

HOMOGENEOUS HERMITIAN HOLOMORPHIC VECTOR BUNDLES AND THE COWEN-DOUGLAS CLASS OVER BOUNDED SYMMETRIC DOMAINS

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ABSTRACT. It is known that all the vector bundles of the title can be obtained by holomorphic induction from representations of a certain parabolic Lie algebra on finite dimensional inner product spaces. The representations, and the induced bundles, have composition series with irreducible factors. Our first main result is the construction of an explicit differential operator intertwining the bundle with the direct sum of its factors. Next, we study Hilbert spaces of sections of these bundles. We use this to get, in particular, a full description and a similarity theorem for homogeneous n -tuples of operators in the Cowen-Douglas class of the Euclidean unit ball in \mathbb{C}^n .

0. INTRODUCTION

A domain in \mathbb{C}^n is said to be *symmetric* if for each of its points z it has an involutive holomorphic automorphism s_z having z as an isolated fixed point. We consider bounded symmetric domains \mathcal{D} in what is known as their standard Harish-Chandra realization. The *irreducible* ones among these (i.e. those that are not product domains) are in one to one correspondence with simple real Lie algebras \mathfrak{g} such that in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the subalgebra \mathfrak{k} has non-zero center. The simply connected group \tilde{G} with Lie algebra \mathfrak{g} acts on \mathcal{D} by holomorphic automorphisms; one has $\mathcal{D} \cong \tilde{G}/\tilde{K}$ with \tilde{K} corresponding to \mathfrak{k} . The complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} has a vector space direct sum decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$. In the realization \mathcal{D} appears as a balanced convex domain in $\mathfrak{p}^+ \cong \mathbb{C}^n$.

By *homogeneous* holomorphic vector bundles (*hhvb*) we mean the ones homogeneous under \tilde{G} . These bundles arise by the process of holomorphic induction from finite dimensional representations (ϱ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$, which is a subalgebra of $\mathfrak{k}^{\mathbb{C}}$. The Hermitian hhvb-s (meaning homogeneous as Hermitian bundles) come from (ϱ, V) such that V has a \tilde{K} invariant inner product. For many questions, only the existence of a Hermitian structure matters, so we will also talk about *Hermitizable* hhvb-s, which can then have many Hermitian structures.

By a well-known theorem of Grauert, every holomorphic vector bundle over a domain is trivial. So, a hhvb is the same thing as a multiplier representation of \tilde{G} on the space of $\text{Hol}(\mathcal{D}, V)$ of V -valued holomorphic functions. We will keep using a certain natural trivialization which we call the *canonical trivialization* (cf. (1.13), (1.14)).

Hermitian hhvb-s jumped into prominence in 1956, when Harish-Chandra used Hilbert spaces of sections of such bundles to construct the holomorphic discrete series of unitary representations of \tilde{G} . In the next three decades, the full scope of this method of constructing unitary representations was explored. All this work was about hhvb-s that are induced by irreducible representations ϱ of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ (which implies ϱ is 0 on \mathfrak{p}^-). In fact, it was clear that more general ϱ can only give direct sums of representations already constructed.

Still, the highly non-trivial more general representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ and the corresponding hhvb-s exist and deserve being studied both for their own sake and for the sake of applications such as theory

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of Cowen-Douglas operators. The general (ϱ, V) has a descending chain of invariant subspaces and the induced hhvb has a chain of homogeneous sub-bundles forming a composition series whose quotients are irreducible representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$, resp. hhvb-s induced by these.

The first half (sections 1 and 2) of this article is devoted to this study. The main result is Theorem 2.5 which (except at some singular values of a parameter) gives an explicit differential operator Γ (which first appeared, in the one variable case in [16]) intertwining in a \tilde{G} -equivariant way a general Hermitian hhvb with the direct sum of the factor bundles of its composition series. The principal elements of the proof are Lemma 1.7, which is essentially an expression for the derivative of the Jacobian matrix of a holomorphic automorphism and Theorem 2.4 which is a less complicated special case of the final Theorem 2.5.

In Section 3, we first discuss whether \tilde{G} -invariant Hilbert spaces, dense in $\text{Hol}(\mathcal{D}, V)$, exist for our bundles. We show that this question can be completely reduced to the case of bundles induced by irreducible (ϱ, V) , where the answer is well-known. Then we investigate whether the gradient type operators making up Γ in Section 2 are bounded as operators from one Hilbert space to another. We can reduce this question to the case of line bundles, but this leads to a completely satisfactory answer only when \mathcal{D} is the Euclidean ball in \mathbb{C}^n .

In Section 4, we consider homogeneous Cowen-Douglas operator n -tuples associated to bounded symmetric domains \mathcal{D} . For the unit disc in \mathbb{C} there is a complete description of these in [17]. Here we extend the two main results of [17] to the case of the unit ball in \mathbb{C}^n ($n \geq 1$); these are our Corollary 4.4 and Theorem 4.5. Whether these results hold for more general \mathcal{D} remains unanswered.

The results of this article were announced previously in [18].

1. HOMOGENEOUS HOLOMORPHIC VECTOR BUNDLES

We consider symmetric domains \mathcal{D} in their standard realization. We assume throughout that \mathcal{D} is irreducible; this is sufficient for our purpose since every bounded symmetric domain is biholomorphically equivalent to a product of such. As Harish-Chandra showed (cf. [9]), every irreducible \mathcal{D} can be constructed as follows.

Let \mathfrak{g} be a simple non-compact real Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that \mathfrak{k} is not semi-simple. Then \mathfrak{k} is the direct sum of its center and of its semisimple part, $\mathfrak{k} = \mathfrak{z} + \mathfrak{k}_{\text{ss}}$, and there is an element \hat{z} which generates \mathfrak{z} and $\text{ad}(\hat{z})$ is a complex structure on \mathfrak{p} .

The complexification $\mathfrak{g}^{\mathbb{C}}$ is then the direct sum $\mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ of the $i, 0, -i$ eigenspaces of $\text{ad}(\hat{z})$. On $\mathfrak{g}^{\mathbb{C}}$, we have the usual inner product $B_{\nu}(X, Y) = -B(X, \nu Y)$, where B is the Killing form and ν is the conjugation with respect to the compact real form $\mathfrak{k} + i\mathfrak{p}$. We let $G^{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and we let $G, K^{\mathbb{C}}, K, P^{\pm}, Z, \dots$ be the analytic subgroups corresponding to $\mathfrak{g}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}, \mathfrak{p}^{\pm}, \mathfrak{z}, \dots$. We denote by \tilde{G} the universal covering group of the group G and by $\tilde{K}, \tilde{K}_{\text{ss}}, \tilde{Z}, \dots$ its analytic subgroups corresponding to $\mathfrak{k}, \mathfrak{k}_{\text{ss}}, \mathfrak{z}, \dots$. Then \tilde{K} is the universal cover of K . \tilde{K} is also contained in $\tilde{K}^{\mathbb{C}}$, the universal cover of $K^{\mathbb{C}}$.

$K^{\mathbb{C}}P^-$ is a parabolic subgroup of $G^{\mathbb{C}}$. $P^+K^{\mathbb{C}}P^-$ is open dense in $G^{\mathbb{C}}$. The corresponding decomposition $g^+g^0g^-$ of any g in $P^+K^{\mathbb{C}}P^-$ is unique and holomorphic. The natural map $G/K \rightarrow G^{\mathbb{C}}/K^{\mathbb{C}}P^-$ is a holomorphic imbedding, its image is in the orbit of P^+ . Applying now $\exp_{\mathfrak{p}^+}^{-1}$ we get the Harish-Chandra realization of G/K as a bounded symmetric domain $\mathcal{D} \subset \mathfrak{p}^+ \cong \mathbb{C}^n$. The kernel of the action is the (finite) center of G . The action of $g \in G$ on $z \in \mathcal{D}$, written $g \cdot z$, is then defined by $\exp(g \cdot z) = (g \exp z)^+$. We will use the notations $k(g, z) = (g \exp z)^0$ and $\exp Y(g, z) = (g \exp z)^-$, so we have

$$(1.1) \quad g \exp z = (\exp(g \cdot z))k(g, z) \exp(Y(g, z)).$$

The \tilde{G} -homogeneous holomorphic vector bundles (hhvb-s) over \mathcal{D} are obtained by holomorphic induction from finite dimensional joint representations of the pair $(\tilde{K}, \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-)$. Now \tilde{K} is simply connected, so this is the same as a pair (ϱ^0, ϱ^-) of representations of $\mathfrak{k}^{\mathbb{C}}$ resp. \mathfrak{p}^- on a vector space V , satisfying

$$(1.2) \quad \varrho^-([Z, Y]) = [\varrho^0(Z), \varrho^-(Y)], \quad Z \in \mathfrak{k}^{\mathbb{C}}, Y \in \mathfrak{p}^-.$$

This condition can also be equivalently written as

$$(1.3) \quad \varrho^-(\text{Ad}(k)Y) = \varrho^0(k)\varrho^-(Y)\varrho^0(k)^{-1}, \quad Y \in \mathfrak{p}^-, k \in \tilde{K}.$$

(We use the same symbols to denote the representations of Lie groups and their Lie algebras.)

We will refer to such a pair simply as the representation (ϱ, V) .

The homogeneous Hermitian holomorphic vector bundles arise from representations (ϱ, V) such that V has an (arbitrary, fixed) $\varrho^0(\tilde{K})$ -invariant inner product. In this case, we call (ϱ, V) a Hermitian representation.

For an invariant inner product to exist on V it is necessary and sufficient (since \hat{z} generates \mathfrak{z} and \tilde{K}_{ss} is compact) that $\varrho^0(\hat{z})$ should be diagonalizable and have purely imaginary eigenvalues. If (ϱ, V) has this property, we say it is a Hermitizable representation, and the holomorphically induced bundle is a Hermitizable homogeneous holomorphic vector bundle (abbreviated *Hhvb*).

Given a Hhvb, it is easy to describe all its possible structures making it homogeneously Hermitian, and most of our results will be independent of the particular structure chosen. This is mainly due to the following well-known consequence of Schur's Lemma: In the direct decomposition of V under $\varrho^0(\tilde{K})$, the isotypic subspaces are orthogonal to each other, no matter which invariant inner product is chosen. Such representations and bundles are the main objects of our study.

