A REMARK ON CHARACTERIZING INNER PRODUCT SPACES VIA STRONG THREE-POINT HOMOGENEITY

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Abstract. We show that a normed linear space is isometrically isomorphic to an inner product space if and only if it is a strongly n-point homogeneous metric space for any (or every) $n \geqslant 3$. The counterpart for $n = 2$ is the Banach–Mazur problem.

Normed linear spaces and inner product spaces are central to much of mathematics, in particular analysis and probability. The goal of this short note is to provide a characterization that we were unable to find in the literature, of when the norm in a linear space squares to an inner product. This characterization is classical in spirit, and is in terms of a "strong" 3-point homogeneity property that holds in all inner product spaces.

Two prevalent themes in the early 20th century involved exploring metric embeddings – e.g. when a (finite or) separable metric space isometrically embeds into the Hilbert space of square-summable sequences $\ell^2(\mathbb{N})$ – and exploring when a normed linear space $(\mathbb{B}, \|\cdot\|)$ is Hilbert, i.e. $\|\cdot\|^2$ arises from an inner product on $\mathbb{B}$. See e.g. [2], [5], [7]–[8], [10]–[13], [15], [19] (three of which are in the same volume of the *Annals*, in contiguous order); additional works on the latter theme can be found cited in [11]. There have also been books – see e.g. [4], [6], [9] – as well as later works.

We begin with some results from the latter theme. Jordan and von Neumann showed in [12] that the norm in a real or complex linear space $\mathbb{B}$ comes from an inner product if and only if the parallelogram law holds in $\mathbb{B}$; they also showed the (real and) complex polarization identity in *loc. cit.* We collect this and other equivalent conditions for the norm $\|\cdot\|$ in a real or complex linear space $\mathbb{B}$ to arise from an inner product:

(1) (Jordan and von Neumann, [12].) $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$ for all $f, g \in \mathbb{B}$. (The parallelogram law.)
(2) (Ficken, [7].) If $f, g \in \mathbb{B}$ with $\|f\| = \|g\|$, then $\|\alpha f + \beta g\| = \|\beta f + \alpha g\|$ for all scalars α, β.
(3) (Day, [5].) If $f, g \in \mathbb{B}$ with $\|f\| = \|g\| = 1$, then $\|f+g\|^2 + \|f-g\|^2 = 4$. (The parallelogram law, but only for rhombi.)
(4) (James, [11], when $\dim \mathbb{B} \geqslant 3$.) For all $f, g \in \mathbb{B}$, $\|f + \alpha g\| \geqslant \|f\|$ for all scalars α, if and only if $\|g + \alpha f\| \geqslant \|g\|$ for all scalars α. (Symmetry of Birkhoff–James orthogonality.)
(5) (Lorch, [15], over $\mathbb{R}$.) There exists a fixed constant $\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$[1] such that whenever $f', g' \in \mathbb{B}$ with $\|f'\| = \|g'\|$, we have $\|f' + \gamma' g'\| = \|g' + \gamma' f'\|$.
(6) (Lorch, [15], over $\mathbb{R}$.) There exists a fixed constant $\gamma \in \mathbb{R} \setminus \{0, \pm 1\}$ such that whenever $f, g \in \mathbb{B}$ with $\|f + g\| = \|f - g\|$, we have $\|f + \gamma g\| = \|f - \gamma g\|$.

Indeed, that every inner product space satisfies these properties is immediate, while the implications (5) $\Longleftrightarrow$ (6) $\Longrightarrow$ (2) were shown by Lorch; and that (1), (2), (3), (4) imply that $\|\cdot\|^2$ arises from an inner product were shown by the respectively named authors above.

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[1]We correct a small typo in Lorch's result [15]: he stated $\gamma' \neq 0, 1$ but omitted excluding -1; but clearly $\gamma' = -1$ "works" for every normed linear space $\mathbb{B}$ and all vectors $f', g' \in \mathbb{B}$, since $\|f' - g'\| = \|g' - f'\|$.

