follows from [40], because such a polynomial was shown in *loc. cit.* to preserve Loewner positivity on $\mathbb{P}_p([0,\infty))$. Theorem 4.9 also admits extensions to power series and more general preservers; we refer the interested reader to [40] for further details.

5. Theorem C: Critical exponent for total positivity of the Jain–Karlin–Schoenberg kernel

We next show Theorem C on the total non-negativity of the powers of the kernel $K_{\mathcal{JKS}}$, and explain how it connects to the (total) positivity results stated before it in the opening section.

Proof of Theorem C. The second part follows from Corollary 3.2. For the first, begin with the basic trigonometric fact: *If $-\pi/2 < \varphi < \theta < \pi/2$, then $\tan(\theta)\tan(\varphi) > -1$ if and only if $\theta - \varphi < \pi/2$.*

Now let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$ and let $u_j := \tan^{-1}(x_j),\ v_j := \tan^{-1}(y_j)$. Then $\mathbf{u}, \mathbf{v} \in (-\pi/2, \pi/2)^{p,\uparrow}$, so:

$$\begin{aligned} K_{\mathcal{JKS}}(x_j, y_k) &= (1 + \tan(u_j)\tan(v_k)) \mathbf{1}_{\tan(u_j)\tan(v_k) > -1} \\ &= (1 + \tan(u_j)\tan(v_k)) \mathbf{1}_{|u_j - v_k| < \pi/2} \\ &= \sec(u_j)\sec(v_k) \left[\cos(u_j - v_k) \mathbf{1}_{|u_j - v_k| < \pi/2} \right] \\ &= \sec(u_j)\sec(v_k) T_W(u_j, v_k). \end{aligned}$$

It follows that

$$K_{\mathcal{JKS}}[\mathbf{x}; \mathbf{y}]^{\circ\alpha} = D_{\mathbf{u}}^{\alpha} T_W[\mathbf{u}; \mathbf{v}]^{\circ\alpha} D_{\mathbf{v}}^{\alpha}, \qquad \forall \alpha \geqslant 0 \tag{5.1}$$

where $D_{\mathbf{u}}$ for a vector $\mathbf{u} \in (-\pi/2, \pi/2)^{p,\uparrow}$ is the diagonal matrix with (j,j) entry $\sec(u_j)$. Theorem 1.5 now implies that this matrix is TN if $\alpha \geqslant p - 2$, proving the first part.

Finally, we show the third part. Since the kernel $K_{\mathcal{JKS}}$ is invariant under the automorphism group generated by the involutions $x \leftrightarrow y$ and $(x, y) \leftrightarrow (-x, -y)$, it suffices to show that the restriction to $[0, \infty) \times \mathbb{R}$ of $K_{\mathcal{JKS}}^{\alpha}$ is TN_p if and only if $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$. This already holds for $\alpha \geqslant p - 2$ from above; and it does not hold for $\alpha \in (0, p-2) \setminus \mathbb{Z}$ by assertion (2)(c) shown above. The final sub-case is when $\alpha \in \mathbb{Z}^{\geqslant 0}$. Let $\mathbf{x} \in [0, \infty)^{p,\uparrow}$ and $\mathbf{y} \in \mathbb{R}^{p,\uparrow}$; we need to show that

$$\det C^{\circ\alpha} \geqslant 0, \quad \text{where} \quad C := (\max(1 + x_j y_k, 0))_{j,k=1}^p.$$

By the continuity of the function $K_{\mathcal{JKS}}$, we may assume $x_1 > 0$. Now,

$$C^{\circ\alpha} = \operatorname{diag}(x_j^{\alpha}) (\max(0, x_j^{-1} - (-y_k))^{\alpha})_{j,k=1}^p = \operatorname{diag}(x_j^{\alpha} e^{\alpha x_j^{-1}}) (\Omega(x_j^{-1} - (-y_k)))^{\circ\alpha} \operatorname{diag}(e^{\alpha y_k}).$$

Reversing the rows and the columns, we are done by Theorem 1.2.[6] ☐

Remark 5.2. We now explain how Theorem C implies many of the results in Section 1.

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[5] In the original setting of entrywise polynomials and power series preserving positivity on $\mathbb{P}_p((0,\infty))$, no examples were known for $p \geqslant 3$, until recent joint work [40].

[6] This part of Karlin's result, for integer powers $\alpha \geqslant 0$, was already shown by Schoenberg in [57]. For the interested reader, his direct proof is included in the Appendix.

(1) Given scalars $0 < x_1 < \cdots < x_p$ and $y_1 < \cdots < y_p$, the Karlin-kernel Ω is a specialization of the Jain–Karlin–Schoenberg kernel, up to multiplying by diagonal matrices and reversing rows and columns:

$$(T_\Omega[\mathbf{x};\mathbf{y}]^{\circ\alpha})^T = D^{\circ\alpha} K_{\mathcal{JKS}}[\mathbf{y}';\mathbf{x}']^{\circ\alpha} D_1^{\circ\alpha}, \tag{5.3}$$

where $\mathbf{y}' = (-y_1, \dots, -y_p)$, $\mathbf{x}' = (1/x_1, \dots, 1/x_p)$, and D_1, D are diagonal matrices

$$D_1 = \operatorname{diag}(x_p e^{-x_p}, \dots, x_1 e^{-x_1}), \qquad D = \operatorname{diag}(e^{y_p}, \dots, e^{y_1}),$$

(2) The proof of Theorem 3.4 has a similar computation as (5.3), with Ω replaced by $\Omega^{(q,r)}$.
(3) Similarly, the proof of Theorem C(1) shows how, via the transformation arctan, the Jain–Karlin–Schoenberg kernel is intimately related to the Schoenberg-kernel T_W. These observations show how Theorem C about (the powers of) the Jain–Karlin–Schoenberg kernel is related to Theorem A, and to Theorems 1.5 and 1.2 of Schoenberg and Karlin, respectively.
(4) Given an integer $n \geqslant 2$, the kernel $K_{\mathcal{JKS}}$ clearly specializes on the set of bi-tuples

$$\{(\mathbf{x},\mathbf{y}) \in (\mathbb{R}^{n,\uparrow})^2 : 1 + x_j y_k > 0 \ \forall j,k = 1,\dots,n\}$$

to Jain's theorem 1.12(1) – in fact, to the stronger TN assertion in Theorem C(2).
(5) Restricting the kernel $K_{\mathcal{JKS}}$ to $(0,\infty)^2$ via the transform u_0^x, we see that Theorem C implies the equivalence (2) $\iff$ (3) in Theorem B.
(6) Our methods have provided an alternate proof above to Karlin's theorem 1.2. Indeed, as discussed during the proof of Theorem C(3), the result is shown in the Appendix for integer powers, and for non-integer powers $\alpha > p-2$ it is a special case of Schoenberg's theorem – transforming the domain from $(-\pi/2, \pi/2)^2$ to $\mathbb{R}^2$ via arctan, then restricting to $\mathbb{R} \times [0,\infty)$. Here we use the identifications of $K_{\mathcal{JKS}}$ with Schoenberg and Karlin's kernels.