Since \hat{z} spans the center \mathfrak{z} of \mathfrak{k} , $\chi_\lambda(\hat{z}) = i\lambda$ defines a character of \mathfrak{k} . By Schur's Lemma V is the orthogonal sum of ϱ^0 -invariant subspaces V^λ on which $\varrho^0(\hat{z}) = i\lambda$ ($\lambda \in \mathbb{R}$). For any λ , we have

$$(1.4) \quad \varrho^-(Y)V^\lambda \subseteq V^{\lambda-1}, \quad Y \in \mathfrak{p}^-$$

because for any v_λ in V^λ ,

$$\begin{aligned} \varrho^0(\exp t\hat{z})(\varrho^-(Y)v_\lambda) &= \varrho^-(\text{Ad}(\exp t\hat{z})Y)\varrho^0(\exp t\hat{z})v_\lambda \\ &= \varrho^-(e^{-it}Y)e^{\lambda it}v_\lambda \\ &= e^{(\lambda-1)it}(\varrho^-(Y)v_\lambda). \end{aligned}$$

It follows immediately that for indecomposable Hermitizable (ϱ, V) we have

$$(1.5) \quad V = \bigoplus_{j=0}^m V_j$$

an orthogonal sum of representations (ϱ_j^0, V_j) of \mathfrak{k} such that $\varrho_j^0(\hat{z}) = i(\lambda - j)$ with some $\lambda \in \mathbb{R}$ determined by ϱ . Writing, for $Y \in \mathfrak{p}^-$, $1 \leq j \leq m$,

$$\varrho_j^-(Y) = \varrho^-(Y)|_{V_{j-1}}$$

we have $\varrho_j^-(Y) \in \text{Hom}(V_{j-1}, V_j)$. These will be our standing notations. We observe that $\varrho^-(Y)$ is just the direct sum of the $\varrho_j^-(Y)$, ($1 \leq j \leq m$). We also note that (1.3) can be written in the concise form

$$(1.6) \quad \varrho_j^- \in \text{Hom}(\mathfrak{p}^-, \text{Hom}(V_{j-1}, V_j))^{\tilde{K}},$$

where the superscript \tilde{K} means the \tilde{K} -invariant elements in the space.

At this point, we note that an indecomposable ϱ determines a real number λ . So we can always write $\varrho = \chi_\lambda \otimes \varrho^{\text{nor}}$, where the real number determined by ϱ^{nor} is 0.

Setting

$$\tilde{V}_j = V_j \oplus \cdots \oplus V_m$$

it is clear that \tilde{V}_j is an invariant subspace for ϱ . The representation induced by ϱ on $\tilde{V}_j/\tilde{V}_{j+1}$ is isomorphic with the representation $(\varrho_j^0, 0)$ (meaning ϱ_j^0 on $\mathfrak{k}^{\mathbb{C}}$ and 0 on \mathfrak{p}^- .)

We write $\text{Ad}_{\mathfrak{p}^+}, \text{Ad}_{\mathfrak{p}^-}$ for the adjoint representation restricted to \tilde{K} or $\tilde{K}^{\mathbb{C}}$ or $\mathfrak{k}^{\mathbb{C}}$ acting on \mathfrak{p}^+ , resp. \mathfrak{p}^- . They are irreducible (since \mathfrak{g} is simple) and they leave invariant the natural Hermitian inner product B_ν of $\mathfrak{g}^{\mathbb{C}}$ restricted to \mathfrak{p}^\pm .

Lemma 1.1. *Let $W \subseteq V_{j-1}$ be an irreducible subspace for ϱ_{j-1}^0 . Then the subspace $\varrho_j^-(\mathfrak{p}^-)W$ of V_j as a \tilde{K} -representation is equivalent to a subrepresentation of $\mathfrak{p}^- \otimes W$.*

Proof. The map $(Y, w) \mapsto \varrho_j^-(Y)w$ of $\mathfrak{p}^- \times W$ (hence also of $\mathfrak{p}^- \otimes W$) into V_j is \tilde{K} -equivariant, since we have

$$\varrho_j^-(\text{Ad}(k)Y)\varrho_{j-1}^0(k)w = \varrho_j^0(k)\varrho_j^-(Y)w$$

by (1.3). The range of the map is then isomorphic to the \tilde{K} invariant complement of its kernel. \square

The analysis of the Hermitian representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ will be continued in Section 2. Here, the following lemmas lead to Proposition 1.5 and Theorem 1.6, a structure theorem of indecomposable Hhhvb-s, which we will actually not use in the rest of this paper.

As usual, for an irreducible \mathcal{D} , we write $p = (r-1)a + b + 2$, where r is the rank and a, b are the multiplicities of the long (resp. short) restricted roots different from the Harish-Chandra strongly orthogonal roots.

Lemma 1.2. *$Z \cap K_{\text{ss}}$ is a finite cyclic group generated by $\exp t\hat{z}$, where $t = 2\pi\frac{p}{n}$ is the smallest positive t such that $\exp t\hat{z} \in K_{\text{ss}}$.*

Proof. The group is finite since it is central in K_{ss} and cyclic because it is a subgroup of Z . Using the computation in [15, Sec. 3] of the relation between \hat{z} and the generator used by Schlichtkrull, [24, Prop 3.4] gives that $\exp t\frac{p}{n}\hat{z} \in K_{\text{ss}}$ if and only if $t \in 2\pi\mathbb{Z}$. This implies the Lemma. \square

We write $\pi : \tilde{G} \rightarrow G$ for the covering map. Corresponding to the direct product $\tilde{K} = \tilde{Z} \cdot \tilde{K}_{\text{ss}}$, every irreducible representation of \tilde{K} is uniquely of the form $\chi_\lambda \otimes \sigma$, where $\chi_\lambda(\exp t\hat{z}) = e^{it\lambda}$ and σ is an irreducible representation of \tilde{K}_{ss} extended trivially to Z . By [24, Cor. 3.2], K_{ss} is simply connected, so $\pi|_{\tilde{K}_{\text{ss}}}$ is an isomorphism. By Lemma 1.2, $(\pi|_{\tilde{K}_{\text{ss}}})^{-1}(\exp_G(2\pi\frac{p}{n}\hat{z}))$ is in the center of \tilde{K}_{ss} , hence of \tilde{K} . So, by Schur's Lemma, there is a well-defined residue class $\Lambda(\sigma)$ in $\mathbb{R}/\frac{n}{p}\mathbb{Z}$ such that

$$(1.7) \quad \sigma((\pi|_{\tilde{K}_{\text{ss}}})^{-1}(\exp_G 2\pi\frac{p}{n}\hat{z})) = e^{2\pi i\frac{p}{n}\Lambda(\sigma)} I$$

(with a little abuse of notation).

We write $\text{Ad}'_{\mathfrak{p}^-}$ for $\text{Ad}_{\mathfrak{p}^-}$ restricted to \tilde{K}_{ss} . By $\text{ad}_{\mathfrak{p}^-}(\hat{z}) = -i$, we have, $\text{Ad}_{\mathfrak{p}^-} = \chi_{-1} \otimes \text{Ad}'_{\mathfrak{p}^-}$, and so

$$(1.8) \quad \Lambda(\text{Ad}'_{\mathfrak{p}^-}) \equiv -1$$

Lemma 1.3. (i) *If $\sigma, \sigma', \sigma''$ are irreducible and σ is contained in $\sigma' \otimes \sigma''$, then $\Lambda(\sigma) = \Lambda(\sigma') + \Lambda(\sigma'')$.*

(ii) *The irreducible representation $\chi_\lambda \otimes \sigma$ of \tilde{K} is the lift under π of a representation of K if and only if $\lambda \in \Lambda(\sigma)$.*

Proof. (i) is trivial. For (ii), we note that π maps a generic element $\tilde{k}_{\text{ss}} \exp_{\tilde{G}} t\hat{z}$ to $(\pi|_{\tilde{K}_{\text{ss}}})(\tilde{k}_{\text{ss}}) \exp_G t\hat{z}$. $\chi_\lambda \otimes \sigma$ is a lift if and only if it is trivial on $\ker(\pi|_{\tilde{K}})$, i.e., if and only if the condition

$$(1.9) \quad \pi(\tilde{k}_{\text{ss}}) = \exp_G -t\hat{z}$$

implies $\sigma(\tilde{k}_{\text{ss}}) = e^{-i\lambda t} I$. By Lemma 1.2, (1.9) holds for some \tilde{k}_{ss} if and only if $t = -2\pi\frac{p}{n}\ell$ with $\ell \in \mathbb{Z}$, and in this case $\tilde{k}_{\text{ss}} = (\pi|_{\tilde{K}_{\text{ss}}})^{-1}(\exp_G 2\pi\frac{p}{n}\ell\hat{z})$. So, finally $\chi_\lambda \otimes \sigma$ is a lift if and only if

$$(1.10) \quad \sigma((\pi|_{\tilde{K}_{\text{ss}}})^{-1}(\exp_G 2\pi\frac{p}{n}\ell\hat{z})) = e^{2\pi i\frac{p}{n}\ell\lambda} I$$

for all $\ell \in \mathbb{Z}$. Clearly, this holds for all ℓ if and only if it holds for $\ell = 1$. For $\ell = 1$, the left hand side is $e^{2\pi i\frac{p}{n}\Lambda(\sigma)} I$ by definition of $\Lambda(\sigma)$. Hence (1.10) holds if and only if $\lambda \in \Lambda(\sigma)$ finishing the proof. \square

Definition 1.4. A Hermitizable representation (ϱ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ (and the Hhhvb induced by it) is said to be *elementary* if for some $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$, it is of the form $\bigoplus_{j=0}^m V_j$ with $\text{ad}(\hat{z}) = i(\lambda - j)$ on V_j and if $\Lambda(\sigma) + j$ is the same for every irreducible component σ of $\varrho_j^{0, \text{ss}}$ (meaning ϱ_j^0 restricted to \mathfrak{k}_{ss}), for every $0 \leq j \leq m$.

Proposition 1.5. *If (ϱ, V) is an indecomposable Hermitian representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ (i.e. the induced holomorphic homogeneous Hermitian vector bundle is irreducible), then it is elementary.*

Proof. Let (ϱ, V) be indecomposable. In (1.5) we have already seen that there is a decomposition $\bigoplus V_j$ as stated. Now, let Λ be a residue class in $\mathbb{R}/\frac{n}{p}\mathbb{Z}$ and let $V(\Lambda)$ denote the direct sum of all the irreducible constituents σ of V_j , for each $1 \leq j \leq m$, such that $\Lambda(\sigma) + j = \Lambda$. It is immediate from Lemmas 1.1 and 1.3 that $V(\Lambda)$ is invariant under both ϱ^0 and ϱ^- . Hence by indecomposability, there can be only one class Λ such that $V(\Lambda) \neq 0$. \square

Theorem 1.6. *Every elementary Hhhvb E can be written as a tensor product $L_{\lambda_0} \otimes E'$, where L_{λ_0} is the line bundle induced by the character χ_{λ_0} and E is the lift to \tilde{G} of a G -homogeneous holomorphic Hermitian vector bundle which is the restriction to G and \mathcal{D} of a $G^{\mathbb{C}}$ -homogeneous vector bundle over $G^{\mathbb{C}}/K^{\mathbb{C}}P^-$ induced in the holomorphic category by a representation of $K^{\mathbb{C}}P^-$.*

Proof. Suppose E is induced by (ϱ, V) , $V = \bigoplus_0^m V^{\lambda-j}$. We can take any irreducible component σ of $\varrho_{\lambda}^{0, \text{ss}}$, choose some $\lambda' \in \Lambda(\sigma)$ and set $\lambda_0 = \lambda - \lambda'$. Then we can write $\varrho^0 = \chi_{\lambda_0} \otimes \varrho'^0$. By Lemma 1.3, $\varrho_{\lambda}^{\prime, 0}$ is a lift of a representation of K to \tilde{K} and also a representation of $\mathfrak{k}^{\mathbb{C}}$. It follows that it extends then to a holomorphic representation of $K^{\mathbb{C}}$. The ϱ^- part which is unchanged gives a representation of P^- since P^- is simply connected. So, we have a representation of the semidirect product $\tilde{K}^{\mathbb{C}}P^-$, and the Theorem follows. \square

We will study our irreducible Hhhvb-s through a natural trivialization which can be obtained in one of two ways. One way is based on Theorem 1.6, putting together the natural trivializations of L_{λ} (where the multiplier is a power of the jacobian, see e.g. [15]) and a trivialization of E' built from $k(g, z)$ (as defined in (1.1)). The other way, which we will actually follow, makes use of the Herb-Wolf local complexification of \tilde{G} (cf. [10]). In either approach, the point is to define a $\tilde{K}^{\mathbb{C}}$ -valued multiplier $\tilde{k}(g, z)$ and prove its properties.