All of these characterizations of an inner product use the norm and the vector space structure (over $\mathbb{R}$ or $\mathbb{C}$) on $\mathbb{B}$. The goal of this note is to isolate the inner product using the metric in $\mathbb{B}$ but *avoiding* both the additive structure and the (real or complex) scalar multiplication. Thus, our result is in the spirit of both of the aforementioned classical themes: characterizing inner products in normed linear spaces, while using metric geometry alone.

1. The main result and its proof

To state our result, first recall from [20] or even [3, pp. 470] that for an integer $n \geqslant 1$, a metric space (X, d) is *n-point homogeneous* if given two finite subsets $Y, Y' \subseteq X$ with $|Y| = |Y'| \leqslant n$, any isometry $T : Y \to Y'$ extends to an isometry : $X \to X$. We will require a somewhat more restrictive notion: a metric space (X, d) will be termed *strongly n-point homogeneous* if each map T as above can be extended to an onto isometry : $X \twoheadrightarrow X$.

We now motivate our main result (and the above definition via onto isometries). It seems to be folklore that Euclidean space $\mathbb{R}^k$ is n-point homogeneous for all n – and more strongly, satisfies that every isometry between finite subsets $Y, Y' \subseteq \mathbb{R}^k$, upon pre- and post- composing with suitable translations in order to send 0 to 0, extends to an orthogonal linear map : $\mathbb{R}^k \to \mathbb{R}^k$.[2] In fact, all inner product spaces satisfy this "orthogonal extension property" for all pairs of isometric finite subsets Y, Y' (see below). Here we are interested in the converse question, i.e.,

(a) if this "orthogonal extension property" (with Y, Y' finite) characterizes inner product spaces (among normed linear spaces); and
(b) if yes, then how much can this property be weakened without disturbing the characterization – and if it can in fact be weakened to use metric geometry alone. (In an arbitrary normed linear space we necessarily cannot use orthogonality or inner products; but we also want to not use the vector space operations either.)

This note shows that indeed (a) holds. Moreover, (b) we can indeed weaken the orthogonal extension property to (i) replacing the orthogonal linear map by merely an onto isometry – not necessarily linear *a priori* – and (ii) working with 3-point subsets Y, Y'. More precisely:

Theorem 1. *Suppose $(\mathbb{B}, \|\cdot\|)$ is a nonzero real or complex normed linear space. Then $\|\cdot\|^2$ arises from an inner product – real or complex, respectively – if and only if $\mathbb{B}$ is strongly n-point homogeneous for any (equivalently, every) $n \geqslant 3$.*

To the best of our ability, we were unable to find such a result proved in the literature. Before proceeding to its proof, we discuss the assertion for $n = 1, 2$. If $n = 1$ then Theorem 1 fails to hold, since every normed linear space $\mathbb{B}$ is strongly one-point homogeneous: given $x, y \in \mathbb{B}$, the translation $z \mapsto z + y - x$ is an onto isometry sending x to y.

If instead $n = 2$ then one is asking if there exists a (real) linear space $(\mathbb{B}, \|\cdot\|)$ with $\|\cdot\|^2$ not arising from an inner product, such that given any $f, f'; g, g' \in \mathbb{B}$ with $\|f' - f\| = \|g' - g\|$, every isometry sending f, f' to g, g' respectively extends to an onto isometry of $\mathbb{B}$. By pre- and post-composing with translations, one can assume $f = g = 0$ and $\|f'\| = \|g'\|$; now one is asking if every real normed linear space with transitive group of onto-isometries fixing 0 (these are called *rotations*) is isometrically isomorphic to an inner product space. Thus we come to the well-known *Banach–Mazur problem* [1] – which was affirmatively answered by Mazur for finite-dimensional $\mathbb{B}$ [16], has counterexamples among non-separable $\mathbb{B}$ [18], and remains open for infinite-dimensional separable $\mathbb{B}$. This is when $n = 2$; and the $n \geqslant 3$ case is Theorem 1.