In fact, it is possible to refine the above results even more. Given integers $1 \leqslant p \leqslant n$, matrices $C = (1 + x_j y_k)_{j,k=1}^n$ with positive entries, and powers $\alpha \geqslant 0$, one can show that all $p \times p$ minors of $C^{\circ\alpha}$ have the same sign – which depends only on n, p, α but not on x_j, y_k. This follows from above for $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$. If $\alpha \in (0, p-2) \setminus \mathbb{Z}$, this follows by using SSR (strictly sign regular) matrices and kernels, found in Karlin's book [39] and Jain's works [34, 35]. In fact, the following holds, e.g. by Propositions 2.6 and 3.1, and [34, Theorem 2.4]:

Corollary 5.4. *Given a scalar $\alpha \geqslant 0$, an integer $n \geqslant 2$, and tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n,\uparrow}$ such that $1 + x_j y_k > 0$ for all j, k, the power-matrix $C^{\circ\alpha}$ studied above is sign regular, with signature given as follows:*

$$\operatorname{signature}((1 + x_j y_k)^\alpha)_{j,k=1}^n = \begin{cases} ((-1)^{\lfloor p/2 \rfloor} \varepsilon_{p,\alpha})_{p=1}^n, & \textit{if } \alpha \notin \{0, 1, \dots, n-2\}, \\ (((-1)^{\lfloor p/2 \rfloor} \varepsilon_{p,\alpha})_{p=1}^{\alpha+1}, 0, \dots, 0), & \textit{otherwise.} \end{cases}$$

That is, the sign of any $p \times p$ minor of $((1 + x_j y_k)^\alpha)_{j,k=1}^n$ depends only on n, p, α; here, $\varepsilon_{p,\alpha}$ equals

$$\varepsilon_{p,\alpha} = \begin{cases} (-1)^{\lfloor p/2 \rfloor}, & \textit{if } \alpha > p-2, \\ (-1)^{p-s+1}, & \textit{if } 2s < \alpha < 2s+1 \leqslant p-2, \ s \in \mathbb{Z}^{\geqslant 0}, \\ (-1)^{s+1}, & \textit{if } 2s+1 < \alpha < 2s+2 \leqslant p-2, \ s \in \mathbb{Z}^{\geqslant 0}, \\ 0, & \textit{if } \alpha = 0, 1, \dots, p-2. \end{cases}$$

To conclude this section, note that Theorem C completely classifies the powers of $K_{\mathcal{JKS}}$ preserving TN_p on $\mathbb{R} \times [0,\infty)$. The same question on the full domain $\mathbb{R}^2$ of $K_{\mathcal{JKS}}$ remains, but only for integers $\alpha \in \{0, 1, \dots, p-2\}$. This is equivalent to the following

Question 5.5. *For an integer $\alpha \geqslant 0$, can the kernel $K_{\mathcal{JKS}}^\alpha$ be shown to not be $\mathrm{TN}_{\alpha+3}$ on $\mathbb{R} \times \mathbb{R}$? More strongly, can it be shown to not be 'positive semidefinite', i.e. using $\mathbf{x} = \mathbf{y} \in \mathbb{R}^{\alpha+3,\uparrow}$?*

A complete resolution of Question 5.5 would complete the classification of powers of the Jain–Karlin–Schoenberg kernel $K_{\mathcal{JKS}}$ that are totally non-negative of each order $p \geqslant 2$. (It would also complete the remaining sub-case of integer $\alpha \leqslant p$ in Theorem 3.4.) Note that this question for $K_{\mathcal{JKS}}$ has a 'positive' answer for $\alpha = 0, 1$, so that $K_{\mathcal{JKS}}$ is not TN_4. Indeed,

$$\mathbf{x} = \mathbf{y} = (-1, 0, 1) \in \mathbb{R}^{3,\uparrow} \qquad \Longrightarrow \qquad \det K_{\mathcal{JKS}}[\mathbf{x}; \mathbf{y}]^{\circ 0} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2,$$

$$\mathbf{x} = \mathbf{y} = \frac{1}{\sqrt{2}}(-2, -1, 1, 2) \in \mathbb{R}^{4,\uparrow} \qquad \Longrightarrow \qquad \det K_{\mathcal{JKS}}[\mathbf{x}; \mathbf{y}] = \det \begin{pmatrix} 3 & 2 & 0 & 0 \\ 2 & 3/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & 2 \\ 0 & 0 & 2 & 3 \end{pmatrix} = -2,$$

where the $\alpha = 0$ case uses $0^0 = 0$.

6. Theorem D: Laplace transform of a compactly supported TN_p function

We now show Theorem D. The first step toward proving the result is to characterize TN_2 functions Λ on a sub-interval $I \subset \mathbb{R}$, instead of on all of $\mathbb{R}$ as is prevalent in the literature. We provide a proof of this result along the lines of [58], but with a few modifications for more general I:

Lemma 6.1. *Suppose $J \subset \mathbb{R}$ is an interval strictly containing the origin, and $\Lambda : J - J \to \mathbb{R}$ is Lebesgue measurable. The following are equivalent:*

(1) The nonzero-locus of Λ is an interval $I \subset J - J$, on which $\Lambda > 0$ and $\log \Lambda$ is concave.
(2) The Toeplitz kernel $T_\Lambda : J \times J \to \mathbb{R}$ is TN_2.

Thus Λ is continuous on the interior of I, whence discontinuous on $J - J$ at most at two points.

In particular, this applies to $I = (-\rho/2, \rho/2) \subset J = (-\widetilde{\rho}/2, \widetilde{\rho}/2)$, as in Theorem D.

Proof. The result is straightforward if Λ does not vanish at most at one point, so we suppose henceforth that $\Lambda \neq 0$ at least at two points.

(1) $\Longrightarrow$ (2): Given scalars $\alpha < \beta$ and $\gamma < \delta$ in J, note that $\alpha - \gamma, \beta - \delta \in (\alpha - \delta, \beta - \gamma)$. If $\alpha - \gamma$ or $\beta - \delta$ lie outside I, the matrix $M := \begin{pmatrix} \Lambda(\alpha - \gamma) & \Lambda(\alpha - \delta) \\ \Lambda(\beta - \gamma) & \Lambda(\beta - \delta) \end{pmatrix}$ has a zero row or zero column. Else $\alpha - \gamma, \beta - \delta \in I$; if now one of $\alpha - \delta, \beta - \gamma$ is not in I then M is triangular, whence again $\det(M) \geqslant 0$. Else M has all positive entries; now the concavity of $\log \Lambda$ implies $\det(M) \geqslant 0$.

(2) $\Longrightarrow$ (1): Since Λ is TN_2, we have $\Lambda \geqslant 0$ on $J - J$. Fix $\delta > 0$ such that J contains either $[0, \delta)$ or $(-\delta, 0]$. Suppose $\Lambda(x_0) > 0$. We claim that if $x_1 > x_0$ in $J - J$ and $\Lambda(x_1) = 0$, then Λ vanishes on $(J - J) \cap [x_1, \infty)$; and similarly for $x_1 < x_0$ in $J - J$. It suffices to show that $\Lambda(y) = 0$ for $y \in (J - J) \cap (x_1, x_1 + \delta)$. If $J \supset (-\delta, 0]$, this is because

$$0 \leqslant \det T_\Lambda[(x_0, x_1); (x_1 - y, 0)] = \det \begin{pmatrix} \Lambda(x_0 - x_1 + y) & \Lambda(x_0) \\ \Lambda(y) & \Lambda(x_1) \end{pmatrix} = -\Lambda(x_0)\Lambda(y);$$

here, $x_0 - x_1 + y \in (x_0, y) \subset J - J$. Similarly, if $J \supset [0, \delta)$, then we instead use

$$0 \leqslant \det T_\Lambda[(x_0 - x_1 + y, y); (0, y - x_1)] = \det \begin{pmatrix} \Lambda(x_0 - x_1 + y) & \Lambda(x_0) \\ \Lambda(y) & \Lambda(x_1) \end{pmatrix} = -\Lambda(x_0)\Lambda(y).$$

This produces the interval I; now given points $y - \epsilon < y < y + \epsilon$ of I, we show that $\Lambda(y) \geqslant \sqrt{\Lambda(y + \epsilon)\Lambda(y - \epsilon)}$ using discrete-time, finite state-space Markov chains. Let $n_0 := 2\lceil \epsilon/\delta \rceil$, so that $\epsilon/n_0 \in (0, \delta)$. Let $z_k := \Lambda(y + k\epsilon/n_0)$ for $-n_0 \leqslant k \leqslant n_0$; then $z_k > 0$. Now if $J \supset (-\delta, 0]$, then

$$0 \leqslant \det T_\Lambda[(y - (k+1)\epsilon/n_0, y - k\epsilon/n_0); (-\epsilon/n_0, 0)] = z_k^2 - z_{k-1}z_{k+1}, \qquad \forall -n_0 < k < n_0.$$