We write $\pi : \tilde{K}^{\mathbb{C}} \rightarrow K^{\mathbb{C}}$ for the universal covering map. As shown in [10], $P^+ \times \tilde{K}^{\mathbb{C}} \times P^-$ can be given a structure of complex analytic local group such that (writing $\pi : \tilde{K}^{\mathbb{C}} \rightarrow K^{\mathbb{C}}$) $\text{id} \times \pi \times \text{id}$ is the universal local group covering of $P^+K^{\mathbb{C}}P^-$. We write \tilde{G}_{loc} for this local group and abbreviate $\text{id} \times \pi \times \text{id}$ to π . By [10], \tilde{G} , $\tilde{K}^{\mathbb{C}}P^-$, $P^+\tilde{K}^{\mathbb{C}}$ are closed subgroups of $\tilde{G}_{\text{loc}}^{\mathbb{C}}$ and $\tilde{G} \exp \mathcal{D} \subset \tilde{G}_{\text{loc}}^{\mathbb{C}}$. π restricted to \tilde{G} is the covering map of G . Defining $g \cdot z = \pi(g) \cdot z$ and $Y(g, z) = Y(\pi(g), z)$ we have the decomposition

$$(1.11) \quad g \exp z = (\exp g \cdot z) \tilde{k}(g, z) \exp Y(g, z), \quad (g \in \tilde{G}, z \in \mathcal{D})$$

in \tilde{G}_{loc} . We write $\tilde{b}(g, z) = \tilde{k}(g, z) \exp Y(g, z)$. Then applying (1.11) twice, we have

$$(\exp gg'z) \tilde{b}(gg', z) = gg' \exp z = g(\exp g'z) \tilde{b}(g', z) = (\exp gg'z) \tilde{b}(g, g'z) \tilde{b}(g', z)$$

which shows that $\tilde{b}(g, z)$ satisfies the multiplier identity

$$(1.12) \quad \tilde{b}(gg', z) = \tilde{b}(g, g'z) \tilde{b}(g', z).$$

Furthermore, we clearly have $\tilde{b}(kp^-, 0) = kp^-$ for $kp^- \in \tilde{K}^{\mathbb{C}}P^-$.

It follows that given a representation (ϱ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ as above, $\varrho(\tilde{b}(g, z))$ is a multiplier, and

$$(1.13) \quad \varrho(\tilde{b}(g, z)) = \varrho^0(\tilde{k}(g, z)) \varrho^-(\exp Y(g, z)).$$

The vector bundle E^ϱ holomorphically induced by ϱ has a trivialization to be called the *canonical trivialization* in which the space of sections is $\text{Hol}(\mathcal{D}, V)$, and the \tilde{G} -action on it is the multiplier representation U^ϱ :

$$(1.14) \quad (U_g^\varrho f)(z) = \varrho(\tilde{b}(g^{-1}, z))^{-1} f(g^{-1}z).$$

The canonical trivialization will be used throughout the rest of this paper.

It is clear from the product expression (1.13) that $\text{Hol}(\mathcal{D}, \tilde{V}_j)$ for each j , is an U^ϱ -invariant subspace of $\text{Hol}(\mathcal{D}, V)$, and the representation induced by U^ϱ on $\text{Hol}(\mathcal{D}, \tilde{V}_j)/\text{Hol}(\mathcal{D}, \tilde{V}_{j+1})$ is the same as the representation on $\text{Hol}(\mathcal{D}, V_j)$ via the multiplier $\varrho_j^0(\tilde{k}(g, z))$. In other words, we have a chain of homogeneous sub-bundles \tilde{E}_j with $E_j = \tilde{E}_j/\tilde{E}_{j+1}$ holomorphically induced by $(\varrho_j^0, 0)$ on V_j .

If $f \in \text{Hol}(\mathcal{D}, V)$, then we write Df for the derivative: $Df(z)X = (D_X f)(z)$ for $X \in \mathfrak{p}^+$. Thus $Df(z)$ is a \mathbb{C} -linear map from \mathfrak{p}^+ to V . The following Lemma is crucial for the computations of Section 2.

Lemma 1.7. *For any holomorphic representation τ of $\tilde{K}^\mathbb{C}$ and any $g \in \tilde{G}$, $z \in \mathcal{D}$, $X \in \mathfrak{p}^+$,*

$$D_X \tau(\tilde{k}(g, z)^{-1}) = -\tau([Y(g, z), X])\tau(\tilde{k}(g, z)^{-1}).$$

Furthermore,

$$D_X Y(g, z) = \frac{1}{2}[Y(g, z), [Y(g, z), X]].$$

Proof. We have, using the exponential map of $\tilde{G}_{\text{loc}}^\mathbb{C}$

$$\begin{aligned} g \exp(z + tX) &= g \exp z \exp tX \\ &= \exp(gz) \tilde{k}(g, z) \exp Y \exp tX \\ &= \exp(gz) \tilde{k}(g, z) \exp t\{X + [Y, X] + \frac{1}{2}[Y, [Y, X]]\} \exp(Y), \end{aligned}$$

where we have used the abbreviated notation $Y = Y(g, z)$. By the Campbell-Hausdorff formula, and since $\tilde{K}^\mathbb{C}$ normalizes P^+ , equating the $\tilde{K}^\mathbb{C}$ parts of the two sides, we get

$$\tilde{k}(g, z + tX) = \tilde{k}(g, z) \exp(t[Y, X] + O(t^2)).$$

Applying τ to the inverse and taking $\frac{d}{dt}|_0$ gives the first statement. Looking at the P^- part of the decomposition we get the second statement. \square

Remark 1.8. Equating the P^+ -parts of the identity above we get

$$\exp g(z + tX) = \exp(gz + t \text{Ad}(\tilde{k}(g, z))X + O(t^2)),$$

whence, slightly extending [23, p. 65], for all $g \in \tilde{G}$, we have

$$(1.15) \quad Dg(z) = \text{Ad}_{\mathfrak{p}^+} \tilde{k}(g, z).$$

Further we note that by the general identity $DF = -F(DF^{-1})F$ we also know $D\tau(\tilde{k}(g, z))$. Taking $\tau = \text{Ad}_{\mathfrak{p}^+}$, and using (1.15), the Lemma also gives an explicit expression for $D^2g(z)$.

2. THE MAIN RESULTS ABOUT VECTOR BUNDLES

For a more detailed description of the indecomposable Hermitizable representations (ϱ, V) of $\mathfrak{k}^\mathbb{C} + \mathfrak{p}^-$ we have to make some normalizations. We already know that ϱ determines a real number λ ; we also keep using the decomposition $V_0 \oplus \cdots \oplus V_m$ and the restrictions ϱ^0, ϱ_i^0 ($1 \leq i \leq m$), ϱ^- of ϱ as in Section 1. We consider the set of all irreducible representations (α, W^α) of $\mathfrak{k}_{\text{ss}}^\mathbb{C}$ and choose a fixed \tilde{K} -invariant inner product $\langle \cdot, \cdot \rangle_\alpha$ on W^α ; when $\alpha = \text{Ad}'_{\mathfrak{p}^-}$ we choose the restriction of B_ν to to \mathfrak{p}^- .

For any α the tensor product $\text{Ad}'_{\mathfrak{p}^-} \otimes \alpha$ is multiplicity free (cf. [11, Corollary 4.4], or in a wider context [13]). For every irreducible component β of it we choose and fix an equivariant partial isometry $P_{\alpha\beta} : \mathfrak{p}^- \otimes W^\alpha \rightarrow W^\beta$. For $Y \in \mathfrak{p}^-$ and $v \in W^\alpha$, we define

$$(2.1) \quad \tilde{\varrho}_{\alpha\beta}(Y)v = P_{\alpha\beta}(Y \otimes v).$$

We start our closer study of the indecomposable Hermitizable representations (ϱ, V) and the corresponding Hhhvb with the case where ϱ is irreducible. Then $m = 0$ (since each $\tilde{V}_j = V_j + \dots + V_m$ is always an invariant subspace). This implies $\varrho^- = 0$ and hence ϱ^0 is irreducible, i.e., $\varrho^0 = \chi_\lambda \otimes \alpha$ with some α as above. We may assume, without restriction of generality, that $V = W^\alpha$ as a vector space; the possible inner products are $\langle \cdot, \cdot \rangle_V = H \langle \cdot, \cdot \rangle_\alpha$ with some number $H > 0$. We denote the corresponding Hhhvb by $E^{\alpha, \lambda}$.

A little more generally, when we have a multiple of an irreducible ϱ , i.e. $\varrho^- = 0$ and $\varrho^0 = \chi_\lambda I_\alpha \otimes \alpha$ on $V = \mathbb{C}^d \otimes W^\alpha$, the \tilde{K} invariant inner products on V are the tensor products of $\langle H \cdot, \cdot \rangle_{\mathbb{C}^d}$ on \mathbb{C}^d and $\langle \cdot, \cdot \rangle_\alpha$ on W^α with some positive definite matrix H .

(We might note that at this point all choices of μ still give isometrically isomorphic Hermitian representations and homogeneous holomorphic vector bundles)

We will now study the case of indecomposable (ϱ, V) such that m is arbitrary and each summand (ϱ_j^0, V_j) is irreducible. We say that such a ϱ and the corresponding Hhhvb are *filiform*. This case is the key to the general case.

Now, $\varrho_j^0 = \chi_{\lambda-j} \otimes \alpha_j$ ($0 \leq j \leq m$) and V_j is W^{α_j} as a vector space with eventual inner product determined by a positive number H_j . We will use the abbreviations $W_j = W^{\alpha_j}$, $P_j = P_{\alpha_{j-1}, \alpha_j}$, $\tilde{\varrho}_j = \tilde{\varrho}_{\alpha_{j-1}, \alpha_j}$. So

$$(2.2) \quad \tilde{\varrho}_j(Y)v = P_j(Y \otimes v)$$

for $Y \in \mathfrak{p}^-$, $v \in V_{j-1}$. Clearly $\tilde{\varrho}_j$ satisfies (1.6). This space of \tilde{K} invariants is isomorphic with the space of \tilde{K} -equivariant maps $\mathfrak{p}^- \otimes V_{j-1} \rightarrow V_j$, hence is 1 dimensional. It follows that

$$\varrho_j^-(Y) = y_j \tilde{\varrho}_j(Y)$$

for each j with a number y_j . Also, $y_j \neq 0$ since indecomposability is part of the definition of filiform. Since the ϱ_j^- ($1 \leq j \leq m$) together form a representation of the Abelian Lie algebra \mathfrak{p}^- , we have

$$(2.3) \quad \tilde{\varrho}_{j+1}(Y') \tilde{\varrho}_j(Y) = \tilde{\varrho}_{j+1}(Y) \tilde{\varrho}_j(Y')$$

for all $Y, Y' \in \mathfrak{p}^-$ and $1 \leq j \leq m-1$. In terms of P_j , this means

$$(2.4) \quad P_{j+1}(Y' \otimes P_j(Y \otimes v)) = P_{j+1}(Y \otimes P_j(Y' \otimes v))$$

for all $Y, Y' \in \mathfrak{p}^-$ and $v \in V_{j-1}$. A third equivalent way to write this condition is

$$(2.5) \quad P_{j+1} \tilde{\varrho}_j(Y) = \tilde{\varrho}_{j+1}(Y) P_j.$$

(Here on the left hand side $\tilde{\varrho}_j(Y)$ is really an abbreviation for $I \otimes \tilde{\varrho}_j(Y)$.)