[2] This property also holds for infinite subsets of $(\mathbb{R}^k, \|\cdot\|_2)$; a weakening of it was termed the *free mobility postulate* by Birkhoff [3, pp. 469–470]. However, this postulate is not satisfied by any infinite-dimensional Hilbert space. This was already pointed out in 1944 by Birkhoff in *loc. cit.*; for a specific counterexample, see e.g. the proof of [21, Theorem 11.4] in ℓ^2, where the left-shift operator sending $Y := \{(x_n)_{n \geqslant 0} \in \ell^2 : x_1 = 0\}$ onto ℓ^2 is an isometry that does not extend to $(1, 0, 0, \dots)$.

Remark 2. Given the preceding paragraph, one can assume in Theorem 1 that $\mathbb{B}$ is infinite-dimensional, since if $\dim \mathbb{B} < \infty$ then strong 3-point homogeneity implies strong 2-point homogeneity, which by Mazur's solution [16] to the Banach–Mazur problem implies $\mathbb{B}$ Euclidean. That said, our proof of Theorem 1 works uniformly over all normed linear spaces, and we believe is simpler than using the Banach–Mazur problem (which moreover cannot be applied for all $\mathbb{B}$).

Proof of Theorem 1. We begin by proving the real case. We first explain the forward implication, starting with $\mathbb{B} \cong (\mathbb{R}^k, \|\cdot\|_2)$ for an integer $k \geqslant 1$. As is asserted (without proof) on [3, pp. 470], $\mathbb{R}^k$ is n-point homogeneous for every $n \geqslant 1$; we now show that it is moreover strongly n-point homogeneous – in fact, that it satisfies the "orthogonal extension property" above. Indeed, let $Y, Y' \subseteq \mathbb{R}^k$, and let an isometry $T : Y \to Y'$. By translating Y and Y', we may assume $0 \in Y \cap Y'$ and $T(0) = 0$. We now claim that T extends to an orthogonal self-isometry $\widetilde{T}$ of $(\mathbb{R}^k, \|\cdot\|_2)$.[3] This linear orthogonal map $\widetilde{T}$ is necessarily injective on $\mathbb{R}^k$, hence surjective as well. This shows the forward implication for $\mathbb{B} \cong \mathbb{R}^k$.

If instead $\mathbb{B}$ is an infinite-dimensional inner product space, then given an isometry T between subsets $Y, Y' \subset \mathbb{B}$ of common size at most n, first pre- and post- compose by translations to assume that $0 \in Y \cap Y'$ and $T(0) = 0$. Now let $\mathbb{B}_0 \cong (\mathbb{R}^k, \|\cdot\|_2)$ be the span of $Y \cup Y'$. By the preceding paragraph, $T : Y \to Y'$ extends to an orthogonal operator on $\mathbb{B}_0$, which we continue to denote by T; as $\mathbb{B}_0$ is a complete subspace of $\mathbb{B}$, the "projection theorem" gives $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_0^\perp$. Now the bijective orthogonal map $T|_{\mathbb{B}_0} \oplus \mathrm{Id}|_{\mathbb{B}_0^\perp}$ completes the proof of the forward implication – for any $n \geqslant 1$.

We next come to the reverse implication; now $n \geqslant 3$. From the definitions, it suffices to work with $n = 3$. Moreover, the cases of $\dim \mathbb{B} = 0, 1$ are trivial since $\mathbb{B}$ is then an inner product space, so the reader may also assume $\dim \mathbb{B} \in [2, \infty]$ in the sequel, if required.

We work via contradiction: let $0 \neq (\mathbb{B}, \|\cdot\|)$ be a 3-point homogeneous normed linear space, with $\|\cdot\|^2$ not induced by an inner product. Then Lorch's (corrected) condition (5) – stated at the start of this note – fails to hold for any $\gamma' \in \mathbb{R} \setminus \{0, \pm 1\}$. Fix such a scalar γ'; then there exist vectors $f', g' \in \mathbb{B}$ with $\|f'\| = \|g'\|$ but $\|f' + \gamma' g'\| \neq \|g' + \gamma' f'\|$. In particular, $0, f', g'$ are distinct. Now let $Y = Y' = \{0, f', g'\}$, and consider the isometry $T : Y \to Y'$ which fixes 0 and interchanges f', g'. By hypothesis, T extends to an onto isometry $: \mathbb{B} \to \mathbb{B}$, which we also denote by T. But then T is affine-linear by the Mazur–Ulam theorem [17], hence is a linear isometry as $T(0) = 0$. This yields

$$\|f' + \gamma' g'\| = \|f' - (-\gamma')g'\| = \|T(f') - T(-\gamma' g')\| = \|g' + \gamma' f'\|,$$

which provides the desired contradiction, and proves the reverse implication.