If instead $J \supset [0, \delta)$ then we use $0 \leqslant \det T_\Lambda[(y - k\epsilon/n_0, y - (k-1)\epsilon/n_0); (0, \epsilon/n_0)]$ for the same values of k, to obtain the same conclusions. From each case, it follows inductively that

$$z_0 \geqslant (z_1 z_{-1})^{1/2} \geqslant (z_2 z_0^2 z_{-2})^{1/4} \geqslant \cdots \geqslant \prod_{j=0}^{n_0} z_{2j-n_0}^{\binom{n_0}{j}/2^{n_0}} \geqslant \cdots$$

At each step, no power of $z_{\pm n_0}$ is changed, while the remaining powers z_j^γ are lower-bounded by $(z_{j-1} z_{j+1})^{\gamma/2}$. The exponents of the z_j give probability distributions on $\mathcal{S} := \{-n_0, \dots, 0, 1, \dots, n_0\}$ corresponding to the symmetric gambler's ruin, i.e. a simple random walk on the state space $\mathcal{S}$ with absorbing barriers $z_{\pm n_0}$. The transition probabilities here for all other states z_j are $1/2$ for $z_j \mapsto z_{j\pm 1}$. Since at each stage we moreover have equal powers of $z_{\pm n_0}$, it follows by Markov chain theory (or one can show via a direct argument)[7] that $z_0 \geqslant \sqrt{z_{n_0} z_{-n_0}}$. Hence $-\log \Lambda$ is midpoint-convex and measurable on I. It follows by Sierpińsky's well-known result [60] that $-\log \Lambda$ is continuous on the interior of I, whence convex, and so Λ is also continuous on the interior of I.

Finally, to show $-\log \Lambda$ is convex on I, it suffices to show for $a, b \in I$ and $\lambda \in (0,1)$ that $\log \Lambda(\lambda a + (1-\lambda) b) \geqslant \lambda \Lambda(a) + (1-\lambda)\Lambda(b)$. But this can be shown by approximating λ by dyadic rationals $\lambda_n \in (0,1)$ for all $n \geqslant 1$. For each of these, the above mid-convexity implies:

$$\log \Lambda(\lambda_n a + (1-\lambda_n) b) \geqslant \lambda_n \log \Lambda(a) + (1-\lambda_n) \log \Lambda(b), \qquad \forall n \geqslant 1.$$

Letting $n \to \infty$, since Λ is continuous on the interior of I, it follows that $-\log \Lambda$ is convex on I. This completes the proof of (2) $\implies$ (1). □

Proof of Theorem D. Let $0 < \epsilon < \rho/2 \leqslant \widetilde{\rho} - \rho/2$, and work with integers $m > (p-1)\rho/\epsilon$. Then the following increasing, equi-spaced arithmetic progressions fall in the specified domains:

$$\mathbf{x} := (0, \frac{\rho}{m+1}, \frac{2\rho}{m+1}, \dots, \frac{(p-1)\rho}{m+1}) \in [0, \epsilon)^{p,\uparrow}$$
$$\mathbf{y} := (\frac{-m\rho}{2m+2}, \frac{-(m-2)\rho}{2m+2}, \dots, \frac{(m+2p-2)\rho}{2m+2}) \in (-\rho/2, (\rho/2) + \epsilon)^{m+p,\uparrow}.$$

Hence the matrix $T_\Lambda[\mathbf{x}; \mathbf{y}]$ is TN; reversing the order of the rows and columns, the matrix

$$A_m := \begin{pmatrix} a_0 & a_1 & \cdots & & \cdots & a_m & 0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & & \cdots & a_{m-1} & a_m & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_0 & \cdots & a_{m-p+1} & a_{m-p+2} & a_{m-p+3} & \cdots & a_m \end{pmatrix}_{p \times (m+p)}$$

is TN, where we define $a_\nu := \Lambda\left(\frac{(2\nu - m)\rho}{2m+2}\right) > 0$ for $\nu = 0, 1, \dots, m$.

Once this matrix is constructed, repeat the proof of [58, Theorem 1].[8] This shows the polynomial

$$p_m(z) := \frac{\rho}{m+1} \sum_{\nu=0}^m \Lambda((2\nu - m)\rho/(2m+2)) z^\nu$$

has no roots in the sector $|\arg(z)| < p\pi/(m+p-1)$.

Now given $s \in \mathbb{C}$ and $m \geqslant 1$, let $z = e^{-s\rho/(m+1)}$, and consider the holomorphic function

$$F_m(s) := \frac{\rho}{m+1} \sum_{\nu=0}^m e^{-s(2\nu-m)\rho/(2m+2)} \Lambda((2\nu - m)\rho/(2m+2)) = e^{sm\rho/(2m+2)} p_m(z), \qquad s \in \mathbb{C}.$$

[7] Indeed, if c_t denotes the sum of the exponents for $z_{-(n_0-1)}, \dots, z_0, z_1, \dots, z_{n_0-1}$ at 'time' t, then one shows via the AM–GM inequality that $c_{t+(2n_0-1)} \leqslant c_t(1 - 2^{1-n_0})$. Now let $t = m(2n_0 - 1)$, with $m \to \infty$.

[8] This proof can be found in Karlin's book – see [39, Chap. 8, Theorem 3.1] – and uses the variation-diminishing property of the TN matrix A_m, as shown by Schoenberg [54].

From above, $F_m(s)$ has no zeros in the strip

$$|\Im(s)| < \frac{p\pi(m+1)}{\rho(m+p-1)} = \frac{p\pi}{\rho}\left(1 - \frac{p-2}{m+p-1}\right)$$

for all m sufficiently large. If $p = 2$ then this concludes the proof; else fixing $\delta \in (0, p\pi/\rho)$, F_m has no zeros s satisfying: $|\Im(s)| < (p\pi/\rho) - \delta$. By Lemma 6.1, the holomorphic Riemann sums $F_m(s)$ converge to $\mathcal{B}\{\Lambda\}(s)$ uniformly on each bounded domain, so by Hurwitz's theorem, $\mathcal{B}\{\Lambda\} \not\equiv 0$ also has no root s with $|\Im(s)| < (p\pi/\rho) - \delta$. As this holds for all $\delta \in (0, p\pi/\rho)$, the proof is complete. □

Remark 6.2. As noted following Theorem D, the hypotheses therein require using that the restriction of Λ to the interval $I(\epsilon) := (-(\rho/2) - \epsilon, (\rho/2) + \epsilon)$ is TN_p. If this can be strengthened to using only $I(0) = (-\rho/2, \rho/2)$, then this would answer Question 5.5 in the affirmative, by specializing to $\Lambda = W$, $\rho = \pi$, and translating from T_W to $K_{\mathcal{JKS}}$ via arctan as above. Indeed, the above strengthening would imply that the following function has no roots s with $|\Im(s)| < p$:

$$\mathcal{B}\{W^\alpha\}(s) = \int_{-\pi/2}^{\pi/2} e^{-sx} \cos(x)^\alpha \, dx.$$

Since $\alpha \in [0, \infty)$ here, the right-hand side can be computed using a well-known, classical formula of Cauchy [11, pp. 40], or directly as in [58, §10], to yield:

$$\mathcal{B}\{W^\alpha\}(s) = \frac{\pi\Gamma(\alpha+1)}{2^\alpha \Gamma(\frac{1}{2}(\alpha+2+si))\Gamma(\frac{1}{2}(\alpha+2-si))},$$

and this has roots at $s = \pm(\alpha+2)i$. It follows that $\alpha + 2 = |\alpha+2| \geqslant p$. This also explains how Schoenberg's work [57, 58] implies that T_{W^α} is not TN_p for $\alpha \in (0, p-2)$.

7. Theorem E: Characterizing TN_p functions; classifying discontinuous PF/TN functions

Finally, we come to Theorem E and a few related variants, which characterize not only TN_p functions $\Lambda : \mathbb{R} \to \mathbb{R}$, but also TN_p kernels $K : X \times Y \to (0, \infty)$ for general $X, Y \subset \mathbb{R}$.

7.1. **Clarifications in the literature: discontinuous PF functions.** We begin by addressing some gaps found in the literature, vis-a-vis TN_3 functions and discontinuous Pólya frequency functions. Recall that a characterization of TN_p functions is known for $p = 2$ by Schoenberg [57] (see Lemma 6.1). For $p = 3$ an analogous result can be found in Weinberger's work [64], but it turns out to have a small gap, owing to the following lemma.