To summarize, any sequence $\alpha = (\alpha_0, \dots, \alpha_m)$ such that α_j is contained in $\text{Ad}_{\mathfrak{p}^-} \otimes \alpha_{j-1}$ and such that the P_j -s satisfy (2.4), together with $\lambda \in \mathbb{R}$ and a sequence $y = (y_1, \dots, y_m)$ of non-zero numbers determine a filiform Hermitizable representation. Its possible Hermitian structures are given by sequences $H = (H_0, \dots, H_m)$ of positive numbers. There is considerable redundancy here. (In fact, all choices of $y_j \neq 0$ ($\forall j$) give isomorphic hhhvb-s, while $y_j > 0$, $H_j = 1$ ($\forall j$) is one possible normalization of the hhhvb-s.) But this is not important at this point.

We proceed towards Theorem 2.4, the main result about the filiform case.

We denote by ι the identification of $(\mathfrak{p}^+)^*$ with \mathfrak{p}^- under the Killing form, and for any vector space W , extend it to a map from $\text{Hom}(\mathfrak{p}^+, W)$ to $\mathfrak{p}^- \otimes W$; that is, for $Y \in \mathfrak{p}^-$, $w \in W$,

$$\iota(B(\cdot, Y)w) = Y \otimes w.$$

For any $T \in \text{Hom}(\mathfrak{p}^+, W)$ and for all $k \in \tilde{K}^\mathbb{C}$, the invariance of B implies

$$(2.6) \quad \iota(T \circ \text{Ad}_{\mathfrak{p}^+}(k^{-1})) = \text{Ad}_{\mathfrak{p}^-}(k)\iota(T).$$

Of course, linear transformations affecting only W commute with ι . In particular, as in our later applications, if W is some space of linear transformations $F_1 \rightarrow F_2$ and $U : F_2 \rightarrow F_3$ and $V : F_0 \rightarrow F_1$ are fixed linear transformations, then

$$(2.7) \quad \iota(UTV) = U\iota(T)V,$$

Lemma 2.1. *Let ϱ be a filiform representation. Then there exist constants u, w independent of λ , such that for all $Y \in \mathfrak{p}^-$, we have*

$$(2.8) \quad P_j \iota \varrho_{j-1}^0([Y, \cdot]) = c_j(\lambda) \tilde{\varrho}_j(Y),$$

where

$$(2.9) \quad c_j(\lambda) = (u + (j-1)w - \frac{\lambda}{2n}).$$

Proof. We have $\varrho_{j-1}^0 = \chi_{\lambda-j+1} \otimes \varrho_{j-1}^0$, with ϱ_{j-1}^0 trivial on \mathfrak{z} (i.e. a representation of \tilde{K}_{ss}). Now

$$(2.10) \quad \varrho_{j-1}^0([Y, \cdot]) = \chi_{\lambda-j+1}([Y, \cdot]) \otimes I_{V_{j-1}} + \varrho_{j-1}^0([Y, \cdot]).$$

The first term, evaluated on $X \in \mathfrak{p}^+$, depends only on the projection of $[Y, X]$ onto $\mathfrak{z}^\mathbb{C}$. This projection is equal to

$$\frac{B_\nu([Y, X], \hat{z})}{B_\nu(\hat{z}, \hat{z})} \hat{z} = \frac{B([\hat{z}, Y], X)}{B(\hat{z}, \hat{z})} \hat{z} = \frac{i}{2n} B(Y, X) \hat{z},$$

where we have used $\nu \hat{z} = \hat{z}$ and $B(\hat{z}, \hat{z}) = -2n$. Hence

$$\chi_{\lambda-j+1}([Y, \cdot]) = -\frac{\lambda-j+1}{2n} B(Y, \cdot),$$

and so, applying $P_j \circ \iota$ to the first term on the right in (2.10) we obtain

$$(2.11) \quad -\frac{\lambda-j+1}{2n} \tilde{\varrho}_j(Y).$$

Next we apply $P_j \circ \iota$ to the second term in (2.10). We get an element of $\text{Hom}(V_{j-1}, V_j)$ which because of (2.6) and the equivariance of P_j depends on $Y \in \mathfrak{p}^-$ in a \tilde{K} -equivariant way. But we already know that every equivariant map from \mathfrak{p}^- to $\text{Hom}(V_{j-1}, V_j)$ is a constant multiple of $\tilde{\varrho}_j$. Putting this together with (2.11) we have (2.8) with

$$(2.12) \quad c_j(\lambda) = c'_j - \frac{\lambda}{2n},$$

where c'_j is some constant independent of λ .

To prove (2.9) it will be enough to prove that $c_{j+1}(\lambda) - c_j(\lambda)$ is independent of j , ($1 \leq j \leq m-1$). For this we give another expression for the left hand side of (2.8). Let $\{e_\beta\}$ be a basis for \mathfrak{p}^+ and $\{e'_{-\beta}\}$ the B -dual basis of \mathfrak{p}^- . Expanding an arbitrary $X \in \mathfrak{p}^+$ in terms of the basis we have

$$\varrho_{j-1}^0([Y, X]) = \sum_{\beta} B(e'_{-\beta}, X) \varrho_{j-1}^0([Y, e_\beta])$$

and

$$\iota \varrho_{j-1}^0([Y, \cdot]) = \sum_{\beta} e'_{-\beta} \otimes \varrho_{j-1}^0([Y, e_\beta])$$

Applying P_j to this, we can rewrite (2.8) as

$$(2.13) \quad c_j(\lambda) \tilde{\varrho}_j(Y) = \sum_{\beta} \tilde{\varrho}_j(e'_{-\beta}) \varrho_{j-1}^0([Y, e_\beta])$$

We choose Y, Y', Y'' in \mathfrak{p}^- such that

$$(2.14) \quad \tilde{\varrho}_{j+2}(Y'') \tilde{\varrho}_{j+1}(Y') \tilde{\varrho}_j(Y) \neq 0.$$

This is possible by the irreducibility of each V_j . Now we write (2.13) with $j + 1$ instead of j , multiply on the right by $\tilde{\varrho}_j(Y')$, then use that ϱ is a representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$:

$$(2.15) \quad \begin{aligned} c_{j+1}(\lambda)\tilde{\varrho}_{j+1}(Y)\tilde{\varrho}_j(Y') &= \\ & \sum_{\beta} \tilde{\varrho}_{j+1}(e'_{-\beta})\tilde{\varrho}_j(Y')\varrho_{j-1}^0([Y, e_{\beta}]) + \sum_{\beta} \tilde{\varrho}_{j+1}(e'_{-\beta})\tilde{\varrho}_j([[Y, e_{\beta}], Y']). \end{aligned}$$

We multiply (2.13) on the left by $\tilde{\varrho}_{j+1}(Y')$ and subtract it from (2.15). Using (2.3) on both sides, we obtain

$$(2.16) \quad (c_{j+1}(\lambda) - c_j(\lambda))\tilde{\varrho}_{j+1}(Y')\tilde{\varrho}_j(Y) = \sum_{\beta} \tilde{\varrho}_{j+1}(e'_{-\beta})\tilde{\varrho}_j([[Y, e_{\beta}], Y']).$$

Now we write this with $j + 1$ in place of j , multiply on the right by $\tilde{\varrho}_j(Y'')$, and compare the resulting equality with (2.15) left multiplied by $\tilde{\varrho}_{j+2}(Y'')$. By (2.3), the right hand sides are equal, and by (2.14) it follows that

$$c_{j+2}(\lambda) - c_{j+1}(\lambda) = c_{j+1}(\lambda) - c_j(\lambda).$$

Since this holds for every j , the proof is complete. \square

Lemma 2.2. *Let ϱ be a filiform representation. For all $0 \leq j \leq m - 1$, and holomorphic $F : \mathcal{D} \rightarrow V_j$,*

$$\begin{aligned} P_{j+1}\iota D^{(z)}\{\varrho_j^0(\tilde{k}(g, z)^{-1})F(gz)\} \\ = -c_{j+1}(\lambda)\tilde{\varrho}_{j+1}(Y(g, z))(\varrho_j^0(\tilde{k}(g, z)^{-1})F(gz)) + \varrho_{j+1}^0(\tilde{k}(g, z)^{-1})((P_{j+1}\iota DF)(gz)), \end{aligned}$$

where $D^{(z)}$ denotes differentiation with respect to z .

Proof. Applying the Leibniz product rule on the left hand side we get

$$P_{j+1}\iota(D^{(z)}\varrho_j^0(\tilde{k}(g, z)^{-1}))F(gz) + P_{j+1}\iota\varrho_j^0(\tilde{k}(g, z)^{-1})D^{(z)}\{F(gz)\}.$$

To the first term we apply Lemma 1.7, then Lemma 2.1 and obtain the first term in the assertion of the Lemma. The second term, by (1.15), (2.6) and the equivariance of P_{j+1} gives the second term in the assertion. \square

Lemma 2.3. *Let ϱ be a filiform representation. For all $1 \leq j \leq m - 1$, with the constant w of Lemma 2.1,*

$$P_{j+1}\iota D^{(z)}\tilde{\varrho}_j(Y(g, z)) = -\frac{w}{2}\tilde{\varrho}_{j+1}(Y(g, z))\tilde{\varrho}_j(Y(g, z)).$$

Proof. We abbreviate $Y = Y(g, z)$. Using Lemma 1.7, the linearity of $\tilde{\varrho}_j$, and that $\tilde{\varrho}_j(Y)$ is the restriction to V_{j-1} of the representation $\varrho_{(1, \dots, 1)}$ of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ we find

$$\begin{aligned} P_{j+1}\iota D^{(z)}\tilde{\varrho}_j(Y) &= \frac{1}{2}P_{j+1}\iota\tilde{\varrho}_j([Y[Y, \cdot]]) \\ &= \frac{1}{2}P_{j+1}\iota\tilde{\varrho}_j(Y)\varrho_{j-1}^0([Y, \cdot]) - \frac{1}{2}P_{j+1}\iota\varrho_{j+1}^0([Y, \cdot])\tilde{\varrho}_j(Y). \end{aligned}$$

Now ι commutes with $\tilde{\varrho}_j(Y)$, and from (2.4) we have $P_{j+1}\tilde{\varrho}_j = \tilde{\varrho}_{j+1}P_j$. Using this and Lemma 2.1, we get that the first term equals

$$\frac{1}{2}c_j(\lambda)\tilde{\varrho}_{j+1}(Y)\tilde{\varrho}_j(Y).$$

For the second term, Lemma 2.1 immediately gives

$$-\frac{1}{2}c_{j+1}(\lambda)\tilde{\varrho}_{j+1}(Y)\tilde{\varrho}_j(Y).$$

The statement now follows from (2.9). \square

For an indecomposable filiform Hhhvb E^ϱ as described above, we will use the notation $E^{y,\lambda}$. Writing $0 = (0, \dots, 0)$, E^0 makes sense, it is the direct sum of the irreducible factor bundles in the composition series of E^y .