This proves the real case; finally, suppose $(\mathbb{B}, \|\cdot\|)$ is a nonzero complex normed linear space. The forward implication follows from the real case shown above, since $(\mathbb{B}, \|\cdot\|)$ is certainly a real normed space – which we denote by $\mathbb{B}|_\mathbb{R}$. For the same reason, in the reverse direction we obtain that $\mathbb{B}|_\mathbb{R}$ is isometrically isomorphic (over $\mathbb{R}$) to a real inner product space: $\|f\| = \sqrt{\langle f, f\rangle_\mathbb{R}}$ for a real inner product $\langle \cdot, \cdot \rangle_\mathbb{R}$ on $\mathbb{B}|_\mathbb{R}$ and all $f \in \mathbb{B}|_\mathbb{R}$. Now the complex polarization trick of Jordan–von Neumann [12] gives that

$$\langle f, g\rangle := \langle f, g\rangle_\mathbb{R} - i\langle if, g\rangle_\mathbb{R} \tag{1.1}$$

is indeed a complex inner product on $\mathbb{B}$ satisfying: $\|f\| = \sqrt{\langle f, f\rangle}$ for all f. $\square$

[3] This claim can be proved from first principles and is likely folklore. However, for self-completeness we present a sketch via a "lurking isometry" argument. Fix a maximal linearly independent subset $\{y_1, \dots, y_r\}$ in Y; then by polarization, $2\langle y_i, y_j\rangle = \|y_i - 0\|_2^2 + \|y_j - 0\|_2^2 - \|y_i - y_j\|_2^2 = \cdots = 2\langle T(y_i), T(y_j)\rangle$ for $1 \leqslant i, j \leqslant r$. So the Gram matrix of the $T(y_i)$ equals that of the y_i, and hence is invertible. Next, one shows that the linear extension $\widetilde{T}$ of T from $\{y_1 \dots, y_r\}$ to $\mathrm{span}_\mathbb{R}(Y)$ (hence mapping into $\mathrm{span}_\mathbb{R}(Y')$) indeed agrees with T on Y, and also preserves lengths, hence is injective. Finally, choose orthonormal bases of the orthocomplements in $\mathbb{R}^k$ of $\mathrm{span}_\mathbb{R}(Y)$ and of $\widetilde{T}(\mathrm{span}_\mathbb{R}(Y))$, and map the first of these bases (within $\mathrm{span}_\mathbb{R}(Y)^\perp$) bijectively onto the second; then extend by linearity. Direct-summing these two orthogonal linear maps yields the desired extension of T to a linear self-isometry $\widetilde{T}$ of $(\mathbb{R}^k, \|\cdot\|_2)$. For full details, see [14, Theorem 22.3] and its proof.

2. A second proof; Lorch's characterizations for complex linear spaces

We next provide a second proof of the "reverse implication" of Theorem 1 for complex linear spaces $\mathbb{B} = \mathbb{B}|_{\mathbb{C}}$, which requires the *complex version* of Lorch's (corrected) condition (5) above. Note, the above argument over $\mathbb{R}$ cannot immediately proceed verbatim over $\mathbb{C}$ for two reasons:

(a) Lorch's condition (5) needs to be verified as characterizing a complex inner product.
(b) The Mazur–Ulam theorem does not go through over $\mathbb{C}$ – e.g., the isometry $\eta : (z, w) \mapsto (\overline{z}, w)$ of the Hilbert space $(\mathbb{C}^2, \| \cdot \|_2)$ is neither $\mathbb{C}$-linear nor $\mathbb{C}$-antilinear.