Lemma 7.1. *For all $d \in [0,1]$, the following 'Heaviside' function is* TN*, whence* TN_3*:*

$$H_d(x) = \begin{cases} 0, & x < 0, \\ d, & x = 0, \\ 1, & x > 0. \end{cases} \tag{7.2}$$

In particular, the function $\lambda_d(x) := e^{-x} H_d(x)$ is a Pólya frequency function.

Weinberger's result [64, Theorem 1] asserts in particular that if $f : \mathbb{R} \to \mathbb{R}$ is TN_3, then either $f(x) = H_1(ax+b)e^{cx+c'}$ for suitable scalars $a, b, c, c' \in \mathbb{R}$, or the nonzero-locus of f is an open interval. However, H_d, λ_d are nonzero on $[0, \infty)$ and are TN for $d \in (0,1)$ as well.

Remark 7.3. In fact, this gap in [64] stems from earlier works. In the 1947–48 announcements [55, 56] of his forthcoming results on Pólya frequency functions, Schoenberg asserts that λ_1 is the only discontinuous PF function, up to changes in scale and origin. In his full paper, in [57, Corollary 2], Schoenberg repeats this, by remarking that the only discontinuous Pólya frequency function is "essentially equivalent to" $\lambda(x) = e^{-x}\mathbf{1}_{x \geqslant 0}$. In particular, it seems Schoenberg was not aware of λ_d for $d \in (0,1)$; similarly, we could not find H_d, λ_d in the text of Karlin [39].

To our knowledge, the functions λ_d were very recently observed to be Pólya frequency functions – in joint work [3], where Lemma 7.1 was stated and used without a proof. Thus, in the interest of future clarity, we quickly record a proof.

Proof of Lemma 7.1. Let $p \geqslant 1$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$; define $M := T_{H_d}[\mathbf{x}; \mathbf{y}]$. We prove that $\det M \geqslant 0$ by induction on p. The base case $p = 1$ is clear; for the induction step, assume $p \geqslant 2$ and consider various sub-cases:

(1) If $x_1 < y_2$, then all entries in the first row vanish, except at most the first entry. Hence,
$$\det T_{H_d}[\mathbf{x}; \mathbf{y}] = H_d(x_1 - y_1) \det T_{H_d}[\mathbf{x}'; \mathbf{y}'], \qquad \text{where } \mathbf{x}' = (x_2, \dots, x_p),\ \mathbf{y}' = (y_2, \dots, y_p).$$
Now the induction hypothesis implies $\det T_{H_d}[\mathbf{x}; \mathbf{y}] \geqslant 0$.

(2) Otherwise, suppose henceforth that $y_1 < y_2 \leqslant x_1$. First suppose $y_2 = x_1$; subtracting the second row of the matrix M from the first yields a matrix with first column $(1-d, 0, \dots, 0)^T$. Now expand along the first column and use the induction hypothesis.

(3) Finally, if $y_1 < y_2 < x_1$, then the first two columns of $T_{H_d}[\mathbf{x}; \mathbf{y}]$ are identical, so $\det M = 0$.

Finally, given any TN_p function $f(x)$ for $p \geqslant 1$, and scalars $a, b \in \mathbb{R}$, the function $e^{ax+b} f(x)$ is also TN_p, since for all $1 \leqslant r \leqslant p$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{r,\uparrow}$, the matrix
$$T_{e^{ax+b} f}[\mathbf{x}; \mathbf{y}] = D \cdot T_f[\mathbf{x}; \mathbf{y}] \cdot D',$$
where D, D' are diagonal $r \times r$ matrices with (j, j) entries $e^{ax_j + b}$ and e^{-ay_j} respectively. In particular, the matrix on the left again has non-negative determinant. Hence λ_d is also TN. □

We next fix the aforementioned results of Schoenberg on discontinuous Pólya frequency functions:

Theorem 7.4 (Classification of discontinuous Pólya frequency and TN functions)**.**
A Pólya frequency function is discontinuous if and only if, up to a change in scale and origin, it is of the form λ_d, where $d \in [0, 1]$.

More generally, a TN *function $\Lambda : \mathbb{R} \to \mathbb{R}$ is discontinuous if and only if it is the Dirac function $\mathbf{1}_{x=0}$, or it is of the form $e^{ax+b} \lambda_d(x)$ for $a, b \in \mathbb{R}$ and $d \in [0, 1]$ – once again, up to a change in scale and origin.*

Proof. This is shown using two results of Schoenberg from [57]:

- His classification of the non-smooth Pólya frequency functions, as those one-sided PF functions Λ whose bilateral Laplace transform has reciprocal $(1 + a_1 s) \cdots (1 + a_m s) e^{\delta s}$ with $a_m, \delta \geqslant 0$ and $\delta + \sum_j a_j > 0$ (and $m > 0$). Schoenberg also shows that if $m > 1$ then Λ is continuous. Thus, it reduces to understanding the inverse Laplace transforms of $1/(1 + s)$. This determines Λ via the property that a PF function is continuous on the interior of the interval where it is positive (via Lemma 6.1). Thus $\Lambda \equiv 0$ on $(-\infty, 0)$ and $\Lambda(x) = e^{-x}$ on $(0, \infty)$. The value at the origin can be $d \in [0, 1]$, by Lemma 7.1; it cannot be negative; and it is at most 1 by considering the 2×2 minor
$$0 \leqslant \det T_\Lambda[(1, 2); (0, 1)] = \det \begin{pmatrix} e^{-1} & \Lambda(0) \\ e^{-2} & e^{-1} \end{pmatrix} \quad \Longrightarrow \quad \Lambda(0) \leqslant 1.$$
- Schoenberg also shows that every TN function is either a Dirac function or a Pólya frequency function, up to an exponential factor e^{ax+b} with $a, b \in \mathbb{R}$. □

We now make a brief digression into another assertion by Weinberger: he used his proposed characterization of TN_3 functions to show that every power x^α for $\alpha \geqslant 1$ preserves the TN_3 functions, just as every power $\alpha \geqslant 0$ preservers the TN_2 functions. (This latter assertion is obvious from Lemma 6.1, and on any interval, not just $\mathbb{R}$.) In light of the above gap in [64], we provide an alternate proof of this latter result:

Proposition 7.5. *Suppose $\alpha \in \mathbb{R}$. Then x^α preserves the class of* TN_3 *functions if and only if $\alpha \geqslant 1$. (Here we use $0^0 := 0$.)*

If we instead use $0^0 := 1$, then clearly x^0 preserves the class of TN_3 functions.

Proof. First recall that the Gaussian kernel $G_1(x) := e^{-x^2}$ is a Pólya frequency function, whence TN_3. (In fact it is TP; see the proof of Corollary 4.2 in the Appendix.) Examining any 'principal 2×2 submatrix' of the associated kernel $T_{G_1^\alpha}$ shows that $\alpha \geqslant 0$ if $T_{G_1^\alpha}$ is TN_2. Now say $\alpha \in [0, 1)$. By Theorem 1.5, $W(x)$ is TN_3, but W^α is not TN_3 (as can be directly inspected by looking at the principal submatrix drawn at $(-\pi/4, 0, \pi/4)$, with $0^0 := 0$).