We denote by U^y resp. U^0 the \tilde{G} -action on the sections of E^y and E^0 defined by (1.14); we observe that in the case of U^0 the second factor in (1.13) is identically the identity.

If $f \in \text{Hol}(\mathcal{D}, V)$, we write f_j for the component of f in V_j , that is, the projection of f onto V_j . We continue using the notations introduced up to this point.

Theorem 2.4. *Let ϱ be a filiform representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+$, and E^y the holomorphically induced vector bundle. Suppose that $\lambda \in \mathbb{R}$ is regular in the sense that*

$$\begin{aligned} c_{ij} &= \frac{2^{i-j}}{(i-j)!} \prod_{k=1}^{i-j} (c_{j+1}(\lambda) + c_{j+k}(\lambda))^{-1} \\ &= \frac{1}{(i-j)!} \prod_{k=1}^{i-j} \left\{ u + \left(j + \frac{k-1}{2} \right) w - \frac{\lambda}{2n} \right\}^{-1} \end{aligned}$$

is meaningful for $0 \leq j < i \leq m$. Then the operator $\Gamma : \text{Hol}(\mathcal{D}, V) \rightarrow \text{Hol}(\mathcal{D}, V)$ given by

$$(\Gamma f_j)_\ell = \begin{cases} c_{\ell j} y_\ell \cdots y_{j+1} (P_\ell \iota D) \cdots (P_{j+1} \iota D) f_j & \text{if } \ell > j, \\ f_j & \text{if } \ell = j, \\ 0 & \text{if } \ell < j \end{cases}$$

intertwines the actions U^0 and U^y of \tilde{G} on the trivialized sections of E^0 and E^y .

Proof. It is helpful to think of f as a (column) vector with entries f_j and of Γ as a lower triangular matrix.

We must show that Γ intertwines the actions of \tilde{G} via the multipliers $\varrho^0(\tilde{k}(g, z))$ respectively $\varrho^0(\tilde{k}(g, z))\varrho^-(\exp Y(g, z))$. The first multiplier acts diagonally. For the second multiplier, we observe that $\varrho^-(Y)$ acts by a subdiagonal matrix

$$\varrho^-(Y)_{j,k} = \delta_{j-1,k} y_j \tilde{\varrho}_j(Y).$$

Hence, by exponentiation, $\varrho^-(\exp Y(g, z))$ is lower triangular and for $i \geq j$,

$$\begin{aligned} \varrho^-(\exp Y(g, z))_{i,j} &= \exp \left(\varrho^-(Y(g, z)) \right)_{i,j} \\ (2.17) \quad &= \frac{1}{(i-j)!} y_i \cdots y_{j+1} \tilde{\varrho}_i(Y(g, z)) \cdots \tilde{\varrho}_{j+1}(Y(g, z)). \end{aligned}$$

The intertwining property to be proved is

$$(2.18) \quad \Gamma \left(\varrho^0(\tilde{k}(g, z)^{-1}) f(gz) \right) = \varrho^-(\exp -Y(g, z)) \varrho^0(\tilde{k}(g, z)^{-1}) (\Gamma f)(gz)$$

We set $f = f_j$ (thinking of f as a “vector” whose only non-zero component is the j^{th} one) and write the ℓ^{th} component of the left hand side, for $\ell \geq j$,

$$(2.19) \quad c_{\ell j} y_\ell \cdots y_{j+1} (P_\ell \iota D) \cdots (P_{j+1} \iota D) \left(\varrho_j^0(\tilde{k}(g, z)^{-1}) (f_j \circ g)(z) \right).$$

Using the abbreviation

$$F^{(i)} = (P_i \iota D) \cdots (P_{j+1} \iota D) f_j,$$

the corresponding component on the right hand side of (2.18) is

$$(2.20) \quad \sum_i \frac{(-1)^{\ell-i}}{(\ell-i)!} y_\ell \cdots y_{i+1} \tilde{\varrho}_\ell(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) c_{i,j} y_i \cdots y_{j+1} \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz)$$

with the terms being non-zero only for $i \geq j$. So, verifying (2.18) amounts to verifying

$$(2.21) \quad c_{\ell,j}(P_\ell \iota D) \cdots (P_{j+1} \iota D) \left(\varrho_j^0(k(g, z)^{-1}) f_j(gz) \right) \\ = \sum_{i=j}^{\ell} \frac{(-1)^{\ell-i}}{(\ell-i)!} c_{i,j} \tilde{\varrho}_\ell(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz)$$

for all $\ell \geq j$. We prove (2.21) by induction on $\ell \geq j$. For $\ell = j$ the identity is trivial. To pass from ℓ to $\ell + 1$, we have to show that applying $(P_{\ell+1} \iota D)$ to the right hand side we get $\frac{c_{\ell,j}}{c_{\ell+1,j}}$ times the analogous expression with $\ell + 1$ in place of ℓ . Using the product rule for ιD , in a first step, we get

$$\sum_i \frac{(-1)^{\ell-i}}{(\ell-i)!} c_{i,j} P_{\ell+1} \left\{ \sum_{k=i+1}^{\ell} \tilde{\varrho}_\ell(Y(g, z)) \cdots (\iota D \tilde{\varrho}_k(Y(g, z))) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz) + \right. \\ \left. \tilde{\varrho}_\ell(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \iota D (\varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz)) \right\}$$

Repeated application of (2.4) moves $P_{\ell+1}$ forward to give $P_{k+1} \iota D(\tilde{\varrho}_k(Y(g, z)))$ in the terms of the sum over k and $P_{i+1} \iota D$ in the last factor of the last term. At this point Lemmas 2.2 and 2.3 can be applied and give, after collecting like terms,

$$\sum_i \frac{(-1)^{\ell-i}}{(\ell-i)!} c_{i,j} \left\{ -\frac{1}{2} (c_{i+1}(\lambda) + c_{\ell+1}(\lambda)) \tilde{\varrho}_{\ell+1}(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz) \right. \\ \left. + \tilde{\varrho}_{\ell+1}(Y(g, z)) \cdots \tilde{\varrho}_{i+2}(Y(g, z)) \varrho_{i+1}^0(\tilde{k}(g, z)^{-1}) F^{(i+1)}(gz) \right\}.$$

This splits naturally into two sums. In the first sum, we slightly rewrite the coefficient in front, in the second sum, we change the summation index i to $i - 1$, and obtain

$$\frac{1}{2} \sum_{i=j}^{\ell} \frac{(-1)^{\ell-i+1}}{(\ell-i+1)!} (\ell - i + 1) c_{i,j} (c_{i+1}(\lambda) + c_{\ell+1}(\lambda)) (\tilde{\varrho}_{\ell+1}(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz) \\ + \sum_{i=j+1}^{\ell+1} \frac{(-1)^{\ell-i+1}}{(\ell-i+1)!} c_{i-1,j} \tilde{\varrho}_{\ell+1}(Y(g, z)) \cdots \tilde{\varrho}_{i+1}(Y(g, z)) \varrho_i^0(\tilde{k}(g, z)^{-1}) F^{(i)}(gz).$$

This can be written as a single sum over i from j to $\ell + 1$. (The two extra terms at the ends are 0 since we may set $c_{j-1,j} = 0$.) This sum will be $\frac{c_{\ell,j}}{c_{\ell+1,j}}$ times the $(\ell + 1)$ - analogous term of the right hand side of (2.21), i.e. our induction will be complete if all corresponding coefficients agree, i.e. if

$$\frac{\ell-i+1}{2} c_{i,j} (c_{i+1}(\lambda) + c_{\ell+1}(\lambda)) + c_{i-1,j} = \frac{c_{\ell,j}}{c_{\ell+1,j}} c_{i,j}$$

for all $0 \leq i \leq j \leq \ell$. One can easily verify that these identities follow from (2.9) finishing the proof. \square

To pass to more general indecomposable Hermitizable (ϱ, V) , it is useful to first consider the ‘‘fliform with multiplicities’’ case, where for $0 \leq j \leq m$,

$$V_j = \mathbb{C}^{d_j} \otimes W_j, \quad \varrho_j^0 = \chi_{\lambda-j} I_{d_j} \otimes \alpha_j$$

with irreducible representations (α_j, W_j) of \tilde{K}_{ss} . Now $\oplus W_j$ is fliform, and $\tilde{\varrho}_j(Y) : W_{j-1} \rightarrow W_j$ is defined as before. Since ϱ_j^- has to satisfy (1.6), it follows by the same argument as before that

$$\varrho_j^-(Y) = y_j \otimes \tilde{\varrho}_j(Y) \quad (Y \in \mathfrak{p}^-)$$

with some linear transformation $y_j \in \text{Hom}(\mathbb{C}^{d_{j-1}}, \mathbb{C}^{d_j})$. We are using here the natural identification $\text{Hom}(\mathbb{C}^{d_{j-1}}, \mathbb{C}^{d_j}) \otimes \text{Hom}(W_{j-1}, W_j) = \text{Hom}(\mathbb{C}^{d_{j-1}} \otimes W_{j-1}, \mathbb{C}^{d_j} \otimes W_j)$.

Hence the formula (2.17) remains correct after putting a \otimes symbol after the product $y_i \cdots y_{j+1}$. We define Γ as in Theorem 2.4, again putting the \otimes symbol after the y - product. The intertwining

property of Γ follows as before from (2.21), in which the y_k -s play no role. So, the analogue of Theorem 2.5 holds.

The possible Hermitian structures, as indicated at the beginning of this section, are given by positive definite linear transformations H_j on \mathbb{C}^{d_j} , ($0 \leq j \leq m$).

Now we consider the most general indecomposable Hermitizable (ϱ, V) . Here for each $0 \leq j \leq m$ there is a set A_j of inequivalent irreducible representations of \tilde{K}_{ss} such that

$$\begin{aligned} V_j &= \bigoplus_{\alpha \in A_j} V_j^\alpha, & \varrho_j^0 &= \bigoplus_{\alpha \in A_j} \varrho_j^{0\alpha} \\ V_j^\alpha &= \mathbb{C}^{d_{j\alpha}} \otimes W^\alpha, & \varrho_j^{0\alpha} &= \chi_{\lambda-j} I_{d_{j\alpha}} \otimes \alpha. \end{aligned}$$

The possible inner products on V_j^α are given by positive definite linear transformations H_j^α .

We call a sequence $(\alpha_j, \dots, \alpha_i)$ with α_k in A_k ($j \leq k \leq i$) *admissible* if each α_k is contained in $\text{Ad}'_{\mathfrak{p}^-} \otimes \alpha_{k-1}$ for $j+1 \leq k \leq i$. When a two term sequence (α, β) is admissible, we have the equivariant map $P_{\alpha, \beta} : \mathfrak{p}^- \otimes W^\alpha \rightarrow W^\beta$ and $\tilde{\varrho}_{\alpha\beta}(Y)$ for $Y \in \mathfrak{p}^-$ as in (2.2). As in the ‘‘filiform with multiplicity’’ case, it follows that

$$\varrho_j^-(Y) = \bigoplus (y_j^{\alpha\beta} \otimes \tilde{\varrho}_{\alpha\beta}(Y))$$

with some $y_j^{\alpha\beta}$ in $\text{Hom}(\mathbb{C}^{d_{j\alpha}}, \mathbb{C}^{d_{j\beta}})$, the direct sum taken over all admissible pairs (α, β) in $A_{j-1} \times A_j$.