However, since η is $\mathbb{R}$-linear on $\mathbb{C}^2$, this reveals how to potentially fix (b) for a second proof of the reverse implication: it suffices to use Lorch's condition (5) for γ' still real – and avoiding $0, \pm 1$ as above – now for all vectors $f', g' \in \mathbb{B}|_{\mathbb{C}}$. If this still characterizes a complex inner product, then one could continue the above proof verbatim, using the Mazur–Ulam theorem for *real* normed linear spaces and replacing the word "linear" twice by "$\mathbb{R}$-linear" in the proof.

Thus we need to verify if Lorch's condition (5) characterizes a complex inner product. More broadly, one can ask which of Lorch's conditions (I_1)–(I_6) and (I_1') in [15] – which characterized an inner product in a real normed linear space – now do the same over $\mathbb{C}$. In fact, these characterizations have since been cited and applied in many papers that work over real linear spaces. In order to work over complex normed linear spaces – and also given that several characterizations in [5, 7, 12] before Lorch held uniformly over both $\mathbb{R}$ and $\mathbb{C}$ – provides a natural additional reason to ask if Lorch's conditions also characterize complex inner products.

As we are not sure if this is recorded, we quickly explain why this does hold. Indeed, Lorch's characterizations themselves involve real scalars, so even if one starts with a complex normed linear space $\mathbb{B}$, one can still obtain a real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ on $\mathbb{B} = \mathbb{B}|_{\mathbb{R}}$. But then the discussion around (1.1) recovers the complex inner product from $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. This yields

Theorem 3 ("Complex Lorch"). *A complex nonzero normed linear space $(\mathbb{B}, \| \cdot \|)$ is isometrically isomorphic to a complex inner product space if and only if any of the following conditions of Lorch* [15] *holds: $(I_1), (I_1')$ (see condition (6) and (the corrected) condition (5) at the start of this note, respectively), or $(I_2), \dots, (I_6)$ – now stated verbatim in $\mathbb{B}$, with the constants γ, γ', α still being real.*

In particular, this provides a second proof of one implication in Theorem 1 over $\mathbb{C}$.

Finally, we note that the arguments in Section 1 in fact show a strengthening of Theorem 1. Namely: continuing the discussion preceding Theorem 1, the "orthogonal extension property" can be weakened to strong 3-point homogeneity, but in fact to an even weaker condition – wherein one only works with $Y = Y'$ the vertices of an isosceles triangle in $\mathbb{B}$ (with the two equal sides of specified length).

Theorem 4. *The following are equivalent for a nonzero real or complex normed linear space $(\mathbb{B}, \| \cdot \|)$.*

(1) *$\| \cdot \|^2$ arises from an (real or complex) inner product on $\mathbb{B}$.*
(2) *If $Y, Y' \subset \mathbb{B}$ are finite subsets, with $|Y| = |Y'|$, then any isometry $: Y \to Y'$ – up to pre- and post- composing by translations – extends to an $\mathbb{R}$-linear onto isometry $: \mathbb{B} \to \mathbb{B}$.*
(3) *$\mathbb{B}$ is strongly n-point homogeneous for any (equivalently, every) $n \geqslant 3$.*
(4) *Given any isosceles triangle in $\mathbb{B}$, say with vertex set $Y = Y' = \{0, f', g'\}$ such that $\|f'\| = \|g'\| = 1$, the isometry $: Y \to Y'$ that fixes 0 and flips f', g' extends to an onto isometry of $\mathbb{B}$.*

Indeed, we showed above that (4) $\implies$ (1) $\implies$ (2) (as the f', g' in Lorch's condition (5) can be simultaneously rescaled), while (2) $\implies$ (3) $\implies$ (4) is trivial. To see why one cannot assert $\mathbb{C}$-linearity in the second statement, let

$$\mathbb{B} = (\mathbb{C}^2, \| \cdot \|_2), \qquad Y = \{(0,0), (1,0), (i,0)\}, \qquad Y' = \{(0,0), (1,0), (0,1)\},$$

and let $T : Y \to Y'$ fix $(0,0)$ and $(1,0)$, and send $(i,0)$ to $(0,1)$. Then T necessarily cannot extend to a $\mathbb{C}$-linear map. Also note that akin to Theorem 1, here too the final three assertions each characterize inner products using (isosceles) metric geometry alone, and without appealing to the vector space structure in $\mathbb{B}$.

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