This shows one implication. Conversely, suppose $\alpha \geqslant 1$, and f is TN_3. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3,\uparrow}$; then the matrix $T_f[\mathbf{x}; \mathbf{y}]$ is TN. By Whitney's density theorem [65], there is a sequence of 3×3 TP matrices A_k converging entrywise to $T_f[\mathbf{x}; \mathbf{y}]$. By [14, Theorem 5.2], $A_k^{\circ\alpha}$ is TP for all $k \geqslant 1$ (since $\alpha \geqslant 1$); now taking limits, $T_{f^\alpha}[\mathbf{x}; \mathbf{y}]$ is TN as desired. ☐

7.2. **Characterizing** TN_p **functions and kernels.** Returning to the above attempt by Weinberger to characterize TN_3 functions, and the preceding result by Schoenberg for TN_2 functions: we now prove the aforementioned Theorem E, characterizing TN_p functions for all $p \geqslant 3$. To our knowledge there are no other such results for TN_p functions in the literature, prior to Theorem E. This result will follow from a more general formulation:

Proposition 7.6. *Let $t_*, \rho \in \mathbb{R}$ and fix a subset $Y \subset \mathbb{R}$ that is not bounded above. Suppose $X \subset \mathbb{R}$ contains $t_* + y$ for all $\rho < y \in Y$. Let $\Lambda : X - Y \to [0, \infty)$ be such that $\Lambda(t_*) > 0$ and*

$$\lim_{y \in Y,\ \rho < y \to \infty} \Lambda(x_0 - y)\Lambda(t_* + y - y_0) \to 0, \qquad \forall x_0 \in X,\ y_0 \in Y.$$

If $\det T_\Lambda[\mathbf{x}; \mathbf{y}] \geqslant 0$ for all $\mathbf{x} \in X^{p,\uparrow}, \mathbf{y} \in Y^{p,\uparrow}$, then the kernel T_Λ is TN_p.

Proposition 7.6 extends a recent result of Förster–Kieburg–Kösters [19] in two ways: first, it works over a large class of domains $X, Y \subset \mathbb{R}$, whereas the result in [19] requires $X = Y = \mathbb{R}$. Second, even assuming $X = Y = \mathbb{R}$, the result in [19] requires Λ to be integrable; however, Proposition 7.6 is strictly more general as it works for all TN_p functions, such as (via Remark A.1)

$$\Lambda(x) = \begin{cases} ce^{\beta(x - x_0)}, & \text{if } x \leqslant x_0, \\ ce^{\alpha(x - x_0)}, & \text{if } x > x_0, \end{cases} \qquad \text{where } -\infty \leqslant \alpha < \beta \leqslant +\infty,\ c > 0. \tag{7.7}$$

If now $\alpha\beta \geqslant 0$, then Λ is not integrable, but the hypotheses in Proposition 7.6 are satisfied.

Proof of Proposition 7.6. We show by downward induction on $1 \leqslant r \leqslant p$ that all $r \times r$ minors of T_Λ on $X \times Y$ are non-negative. The $r = p$ case is obvious, and it suffices to deduce from it the $r = p - 1$ case. Thus, fix $\mathbf{x}' \in X^{p-1,\uparrow}$ and $\mathbf{y}' \in Y^{p-1,\uparrow}$. We are to show that

$$\psi(x_p, y_p) := \det T_\Lambda[(\mathbf{x}', x_p); (\mathbf{y}', y_p)] \geqslant 0\ \forall x_p > x_{p-1}, y_p > y_{p-1} \quad \implies \quad \det T_\Lambda[\mathbf{x}'; \mathbf{y}'] \geqslant 0.$$

We now refine the argument in [19]. Begin by defining the $(p-1) \times (p-1)$ matrix $A := T_\Lambda[\mathbf{x}'; \mathbf{y}']$, and let $A^{(j,k)}$ denote the submatrix obtained by removing the jth row and kth column of A. (Since $p \geqslant 3$, these matrices are at least 1×1.) Now the following scalar does not depend on x_p, y_p:

$$L := \max_{1 \leqslant j,k \leqslant p-1} |\det A^{(j,k)}| \geqslant 0. \tag{7.8}$$

Next, define $t_m \in Y$ for all $m \geqslant 1$ such that $t_m > \max\{x_{p-1} - t_*, y_{p-1}, \rho\}$ and

$$\Lambda(x_j - t_m)\Lambda(t_* + t_m - y_k) < 1/m, \qquad \forall 0 < j, k < p.$$

With these choices made, we turn to the proof. Begin by expanding $\psi(x_p, y_p)$ along the final row, and excluding the cofactor for (p,p), expand all other cofactors along the final column, to get:

$$\psi(x_p, y_p) = \Lambda(x_p - y_p)\det(A) + \sum_{j,k=1}^{p-1} (-1)^{j+k-1}\Lambda(x_j - y_p)\Lambda(x_p - y_k)\det A^{(j,k)}.$$

Define $y_p^{(m)} := t_m$ and $x_p^{(m)} := t_* + t_m$, with t_*, t_m as above. Then

$$x_p^{(m)} \in X, \quad x_p^{(m)} > x_{p-1}, \quad y_p^{(m)} \in Y, \quad y_p^{(m)} > y_{p-1}.$$

Moreover, since $\psi(x_p^{(m)}, y_p^{(m)}) \geqslant 0$, we compute for $m \geqslant 1$:

$$\Lambda(t_*)\det(A) \geqslant \psi(x_p^{(m)}, y_p^{(m)}) - L\sum_{j,k=1}^{p-1} \Lambda(x_j - y_p^{(m)})\Lambda(x_p^{(m)} - y_k) \geqslant -L\frac{(p-1)^2}{m}.$$

Now taking $m \to \infty$ concludes the proof, since $\Lambda(t_*) > 0$ by assumption. □

Remark 7.9. Proposition 7.6 specializes to $X = Y = G$, an arbitrary additive subgroup of $(\mathbb{R}, +)$. E.g. for $G = \mathbb{Z}$, we obtain a result – whence a characterization, akin to Theorem E and results below – for 'Pólya frequency sequences of order p' that vanish at $\pm\infty$. Here, t_* would be an integer.

With Proposition 7.6 at hand, the final outstanding proof follows.

Proof of Theorem E. If $\Lambda \equiv 0$ then the result is immediate. If $\Lambda(x) = e^{ax+b}$ then the result is again easy, since by the argument to show Lemma 7.1, it suffices to show the case of $a = b = 0$, which is obvious. Now suppose Λ is not of the form ce^{ax} for $a \in \mathbb{R}$ and $c \geqslant 0$. Then (2) follows by Proposition 7.6 with arbitrary $\rho \in \mathbb{R}$.

Conversely, suppose Λ is not of the form ce^{ax} for $a \in \mathbb{R}$ and $c \geqslant 0$. Since it is TN_p, clearly (1)(a),(c) follow. In particular, since Λ is also TN_2, $g(x) := \log\Lambda(x)$ is concave on $\mathbb{R}$ (in the generalized sense, i.e., it is allowed to take the value $-\infty$), by Lemma 6.1. Now let I be the nonzero-locus of Λ. If I is not all of $\mathbb{R}$, then (1)(b) is immediate. If instead $\Lambda(x) > 0$ for all $x \in \mathbb{R}$, then since Λ is not an exponential, $g(x)$ is not linear from above. Hence a short argument of Schoenberg [57] shows that there exist $\beta, \gamma \in \mathbb{R}$ and $\delta > 0$ such that[9]

$$e^{-\gamma x}\Lambda(x) \leqslant e^{\beta - \delta|x|}, \qquad \text{as } x \to \pm\infty.$$

From this, the decay property (1)(b) immediately follows. □

We conclude by extending the above result to arbitrary positive-valued kernels on $X \times Y$:

Proposition 7.10. *Let $X, Y \subset \mathbb{R}$ be non-empty, and $K : X \times Y \to (0,\infty)$ a kernel satisfying any of the following decay conditions:*

$$\begin{aligned}
\sup Y \notin Y, &\qquad \lim_{y \in Y,\ y \to (\sup Y)^-} K(x_0, y) = 0, \qquad \forall x_0 \in X,\\
\inf Y \notin Y, &\qquad \lim_{y \in Y,\ y \to (\inf Y)^+} K(x_0, y) = 0, \qquad \forall x_0 \in X,\\
\sup X \notin X, &\qquad \lim_{x \in X,\ x \to (\sup X)^-} K(x, y_0) = 0, \qquad \forall y_0 \in Y,\\
\inf X \notin X, &\qquad \lim_{x \in X,\ x \to (\inf X)^+} K(x, y_0) = 0, \qquad \forall y_0 \in Y.
\end{aligned}$$

Given an integer $p \geqslant 2$, the kernel K is TN_p on $X \times Y$, if and only if every $p \times p$ minor of K is non-negative.