Knowing the set $y = \{y_j^{\alpha\beta}\}$ implies knowing the sets A_j and the multiplicities $d_{j\alpha}$. So, our general irreducible Hhhvb is determined by y and λ ; we may denote it by $E^{y, \lambda}$ or just E^y when λ is taken for granted. Changing all the y -s in E^y to 0 we get a Hhhvb E^0 which is the direct sum of the factors in a composition series of E^y :

$$(2.22) \quad E^0 = \bigoplus_{j=0}^m E_j$$

$$(2.23) \quad E_j = \bigoplus_{\alpha \in A_j} d_{j\alpha} E^{\alpha, \lambda-j}.$$

(By dE we mean the direct sum of d copies of E .)

From (2.3), it follows that

$$(2.24) \quad y_{j+1}^{\beta\gamma} y_j^{\alpha\beta} = 0$$

unless (2.4) is satisfied for $P_{\alpha\beta}$ and $P_{\gamma\beta}$ in place of P_j and P_{j+1} . If (2.4) is satisfied, we say that (α, β, γ) is a *filiform sequence*; we call an admissible sequence $\alpha = (\alpha_j, \dots, \alpha_i)$ ($\alpha_k \in A_k$) of any length filiform if it has only two terms or if every three term part of it is filiform. This is equivalent to saying that $W^\alpha = W^{\alpha_j} \oplus \dots \oplus W^{\alpha_i}$ with \mathfrak{p}^- acting via $\tilde{\varrho}_{\alpha_j \alpha_{j+1}} \oplus \dots \oplus \tilde{\varrho}_{\alpha_{i-1} \alpha_i}$ is a filiform representation.

For W^α , Lemma 2.1 holds and defines the numbers $c_\ell^\alpha(\lambda)$ (which depend only on $\alpha_{\ell-1}$ and α_ℓ , not on other terms of α), u^α , w^α . Then we define

$$(2.25) \quad c_{ij}^\alpha = \frac{2^{i-j}}{(i-j)!} \prod_{k=1}^{i-j} (c_{j+1}^\alpha(\lambda) + c_{j+k}^\alpha(\lambda))^{-1}$$

for all $\lambda \in \mathbb{R}$ that are regular for α , in the sense that the right hand side is meaningful.

We introduce some abbreviations. For a filiform $\alpha = (\alpha_j, \dots, \alpha_i)$ we write

$$\begin{aligned} y^\alpha &= y_i^{\alpha_{i-1}, \alpha_i} \dots y_{j+1}^{\alpha_j, \alpha_{j+1}} \\ \varrho^\alpha(Y) &= \tilde{\varrho}_i^{\alpha_{i-1}, \alpha_i}(Y) \dots \tilde{\varrho}_{j+1}^{\alpha_j, \alpha_{j+1}}(Y) \\ D^\alpha &= (P_{\alpha_{i-1}, \alpha_i} \iota D) \dots (P_{\alpha_j, \alpha_{j+1}} \iota D) \end{aligned}$$

Furthermore, for $\alpha \in A_j$, $\beta \in A_i$, ($j < i$), we denote by $A_{ji}(\alpha, \beta)$ the set of all filiform sequences (associated to ϱ) $\alpha = (\alpha_j, \dots, \alpha_i)$ such that $\alpha_j = \alpha, \alpha_i = \beta$.

For $f \in \text{Hol}(D, V)$, we write f_j^α for its projection onto V_j^α .

Theorem 2.5. *Let (ϱ, V) be indecomposable Hermitian and let $E^\varrho = E^y$ be the corresponding Hhhvb. Suppose that $\lambda \in \mathbb{R}$ is regular for every α occurring in ϱ . Then the operator $\Gamma^{y,\lambda} : \text{Hol}(\mathcal{D}, V) \rightarrow \text{Hol}(\mathcal{D}, V)$ given by*

$$(\Gamma^{y,\lambda} f_j^\alpha)_\ell^\beta = \begin{cases} \sum_{\alpha \in A_{j\ell}(\alpha,\beta)} c_{\ell j}^\alpha y^\alpha \otimes D^\alpha f_j^\alpha & \text{if } \ell > j, \\ f_j^\alpha & \text{if } \ell = j, \text{ and } \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

intertwines the actions of \tilde{G} on the trivialized sections of E^0 and E^ϱ .

Proof. We have to prove (2.18) in our more general situation. $\varrho^0(\tilde{k}(g, z)^{-1})$ acts diagonally by $\chi_{\lambda-j} I \otimes \alpha_j$ on each V_j^α . For the other multiplier, we use (2.17) and get, for the V_i^γ - component of the image of any $v_j \in V_j^\alpha$, ($j < i$)

$$(\varrho^-(\exp -Y(g, z))v_j^\alpha)_i^\gamma = \frac{(-1)^{i-j}}{(i-j)!} \sum_{\alpha \in A_{ji}(\alpha,\gamma)} y^\alpha \otimes \varrho^\alpha(Y(g, z))v_j^\alpha.$$

We write down the V_ℓ^β - component of the left hand side of (2.18) applied to f_j^α , for $\ell > j$:

$$(2.26) \quad \sum_{\alpha \in A_{j\ell}(\alpha,\beta)} c_{\ell j}^\alpha y^\alpha \otimes D^\alpha (\chi_{\lambda-j} I \otimes \alpha_j(\tilde{k}(g, z)^{-1})f_j^\alpha(gz))$$

and the corresponding right hand side:

$$(2.27) \quad \sum_{i=j}^{\ell} \frac{(-1)^{\ell-i}}{(\ell-i)!} \sum_{\gamma \in A_i} \left(\sum_{\sigma \in A_{i\ell}(\gamma,\beta)} y^\sigma \otimes \varrho^\sigma(Y(g, z)) \right) \left(\chi_{\lambda-i} I \otimes \gamma(\tilde{k}(g, z)^{-1}) \sum_{\tau \in A_{ji}(\alpha,\gamma)} c_{\ell j}^\tau y^\tau \otimes (D^\tau f_j^\alpha)(gz) \right)$$

By (2.24) we have $y^\sigma y^\tau = 0$ unless the sequence (τ, σ) (i.e σ following τ) is filiform. The triple sum gives then all sequences in $A_{j\ell}(\alpha, \beta)$ exactly once. γ is α_i and τ, σ are the parts of α up to resp. beyond α_i . So (2.26) is equal to

$$(2.28) \quad \sum_{\alpha \in A_{j\ell}(\alpha,\beta)} \sum_{i=j}^{\ell} \frac{(-1)^{\ell-i}}{(\ell-i)!} c_{i j}^\alpha y^\alpha \otimes \varrho^\alpha(Y(g, z))(\chi_{\lambda-i} \otimes \alpha_i)(\tilde{k}(g, z)^{-1})(D^\tau f_j^\alpha)(gz).$$

By (2.21), which was proved in Theorem 2.4, the terms of (2.28) for each α agree with the corresponding term in (2.26), finishing the proof. \square

Remark 2.6. (1) When looking for examples of Hermitian representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ one finds a large class by taking representations of the simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and restricting them to $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$. By an unpublished result of R. Parthasarathy [22] taking sub-quotients of such (not necessarily irreducible) representations and tensoring with one dimensional representations one gets all possible indecomposable Hermitian representations. The proof (originally given for $K^{\mathbb{C}}P^-$) uses the Borel-Weil theorem.

- (2) A filiform representation can be constructed by taking an arbitrary irreducible (ϱ^0, V_0) and defining inductively (ϱ_j^0, V_j) as the irreducible piece of $\mathfrak{p}^- \otimes V_{j-1}$ whose highest weight is the sum of the highest weights of $\text{Ad}_{\mathfrak{p}^-}$, resp. ϱ_{j-1}^0 (the ‘‘Cartan product’’). To see that (2.4) holds in this case, we note that $\mathfrak{p}^- \otimes \mathfrak{p}^- \otimes V_{j-1}$ now contains V_{j+1} , whose highest weight is $2\beta + \Lambda$ with multiplicity 1. So the two sides of (2.4), which are images of $Y \otimes Y' \otimes v$ under \tilde{K} -equivariant maps must coincide up to constant. The case $Y = Y'$ shows that the constant is 1.
- (3) In the case where \mathcal{D} is the one variable disc, we have $\tilde{K}_{\text{ss}} = \{1\}$ and its only representation is the trivial one. In this case Theorem 2.5 reduces to [17, Theorem 3.1].

- (4) When \mathcal{D} is the unit ball in \mathbb{C}^2 , \tilde{K}_{ss} is $\text{SU}(2)$ whose irreducible representations we denote by τ_0, τ_1, \dots (with $\dim \tau_k = k + 1$). We have $\text{Ad}'_{\mathfrak{p}^-} \cong \tau_1$, and by the Clebsch-Gordan formula $\text{Ad}'_{\mathfrak{p}^-} \otimes \tau_k = \tau_{k-1} \oplus \tau_{k+1}$. The construction in (2) together with a χ_λ gives the filiform sequences τ_j, \dots, τ_ℓ (for any fixed $(0 \leq j \leq \ell)$) and the corresponding filliform representations. The contragredients of these namely τ_ℓ, \dots, τ_j are also filiform. There are no others since one can easily show that $(\tau_k, \tau_{k\pm 1}, \tau_k)$ is never a filiform sequence.
- (5) In general, the irreducible Hhhvb-s $E^{y,\lambda}, E^{y',\lambda}$ are isomorphic if there exists a family of invertible linear transformations $\{a_j^\alpha\}$ such that $y'^{\alpha\beta} = (a_j^\beta)^{-1} y_j^{\alpha\beta} a_{j-1}^\alpha$. Given also the Hermitian structures H (resp. H'), they are isomorphic if, in addition, $H_j'^\alpha = (a_j^\alpha)^* H_j^\alpha a_j^\alpha$.

3. HILBERT SPACES OF SECTIONS

Some relatively simple known facts about vector-valued reproducing kernel spaces are fundamental for this section. We start by listing these in the exact form we will need them. They are not difficult to prove in the order given. Most of the statements can be found, for instance in [19, Chapter I], although with rather different notations.

We consider (complete) Hilbert spaces $\mathcal{H} \subseteq \mathfrak{F}(D, V)$, where V is a finite dimensional Hilbert space and $\mathfrak{F}(D, V)$ is the set all V -valued functions on a set D . The inner product on \mathcal{H} is denoted by $(\cdot | \cdot)$, on V by $\langle \cdot, \cdot \rangle$. The adjoint of an element A in $\text{Hom}(V, V)$ (and more generally, of a linear transformation between two finite dimensional Hilbert spaces) is denoted $A^\#$, while $*$ is used in the case of infinite dimensions, e.g. for \mathcal{H} .

For $v \in V$, we define $v^\#$ in the linear dual of V by $\langle \cdot, v \rangle$. (This is actually the adjoint if we identify v with the map $z \rightarrow zv$ in $\text{Hom}(\mathbb{C}, V)$.) We have $v^\# A^\# = (Av)^\#$ for A as above.

If $K(z, w)$ is a “kernel”, i.e., a $\text{Hom}(V, V)$ -valued function of z and w in D , we write, for any $v \in V$,

$$(3.1) \quad K_w(z) = K(z, w),$$

$$(3.2) \quad (K_w v)(z) = K_w(z)v.$$

Given $\mathcal{H} \subseteq \mathfrak{F}(D, V)$, we say K (or $K(z, w)$) is a reproducing kernel for \mathcal{H} if $K_w v \in \mathcal{H}$ for all w and v , and if

$$(3.3) \quad (f | K_w v) = \langle f(w), v \rangle$$

for all $f \in \mathcal{H}$. (It is obvious that K is unique and that it exists if and only if the “evaluation maps” $\text{ev}_w f = f(w)$ from \mathcal{H} to V are continuous for all w . As a linear map $V \rightarrow \mathcal{H}$, K_w is just the adjoint ev_w^* .) The reproducing kernel is positive definite, denoted $K \succ 0$, in the sense that

$$\sum_{j,k} \langle K(z_j, z_k) v_k, v_j \rangle \geq 0$$

for any z_1, \dots, z_N in D and v_1, \dots, v_N in V . In particular, this implies $K(z, w)^\# = K(w, z)$. For any two kernels we write $K_0 \prec K_1$ if $K_1 - K_0 \succ 0$.

We mention that if $\{e_\nu\}$ is any orthonormal basis for \mathcal{H} and K is the reproducing kernel, then

$$(3.4) \quad K(z, w) = \sum_\nu e_\nu(z) e_\nu(w)^\#$$

the sum being convergent both in $\text{Hom}(V, V)$ and also in \mathcal{H} when it is applied to a $v \in V$, and regarded as a function of z with w fixed.

Suppose $T : \mathcal{H} \rightarrow \mathfrak{F}(D, V)$ is a linear transformation. Then $T K_w$ (for the reproducing kernel K , or any other kernel K_w such that $K_w v$ is in \mathcal{H} for all w, v) is naturally defined by

$$(T K_w) v = T(K_w v) \quad (v \in V).$$

Depending on the context, we will also use the notation $T^{(z)}K(z, w)$ and $T_1K(z, w)$ for $(TK_w)(z)$ to indicate that the operator is applied to $K(z, w)$ as a function of z , i.e. the first variable, with w held fixed.

For every $f \in \mathcal{H}$ we define $f^\#$ by $f^\#(z) = f(z)^\#$ ($z \in D$). For a linear transformation T of \mathcal{H} we define $T^\#$ by

$$(3.5) \quad T^\# f^\# = (Tf)^\#.$$

Now $T^{(w)\#}K(z, w)$ makes sense, by

$$(3.6) \quad T^{(w)\#}K(z, w) = T^{(w)\#}K(w, z)^\# = (T^{(w)}K(w, z))^\#.$$

Note that if A is a $\text{Hom}(V, V)$ - valued function on D , and T_A on \mathcal{H} is defined by $(T_A f)(z) = A(z)f(z)$ (a kind of multiplication operator), then

$$(T_A^\#)^{(w)}K(z, w) = K(z, w)A(w)^\#,$$

and using a natural abbreviation,

$$T_A T_A^\# K = A K A^\#.$$

For easier reference we give numbers to the following statements.

Proposition 3.1. *Suppose that $\mathcal{H} \subseteq \mathfrak{F}(D, V)$ has reproducing kernel K , that V_1 is another finite dimensional Hilbert space and $T : \mathcal{H} \rightarrow \mathfrak{F}(D, V_1)$ is a linear map such that $f \mapsto (Tf)(z)$ is bounded for every $z \in D$. Then $\ker T$ is closed and the range $T\mathcal{H}$ with the Hilbert space structure of $\mathcal{H}/\ker T$ transferred to it via T has $T^{(z)}T^{(w)\#}K(z, w)$ as its reproducing kernel.*

This can be proved e.g. from (3.4).

Proposition 3.2. *Suppose that in addition to the hypothesis of Proposition 3.1 there is given a Hilbert space $\mathcal{H}_1 \subseteq \mathfrak{F}(D, V_1)$ with reproducing kernel K_1 . Then T maps \mathcal{H} into \mathcal{H}_1 and is bounded by $c > 0$ if and only if*

$$(3.7) \quad T^{(z)}T^{(w)\#}K(z, w) \prec c^2 K_1(z, w).$$

A proof can be based on the preceding proposition.

An immediate consequence of Proposition 3.1 is the following well known fact.

Remark 3.3. If G is a group of transformations of D and $m(g, z)$ is a $\text{Hom}(V, V)$ - valued multiplier (i.e. $m(gg', z) = m(g, g'z)m(g', z)$ for all g, g', z .) and \mathcal{H} has reproducing kernel K , then U_g defined by

$$(3.8) \quad (U_g f)(z) = m(g^{-1}, z)^{-1} f(g^{-1}z)$$

preserves \mathcal{H} and is a unitary representation on it if and only if K is quasi-invariant, i.e.

$$K(gz, gw) = m(g, z)K(z, w)m(g, w)^\#$$

for all g, z, w .

Proposition 3.4. *If D is a domain \mathbb{C}^n and $\mathfrak{F}(D, V)$ is changed in the statements to $\text{Hol}(D, V)$, the holomorphic V - valued functions, and if T is a holomorphic differential operator, then the hypothesis of Propositions 3.1 and 3.2 about the boundedness of $f \mapsto Tf(z)$ are automatically satisfied.*

This follows from the Cauchy estimates.

We turn to the main subject of this section. Given an indecomposable $E^e = E^y$ as in Theorem 2.5, a regular unitary structure on it is a Hilbert space $\mathcal{H} \subseteq \text{Hol}(D, V)$ with inner product invariant under U^e and containing the space $\mathcal{P} = \mathcal{P}(\mathfrak{p}^+, V)$ of all V - valued polynomials. If such a structure exists, we say that E^e is regularly unitarizable.

Our first goal is to describe all regular unitary Hhhvb-s and all regularly unitary structures on them. But first we reformulate this definition in an intrinsic trivialization independent way. For this, and also for later use, we recall the following facts of representation theory.

Given a continuous representation U of \tilde{G} on a topological vector space with some minimal good properties, the \tilde{K} -finite vectors, i.e. those f for which $\{U_k f \mid k \in \tilde{K}\}$ span a finite dimensional space, form a dense subspace. On this subspace, U induces a representation \mathfrak{u} of \mathfrak{g} defined by $\mathfrak{u}_X f = \left. \frac{d}{dt} \right|_0 U_{\exp tX} f$. So the \tilde{K} -finite vectors form a $(\mathfrak{g}, \tilde{K})$ -module, i.e. a joint representation of \mathfrak{g} and \tilde{K} . (cf. [25, Proposition 2.5]).

A regular unitary structure can be intrinsically defined as a Hilbert space of holomorphic sections with inner product invariant under the action of \tilde{G} , such that it contains all \tilde{K} -finite sections. It is equivalent to the definition first given since it is not hard to see that in the canonical trivialization the \tilde{K} -finite vectors are exactly the polynomials [19, Proposition XII.2.1]. These remarks also make it clear that the condition $\mathcal{P} \subseteq \mathcal{H}$ can be equivalently replaced by “ \mathcal{H} dense in $\text{Hol}(\mathcal{D}, V)$ ” (in the topology of uniform convergence on compact subsets of \mathcal{D}).

We will need the following non-trivial fact (cf. [25, Theorem 2.12]). If U is a unitary representation of \tilde{G} on a Hilbert space \mathcal{H} , and the \tilde{K} -finite subspace $\mathcal{H}_{\tilde{K}}$ is given the induced $(\mathfrak{g}, \tilde{K})$ -module structure, then the $(\mathfrak{g}, \tilde{K})$ -sub-modules of $\mathcal{H}_{\tilde{K}}$ under closure in \mathcal{H} are in one to one correspondence with the U -invariant subspaces of \mathcal{H} .

One important consequence of this is that if E^ϱ is regularly unitarizable, then automatically \mathcal{P} is dense in \mathcal{H} .

In the case of irreducible ϱ , i.e. when $\varrho^0 = \chi_\lambda \otimes \alpha$ with an irreducible representation α of $K_{\text{ss}}^{\mathbb{C}}$ and $\varrho^- = 0$, the situation is very well known, it is part of the theory of the holomorphic discrete series of representations. For every α , there is a set $\mathcal{W}_c(\alpha)$ of the form $\lambda < \lambda_\alpha$ with λ_α explicitly known such that $E^{\alpha, \lambda}$ is regularly unitarizable if and only if $\lambda \in \mathcal{W}_c(\alpha)$ (cf. [7, 12]). In such a case \mathcal{P} is an irreducible $(\mathfrak{g}, \tilde{K})$ -module, hence it has a unique (up to constant) invariant Hermitian form, which is, in this case, non-degenerate, positive and gives the inner product of the corresponding Hilbert space $\mathcal{H}_1^{(\alpha, \lambda)}$, which is thereby uniquely determined up to constant. We normalize it, as usual, by the condition $\|v\|_{\mathcal{H}} = \|v\|_V$ for $v \in V$. (Note that v regarded as a constant function is in $\mathcal{H}_1^{(\alpha, \lambda)}$.)

Each $\mathcal{H}_1^{(\alpha, \lambda)}$ ($\lambda \in \mathcal{W}_c(\alpha)$) has a reproducing kernel $K^{(\alpha, \lambda)}$ (cf. [19], Theorem XII.2.6 and Remarks to Sec XII.2) which can be explicitly described as follows. Exactly as in [23, p. 64] but working in \tilde{G}_{loc} instead of G , we set, for $z, w \in \mathcal{D}$

$$\tilde{\mathcal{K}}(z, w) = \tilde{k}(\exp -\bar{w}, z)^{-1},$$

the bar denoting conjugation with respect to \mathfrak{g} in $\mathfrak{g}^{\mathbb{C}}$, and also the lift of this map to $\tilde{G}_{\text{loc}}^{\mathbb{C}}$. For later use we also introduce the abbreviation

$$Y_{z, w} = Y(\exp -\bar{w}, z)$$

so the decomposition (1.11) appears now as

$$(3.9) \quad (\exp -\bar{w})(\exp z) = (\exp(\exp -\bar{w}) \cdot z) \tilde{\mathcal{K}}(z, w)^{-1} (\exp Y_{w, z}).$$

Interchanging z and w , taking inverses and conjugating gives another expression for the left hand side. By uniqueness in (1.11) this implies that

$$(3.10) \quad (\exp -\bar{w}) \cdot z = \overline{-Y_{z, w}}$$

$$(3.11) \quad \tilde{\mathcal{K}}(w, z) = \overline{\tilde{\mathcal{K}}(w, z)}^{-1}$$

Also, clearly $\tilde{\mathcal{K}}(z, 0) = \tilde{\mathcal{K}}(0, z) \equiv e$, and as in [23],

$$(3.12) \quad \tilde{\mathcal{K}}(gz, gw) = \tilde{k}(g, z) \tilde{\mathcal{K}}(z, w) \overline{\tilde{k}(g, w)}^{-1}.$$

We can now verify that

$$K^{(\alpha,\lambda)}(z, w) = (\chi_\lambda \otimes \alpha)(\tilde{\mathcal{K}}(z, w)).$$

In fact, for $k \in \tilde{K}^{\mathbb{C}}$ and any Hermitian representation ϱ^0 , we have $\varrho^0(\bar{k}) = \overline{\varrho^0(k)}^{-1}$. So if we apply $\varrho^0 = \chi_\lambda \otimes \alpha$ to (3.12) we get the quasi-invariance (3.8) of $K^{(\alpha,\lambda)}$ with respect to the canonical multiplier. (3.12) also shows that $K^{(\alpha,\lambda)}(z, 0) \equiv 1$ which corresponds to the normalization we fixed on $\mathcal{H}_1^{(\alpha,\lambda)}$. These properties characterize $K^{(\alpha,\lambda)}$.