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[9]Since g is concave, g' exists and is non-increasing on a co-countable subset of $\mathbb{R}$. Since g' is not constant, there exist scalars $x_- < x_+$ and $c_\pm$ such that $g'(x_-) > g'(x_+)$ and $\log\Lambda(x) \leqslant g'(x_\pm)x + c_\pm$. Choose $\gamma, \delta \in \mathbb{R}$ such that $g'(x_+) < \gamma - \delta < \gamma + \delta < g'(x_-)$. Then $\log\Lambda(x) - \gamma x$ is bounded above by $(g'(x_\pm) - \gamma)x + c_\pm$, for $\pm x > 0$.

For instance, this can be specialized to kernels over $X = Y = G$, an additive subgroup of $(\mathbb{R}, +)$.

Proof. One implication is immediate. Conversely, as in the preceding proofs it suffices to show that $K[\mathbf{x}'; \mathbf{y}'] \geqslant 0$ for all tuples $\mathbf{x}' \in X^{p-1,\uparrow}, \mathbf{y}' \in Y^{p-1,\uparrow}$. We show this under the fourth decay condition; the other cases are similar to this proof and the proofs above. Fix increasing tuples

$$\mathbf{x}' := (x_2, \dots, x_p) \in X^{p-1,\uparrow}, \quad \mathbf{y}' := (y_2, \dots, y_p) \in Y^{p-1,\uparrow}$$

as well as $y_1 \in (-\infty, y_2) \cap Y$. Let $A = K[\mathbf{x}'; \mathbf{y}']$ and define $L \geqslant 0$ as in (7.8) above. Also choose for each $m \geqslant 1$ an element $x_1^{(m)} \in X$, such that $x_1^{(m)} < x_2$ and $K(x_1^{(m)}, y_k) < 1/m$ for $2 \leqslant k \leqslant p$. Now compute as in the proof of Proposition 7.6, this time expanding the determinant along the first row and column:

$$\begin{aligned} K(x_1^{(m)}, y_1) \det(A) &\geqslant \det K[(x_1^{(m)}, \mathbf{x}'); (y_1, \mathbf{y}')] - L \sum_{j,k=2}^{p} K(x_j, y_1) K(x_1^{(m)}, y_k) \\ &\geqslant \det K[(x_1^{(m)}, \mathbf{x}'); (y_1, \mathbf{y}')] - \frac{L(p-1)}{m} \sum_{j=2}^{p} K(x_j, y_1). \end{aligned}$$

As $\det K[(x_1^{(m)}, \mathbf{x}'); (y_1, \mathbf{y}')] \geqslant 0$ and $K(x_1^{(m)}, y_1) > 0$, the result follows by letting $m \to \infty$. $\square$

Remark 7.11. We have tried to keep the proofs of the results in our main theme self-contained (modulo the Appendix) – specifically, for the results related to powers preserving TN_p. The only four such proofs that use prior results are those of Corollary 3.5, Theorems B; Theorem C(1); and Theorem D, which use Schoenberg's characterization of PF functions [57]; Lemma 4.1 (Fekete); Theorem 1.5 (Schoenberg); and Schoenberg's [58, Theorem 1] plus Sierpińsky's result [60], respectively.

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Appendix A. Proofs from previous papers

In the interest of keeping this paper as self-contained as possible, this Appendix contains short proofs (from the original papers) of the results which are stated above and are used in proving our main theorems. The reader is welcome to skip these proofs (certainly in a first reading).

Proof of Theorem 1.1(1). We show the 'if' part; the converse was shown in the proof of Theorem 1.12(1). If $\alpha \in \mathbb{Z}^{\geqslant 0}$ then x^α preserves Loewner positivity by the Schur product theorem [59]. If $\alpha \geqslant n-2$, we show the result by induction on $n \geqslant 2$, with the $n=2$ case obvious. Suppose $n \geqslant 3$ and $A \in \mathbb{P}_n((0,\infty))$. Let ζ denote the last column of A, and $B := a_{nn}^{-1} \zeta \zeta^T$. Then $B \geqslant 0$; moreover, $A - B$ has last row and column zero, and is itself positive semidefinite via Schur complements. Now FitzGerald–Horn employ a useful 'integration trick': by the Fundamental Theorem of Calculus,

$$A^{\circ\alpha} = B^{\circ\alpha} + \alpha \int_0^1 (A-B) \circ (\lambda A + (1-\lambda) B)^{\circ(\alpha-1)} \, d\lambda.$$

But $A - B$ has last row/column zero, and the leading principal $(n-1) \times (n-1)$ submatrix of the integrand is in $\mathbb{P}_{n-1}(\mathbb{R})$ by the induction hypothesis. We are done by induction. $\square$

Proof of Theorem 1.2 for integer powers. For integers $\alpha \geqslant 0$, the proof that $x^\alpha e^{-\alpha x} \mathbf{1}_{x \geqslant 0}$ is a Pólya frequency function is in steps. We first show that the kernel $K(x,y) := \mathbf{1}_{x \geqslant y}$ is TN on $\mathbb{R} \times \mathbb{R}$. This is a direct calculation; e.g., Karlin [39, pp. 16] checks for the 'transpose' kernel $K(x,y) := \mathbf{1}_{x \leqslant y}$:

$$\det K[\mathbf{x}; \mathbf{y}] = \mathbf{1}(x_1 \leqslant y_1 < x_2 \leqslant y_2 < \cdots < x_p \leqslant y_p),$$

for all $p \geqslant 1$ and tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{p,\uparrow}$. (Alternately, use Lemma 7.1.) Now pre- and post-multiplying with diagonal matrices with (k,k) entries e^{-x_k} and e^{y_k} respectively, shows that the kernel $\Omega_0(x) := e^{-x} \mathbf{1}_{x \geqslant 0}$ is a Pólya frequency function. Next, the 'Basic Composition Formula' of Pólya–Szegö (see e.g. [39, pp. 17]) shows that the class of Pólya frequency functions is closed under convolution. But for any integer $\alpha \geqslant 1$, the α-fold convolution of $\Omega_0(x)$ with itself, yields precisely $x^{\alpha-1} e^{-x} \mathbf{1}_{x \geqslant 0}$. Finally, multiplying with a suitable exponential function shows Ω^α is still integrable, so also a Pólya frequency function. $\square$

Remark A.1. Let $\Lambda(x)$ be as in (7.7). If $|\alpha|$ or $|\beta|$ is infinite, Λ equals λ_0 or λ_1 (up to a linear change of variables), hence is TN. Else if $\alpha = \beta$ then Λ is an exponential – up to rescaling – so any submatrix drawn from has rank one, whence Λ is TN. Finally, suppose $\alpha < \beta \in \mathbb{R}$. As explained in Lemma 7.1, $\lambda_1(x) = e^{-x} \mathbf{1}_{x \geqslant 0}$ is TN, whence so is $\lambda_1(-x)$. As in the preceding proof, the Basic Composition Formula implies that $\lambda_1(x) * \lambda_1(-x) = e^{-|x|}/2$ is also TN. By a linear change of variables, the function $e^{(\alpha-\beta)|x|/2}$ is TN. Multiplying by $e^{(\alpha+\beta)x/2}$, the function in (7.7) is also TN.

Proof of Proposition 2.2. This Descartes-type result is proved in the spirit of Laguerre and Poulain's classical arguments, via Rolle's theorem. In this proof-sketch, we also address a small gap in [35]. The first step is to observe that $1 + u x_j > 0$ for all j if and only if $u \in (A_{\mathbf{x}}, B_{\mathbf{x}})$. Also note that

$$A_{-\mathbf{x}} = -B_{\mathbf{x}}, \quad \text{and} \quad A_{\mathbf{x}} < 0 < B_{\mathbf{x}}, \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{A.2}$$

We now sketch the proof in [35]. If $r = 0$ then the result is immediate, so we suppose henceforth that $r \neq 0$. Denote by $s \leqslant n-1$ the number of sign changes in $\mathbf{c}$ after removing the zero coordinates. We then claim that the number of zeros is at most s; the proof is by induction on $n \geqslant 1$ and then on $s \geqslant 0$. The base cases of $n = 1$, and $s = 0$ for any $n \geqslant 1$, are easy to show. For the induction step, we may suppose all c_j are non-zero, and the x_j are in increasing order.

The first case is that whenever there is a sign change in $\mathbf{c}$, i.e. $c_{k-1} c_k < 0$, we always have $x_k \leqslant 0$. (This is a small clarification that was not addressed in [35]; on a related note, (A.2) does not appear there.) In this case we simply replace $\mathbf{x}$ by $-\mathbf{x}$ and $\mathbf{c}$ by $\mathbf{c}' := (c_n, \dots, c_1)$. So the assertion for $\varphi_{-\mathbf{x}, \mathbf{c}', r} : (-B_{\mathbf{x}}, -A_{\mathbf{x}}) \to \mathbb{R}$ (via (A.2)) would show the result for $\varphi_{\mathbf{x}, \mathbf{c}, r}$.

Thus there exists k with $c_{k-1} c_k < 0 < x_k$. In turn, there exists $v > 0$ with $1 - v x_k < 0 < 1 - v x_{k-1}$, so that the sequence $c_j(1 - v x_j)$, $j = 1, \dots, n$ has one less sign change than $\mathbf{c}$. Now define

$$\psi(u) := \sum_{j=1}^n c_j (1 - v x_j)(1 + u x_j)^{r-1}, \qquad h(u) := (u+v)^{-r} \varphi_{\mathbf{x}, \mathbf{c}, r}(u), \qquad u \in (A_{\mathbf{x}}, B_{\mathbf{x}}),$$

so the induction hypothesis applies to ψ. But a straightforward computation yields

$$\psi(u) = \frac{-(u+v)^{r+1}}{r} h'(u), \quad \text{and} \quad u + v > 0, \qquad \forall u \in (A_{\mathbf{x}}, B_{\mathbf{x}}),$$

so by the induction hypothesis, h' has at most $s-1$ zeros. We are done by Rolle's theorem. $\square$

Proof-sketch of Proposition 3.1. Suppose $\alpha \in \mathbb{R} \setminus \{0, 1, \dots, n-2\}$, and $S^{\circ\alpha} \mathbf{c}^T = 0$ for a tuple $\mathbf{c} = (c_1, \dots, c_n) \neq 0$. Rewriting this in the language of Proposition 2.2 yields:

$$\varphi_{\mathbf{x}, \mathbf{c}, \alpha}(y_k) = \sum_{j=1}^n c_j (1 + y_k x_j)^\alpha = 0, \qquad \forall 1 \leqslant k \leqslant n.$$

By assumption, $y_k \in (A_{\mathbf{x}}, B_{\mathbf{x}})$ for all k (see the line preceding (A.2)), so Proposition 2.2 implies $\varphi_{\mathbf{x},\mathbf{c},\alpha} \equiv 0$ on $(A_{\mathbf{x}}, B_{\mathbf{x}})$. By (A.2), $\varphi^{(k)}_{\mathbf{x},\mathbf{c},\alpha}(0) = 0,\ \forall 0 \leqslant k \leqslant n-1$. This system can be written as

$$W_{\mathbf{x}}^{(n-1)} D\mathbf{c}^T = 0, \quad \text{where } W_{\mathbf{x}}^{(r)} := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^r & x_2^r & \cdots & x_n^r \end{pmatrix},\ r \in \mathbb{Z}^{\geqslant 0}$$

and D is the diagonal matrix with diagonal entries $1, \alpha, \alpha(\alpha-1), \dots, \alpha(\alpha-1)\cdots(\alpha-n+2)$. By assumption on α, the matrix D is non-singular, as is the (usual) Vandermonde matrix $W_{\mathbf{x}}^{(n-1)}$. Hence $\mathbf{c} = 0$, and so $S^{\circ\alpha}$ is non-singular.

Finally, if $\alpha \in \{0, \dots, n-2\}$, then $S^{\circ\alpha} = (W_{\mathbf{y}}^{(\alpha)})^T D W_{\mathbf{x}}^{(\alpha)}$, where $W_{\mathbf{x}}^{(\alpha)}$ was defined above, and D is a diagonal $(\alpha+1)\times(\alpha+1)$ matrix with (k,k) entry $\binom{n}{k}$. Since these matrices are each of maximal possible rank, the result follows. ☐

Proof of Corollary 3.3. Here we reproduce Karlin's proof of the assertion (2) $\implies$ (1). Since $\Lambda(x)$ is a one-sided Pólya frequency function if and only if $\Lambda(-x)$ is, we may assume without loss of generality that $\Lambda(x) = 0$ for sufficiently small $x < 0$. Now $\mathcal{B}\{\Lambda\}(s)$ is of the form

$$\mathcal{B}\{\Lambda\}(s) = e^{-\delta s} \prod_{j=1}^{\infty} (1 + a_j s)^{-1}, \quad \text{where } a_j \geqslant 0,\ \delta \in \mathbb{R},\ \sum_j a_j < \infty, \tag{A.3}$$

by foundational results of Schoenberg [57]. If $\alpha \in \mathbb{Z}^{>0} \cup (p-1, \infty)$, and $a_j > 0$, then

$$(1 + a_j s)^{-\alpha} = \mathcal{B}\{\Lambda_{j,\alpha}\}(s), \qquad \text{where } \Lambda_{j,\alpha}(x) = \frac{e^{(a_j^{-1} + \alpha - 1)x}}{\Gamma(\alpha) a_j^{\alpha}} \Omega(x)^{\alpha-1},$$

where $\Omega(x)$ is Karlin's kernel from [38]. By choice of α, we have $\alpha - 1 \in \mathbb{Z}^{\geqslant 0} \cup [p-2, \infty)$, so $\Omega(x)^{\alpha}$ is TN_p, whence so is $\Lambda_{j,\alpha}(x)$ by the concluding argument in the proof of Lemma 7.1. Now the convolution of finitely many of the one-sided integrable TN_p functions $\Lambda_{j,\alpha},\ j \geqslant 1$ is still an integrable TN_p function, by the Basic Composition Formula (see above in this Appendix).

Finally, suppose all $a_j > 0$. Since Karlin's proof of [39, Chapter 7, Theorem 12.2] does not address this case explicitly, we add a few lines for completeness. Since $1 + a_j s \leqslant e^{a_j s}$ and $\sum_j a_j < \infty$, we have $\prod_{j \geqslant 1} |1 + a_j s| \leqslant \prod_{j \geqslant 1} (1 + a_j |s|) \leqslant e^{|s| \sum_{j \geqslant 1} a_j} < \infty$. Hence,

$$\phi_n(s) := \frac{e^{-\delta s}}{\prod_{j=1}^n (1 + a_j s)^{\alpha}} \quad \text{converges to} \quad \phi(s) := \frac{e^{-\delta s}}{\prod_{j \geqslant 1} (1 + a_j s)^{\alpha}}$$

on the strip $\Re(s) > \max_j -a_j^{-1}$. Moreover, for $x \in \mathbb{R}$ the functions $\phi_n(ix)$ are bounded above – uniformly for all $n \geqslant 2/\alpha$ – by an integrable function of the form $1/(1 + a^2 x^2)$. More precisely,

$$|\phi_n(ix)| \leqslant \prod_{j=1}^{\lceil 2/\alpha \rceil} |1 + i a_j x|^{-\alpha} \leqslant \sqrt{1 + (ax)^2}^{\,-\alpha \cdot \lceil 2/\alpha \rceil} \leqslant \frac{1}{1 + (ax)^2}, \qquad \forall x \in \mathbb{R},\ n \geqslant 2/\alpha,$$

where $a = \min\{a_1, \dots, a_{\lceil 2/\alpha \rceil}\} > 0$. Now apply the Lebesgue dominated convergence theorem and repeat the argument on [39, pp. 334], to show that the Fourier–Mellin integrals of ϕ_n, which are TN_p functions vanishing on $(-\infty, \delta)$, converge to that of ϕ, which function therefore possesses the same properties. ☐

Proof of Lemma 3.8. First suppose $0 \leqslant B \leqslant A$ are as claimed. For $\lambda \in (0,1)$, the Loewner convexity condition can be reformulated in two ways:

$$\frac{f[B + \lambda(A - B)] - f[B]}{\lambda} \leqslant f[A] - f[B],$$

$$\frac{f[A+(1-\lambda)(B-A)]-f[A]}{1-\lambda} \leqslant f[B]-f[A].$$

Now let $\lambda \to 0^+$ and $\lambda \to 1^-$, respectively. We obtain:

$$(A-B)\circ f'[B] \leqslant f[A]-f[B], \qquad (B-A)\circ f'[A] \leqslant f[B]-f[A].$$

Summing these inequalities gives $(A-B)\circ(f'[A]-f'[B]) \geqslant 0$. Since $A-B$ has only non-zero entries, it has a positive semidefinite 'Schur-inverse'. Take the Schur product with this matrix to obtain $f'[A] \geqslant f'[B]$, as claimed. Adapting the same argument shows that $f'[A_\lambda] \geqslant f'[A_\mu]\ \forall 0 \leqslant \mu \leqslant \lambda \leqslant 1$, where $A_\lambda := \lambda A + (1-\lambda)B$.

Conversely, suppose $0 \leqslant B \leqslant A$ in $\mathbb{P}_n((0,\infty))$ are arbitrary, and f' preserves Loewner monotonicity on $[B,A]$. In the spirit of previous proofs for powers preserving Loewner positivity and monotonicity (see above), another 'integration trick' yields:

$$\begin{aligned} f[(A+B)/2]-f[B] &= \frac{1}{2}\int_0^1 (A-B)\circ f'\left[\lambda\frac{A+B}{2}+(1-\lambda)B\right]\, d\lambda,\\ \frac{f[A]+f[B]}{2}-f[B] = \frac{f[A]-f[B]}{2} &= \frac{1}{2}\int_0^1 (A-B)\circ f'\left[\lambda A+(1-\lambda)B\right]\, d\lambda. \end{aligned} \tag{A.4}$$

Using the Schur product theorem and the hypotheses on f',

$$(A-B)\circ f'\left[\lambda\frac{A+B}{2}+(1-\lambda)B\right] \leqslant (A-B)\circ f'[\lambda A+(1-\lambda)B].$$

Together with (A.4), this yields $f[(A+B)/2] \leqslant \frac{1}{2}(f[A]+f[B])$. Now an easy induction argument, first on $m \geqslant 1$ and then on $k \in [1,2^m]$, yields

$$f\left[\frac{k}{2^m}A+\left(1-\frac{k}{2^m}\right)B\right] \leqslant \frac{k}{2^m}f[A]+\left(1-\frac{k}{2^m}\right)f[B], \qquad \forall m \geqslant 1,\ 1 \leqslant k \leqslant 2^m.$$

Finally, given $\lambda \in (0,1)$ we approximate λ by a sequence of dyadic rationals of the form $k/2^m$. Now the preceding inequality and the continuity of f allows us to deduce that f preserves Loewner convexity on $\{A,B\}$. The same arguments can be adapted, as in the preceding half of this proof, to show that f preserves Loewner convexity on $\{A_\lambda, A_\mu\}$ for $0 \leqslant \mu \leqslant \lambda \leqslant 1$. □

Proof of Corollary 4.2. For the 'if' part, note that every contiguous minor of a Hankel matrix A is a contiguous principal minor of either A or $A^{(1)}$. This shows the result for TP_p by Fekete's lemma 4.1. For TN_p, first let B be a matrix drawn from the Gaussian kernel, say $B = (e^{-(x_j-y_k)^2})_{j,k=1}^n$, with $\mathbf{x},\mathbf{y} \in \mathbb{R}^{n,\uparrow}$. Then $B = D_{\mathbf{x}} V D_{\mathbf{y}}$, where $D_{\mathbf{x}}$ for a vector $\mathbf{x}$ is the diagonal matrix with (k,k) entry $e^{-x_k^2}$, and V is the generalized Vandermonde matrix with (j,k) entry $e^{2x_j y_k} = (e^{2x_j})^{y_k}$, whence non-singular. As every submatrix of B is of this form, it follows that B is TP.

Now given $A_{n\times n}$ Hankel as specified, we have that all contiguous minors of A of order $\leqslant p$ are non-negative. Since the corresponding submatrices are symmetric (and Hankel), it follows that they are all positive semidefinite. Let $B := (e^{-(j-k)^2})_{j,k=1}^n$; then B is TP from above. It follows for $\epsilon > 0$ that every contiguous submatrix of $A+\epsilon B$ of order $\leqslant p$ is positive definite. By Fekete's result, $A+\epsilon B$ is TP_p. Letting $\epsilon \to 0^+$, A is TN_p. The 'only if' part follows by definition. □

Proof of continuity in Proposition 4.3. We claim that $f \equiv 0$ or $f > 0$ on $(0,\infty)$. Indeed, suppose $f(x_0) = 0$ for some $x_0 > 0$. Choose $0 < x < x_0 < y$, apply f entrywise to the Hankel TN matrices $\begin{pmatrix} x_0 & x \\ x & x_0 \end{pmatrix}$, $\begin{pmatrix} x_0 & y \\ y & y^2/x_0 \end{pmatrix}$, and take determinants. It follows that $f(x) = f(y) = 0$, as desired. Using the first of the above test matrices also shows that f is non-decreasing on $(0,\infty)$.

Now suppose $f > 0$ on $(0,\infty)$, and fix $t > 0$. We present Hiai's argument from [28] to show f is continuous at t. For $\epsilon \in (0, t/5)$, we have $0 < t+\epsilon \leqslant \sqrt{(t+4\epsilon)(t-\epsilon)}$. It follows that

$$f(t+\epsilon) \leqslant f\left(\sqrt{(t+4\epsilon)(t-\epsilon)}\right) \leqslant \sqrt{f(t+4\epsilon)f(t-\epsilon)},$$

where the second inequality follows by taking the determinant, after applying f entrywise to the matrix

$$\begin{pmatrix} t+4\epsilon & \sqrt{(t+4\epsilon)(t-\epsilon)} \\ \sqrt{(t+4\epsilon)(t-\epsilon)} & t-\epsilon \end{pmatrix}.$$

Now take $\epsilon \to 0^-$; then continuity follows, since f is positive and non-decreasing on $(0,\infty)$:

$$0 < f(t) \leqslant f(t^+) \leqslant f(t^-) \leqslant f(t), \qquad \forall t > 0. \qquad \square$$

Proof of Lemma 4.4. Let the function $M(x) = 2e^{-|x|} - e^{-2|x|}$ for $x \in \mathbb{R}$. For all integers $n \geqslant 1$,

$$\mathcal{B}\{M^n\}(s) = 2\sum_{k=0}^n (-1)^{k+1}\binom{n}{k}\frac{2^{n-k}(n+k)}{s^2-(n+k)^2} = \frac{p_n(s)}{q_n(s)},$$

say, is the bilateral Laplace transform of $M(x)^n$. Here the polynomial $q_n(s) = \prod_{k=0}^n (s^2-(n+k)^2)$ has all simple roots, and degree $2n+2$. It is easy to check that $\deg(p_n) \leqslant 2n$.

Now for $n=1$ this yields $12/((s^2-1)(s^2-4))$, whose reciprocal is a polynomial, so classical results of Schoenberg [57] imply that $M(x)$ is a Pólya frequency function. Also note that $\deg(p_n) \leqslant 2n$, and one checks by direct evaluation that $p_n(\pm(n+k))$ is non-zero for $0 \leqslant k \leqslant n$, so p_n does not vanish at any root $\pm(n+k)$ of q_n. Finally, $p_n(n)/p_n(2n)$ is also checked to be > 1. Hence the rational function q_n/p_n is not a polynomial for $n > 1$ – in fact, not in the Laguerre–Pólya class. The aforementioned results of Schoenberg now imply that $M(x)^n$ is not a Pólya frequency function. As $M(x)^n$ is integrable and non-vanishing at two points, it follows that $M(x)^n$ is not TN for $n > 1$. $\square$

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