Changing the normalization of the invariant inner product on $\mathcal{H}_1^{(\alpha,\lambda)}$ we get different regular unitary structures on the $E^{\alpha,\lambda}$. We consider this question in the greater generality of $dE^{\alpha,\lambda}$, a direct sum of d copies of $E^{\alpha,\lambda}$.

Here the space of sections is

$$\text{Hol}(\mathcal{D}, \mathbb{C}^d \otimes W^\alpha) \cong \mathbb{C}^d \otimes \text{Hol}(\mathcal{D}, W^\alpha).$$

(We identify the two sides. In practice, this only amounts to writing \mathbb{C}^d -valued functions in terms of a basis in \mathbb{C}^d .) The \tilde{G} action is now by $I_d \otimes U^{\alpha,\lambda}$. It follows that regular unitary structures are gotten by tensoring the inner product in $\mathcal{H}^{(\alpha,\lambda)}$ with an arbitrary inner product on \mathbb{C}^d . We write this latter in terms of the standard inner product of \mathbb{C}^d as $\langle \mu \cdot, \cdot \rangle_{\mathbb{C}^d}$ with a positive definite linear transformation μ on \mathbb{C}^d .

We denote the regular unitary structure so obtained by $\mathcal{H}_\mu^{(\alpha,\lambda)}$. It is trivial to check that it has a reproducing kernel, namely,

$$\mu^{-1} \otimes K^{(\alpha,\lambda)}(z, w).$$

This now includes the case $d = 1$, where μ is scalar.

In the following we keep using the notations involved in Theorem 2.5. We consider an indecomposable Hhhvb $E^\varrho = E^y$; ϱ is understood to determine $\lambda \in \mathbb{R}$. We have E^0 , which is given by (2.22) and (2.23).

Lemma 3.5. *If the irreducible Hhhvb $E^\varrho = E^y$ is regularly unitarizable, then so is E^0 .*

Proof. Suppose \mathcal{H} is a regular unitary structure on E^ϱ and let $\mathcal{H}_j = \mathcal{H} \cap \text{Hol}(\mathcal{D}, \tilde{V}_j)$. By U^ϱ invariance of $\text{Hol}(\mathcal{D}, \tilde{V}_j)$ (cf. Sec. 1), each \mathcal{H}_j is an invariant subspace of \mathcal{H} , closed because point evaluations are continuous on \mathcal{H} . The space of sections of the bundle E_j holomorphically induced by $(\varrho_j^0, 0)$ is $\text{Hol}(\mathcal{D}, V_j)$. A representation U_j of \tilde{G} acts on it via the multiplier $\varrho_j^0(\tilde{k}(g, z))$. The one-to-one linear map L of $\mathcal{H}_j/\mathcal{H}_{j+1}$ into $\text{Hol}(\mathcal{D}, V_j)$ defined by $L(f + \mathcal{H}_j) = f_j$ intertwines the quotient action of U^ϱ with U_j . The image of L (which does contain all V_j -valued polynomials) with the inner product transferred from $\mathcal{H}_j/\mathcal{H}_{j+1}$ is then a regular unitary structure on E_j . Together with E_j then $E_0 = \oplus E_j$ is also regularly unitarizable. \square

The logical order would now require us to first prove Proposition 3.7, because the proof of Theorem 3.6 uses one of its corollaries. We invert this order because the main significance of Proposition 3.7 (whose proof depends only on computations done in Section 2) lies in a different direction.

Theorem 3.6. *Let $E^y = E^{y,\lambda}$ be an indecomposable Hhhvb. Then E^y is regularly unitarizable if and only if E^0 is, which is the case if and only if $\lambda < \lambda_\alpha + j$ for all $\alpha \in A_j$, $0 \leq j \leq m$ in the decomposition of E^0 as $\bigoplus_{j,\alpha} E^{\alpha,\lambda-j}$. The regular unitary structures of E^0 are*

$$\mathcal{H}_\mu^0 = \bigoplus_{j=0}^m \bigoplus_{\alpha \in A_j} \mathcal{H}_{\mu_{j\alpha}}^{(\alpha,\lambda-j)},$$

where $\mu = \{\mu_{j\alpha}\}$ and each $\mu_{j\alpha}$ is a positive definite linear transformation on $\mathbb{C}^{d_{j\alpha}}$. The reproducing kernel of \mathcal{H}_μ^0 is

$$K_\mu^0 = \bigoplus \bigoplus \mu_{j\alpha}^{-1} \otimes K^{(\alpha,\lambda-j)}.$$

The regular unitary structures of E^y are the spaces $\mathcal{H}_\mu^y = \Gamma^{\lambda,y} \mathcal{H}_\mu^0$ with $\Gamma^{\lambda,y}$ a unitary isomorphism. The corresponding reproducing kernel is

$$K_\mu^y = \Gamma^{\lambda,y} \Gamma^{\lambda,y\#} K_\mu^0.$$

For a fixed y (and λ) all \mathcal{H}_μ^y are equal as sets, and their Hilbert norms are equivalent.

Proof. It is clear that every \mathcal{H}_μ^0 is a regular unitary structure on E^0 . Conversely, if \mathcal{H} is a regular unitary structure, then it contains \mathcal{P} , which is now the direct sum of the spaces \mathcal{P}_j^α of V_j^α -valued polynomials. Each \mathcal{P}_j^α is \mathfrak{u}^0 -invariant because \mathfrak{u}^0 (like U^0) respects the direct sum structure of E^0 . By a general result quoted above, \mathcal{H} is therefore the direct sum of closures (in \mathcal{H}) of the spaces $\mathcal{P}^{\alpha,\lambda-j}$. These closures are all of the form $\mathcal{H}_{\mu_j^\alpha}^{(\alpha,\lambda-j)}$ because as mentioned before, in the irreducible case the $(\mathfrak{g}, \tilde{K})$ -module structure determines the inner product up to constant.

The statement about the reproducing kernel K^0 is immediate from the direct sum structure.

By Theorem 2.5 and Corollary 3.8, Γ is an invertible map of $\text{Hol}(\mathcal{D}, V)$ intertwining U^0 with $U^y = U^\varrho$. Clearly, Γ also maps \mathcal{P} onto \mathcal{P} . So $\mathcal{P} \subseteq \Gamma \mathcal{H}^0 \subseteq \text{Hol}(\mathcal{D}, V)$ with an U^ϱ -invariant inner product on \mathcal{H}^0 . Furthermore by Propositions 3.4 and 3.1, $\Gamma \mathcal{H}^0$ is a complete Hilbert space with reproducing kernel $\Gamma \Gamma^\# K^0$.

As for the last statement, it clearly holds for every $E^{\alpha,\lambda}$, hence also for direct sums of such. So \mathcal{H}_μ^0 is the same set for every μ and the norms are equivalent. Since $\Gamma^{y,\lambda}$ is by definition a unitary isomorphism of \mathcal{H}_μ^0 onto \mathcal{H}_μ^y , the same statement is true for the spaces \mathcal{H}_μ^y . \square

If some E^y has a regular unitary structure \mathcal{H}_μ^y , then it has a canonically associated Hermitian structure given by $H = K_\mu^y(0,0)^{-1}$. (The inverse exists by Theorem 3.6 and by $K^{(\alpha,\lambda)}(0,0) = I$ for all α, λ .) The regular unitary structure can be reconstructed from the Hermitian structure by the quasi-invariance of K_μ^y . For Hermitian hhvb-s arising this way, we say that their metric *comes from a regular unitary structure*.

The fact that U^ϱ on \mathcal{H}^ϱ is equivalent to the direct sum of irreducibles is well known in the theory of the holomorphic discrete series; in Theorem 3.6 the equivalence is realized by the explicit differential operator Γ .

In the second half of this section we will be looking at a filiform Hhhvb of two terms (i.e. with $m = 1$). In the arguments, we need an expression for the adjoint of the map $\tilde{\varrho}_1(Y)$ defined by (2.2). We derive this now as a preparation.

Using notation of Section 2, but writing P instead of P_1 , we define for any fixed $Y \in \mathfrak{p}^-$ the map $T_Y : V_0 \rightarrow \mathfrak{p}^- \otimes V_0$ by

$$T_Y v = Y \otimes v$$

so that $\tilde{\varrho}_1(Y) = P T_Y$. A simple computation gives

$$T_Y^\# = Y^\# \otimes I_{V_0}$$

and therefore, using the fact that $\iota Y = -\bar{Y}$ for $Y \in \mathfrak{p}^-$,

$$(3.13) \quad \tilde{\varrho}_1(Y)^\# = (Y^\# \otimes I_{V_0}) P^\# = (B(\cdot, \bar{Y}) \otimes I_{V_0}) P^\#.$$

We consider the following situation. We set $\varrho_0^0 = \chi_\lambda \otimes \alpha$ with some $\lambda \in \mathbb{R}$ and an irreducible Hermitian representation α of $\mathfrak{k}_{\text{ss}}^{\mathbb{C}}$. We take an irreducible component β of $\text{Ad}'_{\mathfrak{p}^-} \otimes \alpha$ and set $\varrho_1^0 = \chi_{\lambda-1} \otimes \beta$. We write P for $P_{\alpha\beta}$ fixed as in Section 2, and define $\tilde{\varrho}_1(Y)$ by (2.2). These data give a filiform representation with $m = 1$ and we can use the corresponding notations and formulas of Section 2; in particular, we have the operator $P \iota D$ mapping sections of $E^{\alpha,\lambda}$ to sections of $E^{\beta,\lambda-1}$. Based on Proposition 3.2 we will discuss the question whether $P \iota D$ is a bounded map of Hilbert spaces in case the two bundles are regularly unitarizable.

Proposition 3.7. *For any \mathcal{D} and any λ, α, β as above,*

$$(3.14) \quad (P\iota D^{(z)})(P\iota D^{(w)})^\# \varrho_0^0(\tilde{\mathcal{K}}(z, w)) = |c_1(\lambda)|^2 A(z, w) + \overline{c_1(\lambda)} \varrho_1^0(\tilde{\mathcal{K}}(z, w)),$$

where $c_1(\lambda)$ is defined by Lemma 2.1 and

$$(3.15) \quad A(z, w) = \tilde{\varrho}_1(Y_{zw}) \varrho_0^0(\tilde{\mathcal{K}}(z, w)) \tilde{\varrho}_1(Y_{zw})^\# = P(Y_{zw} Y_{wz}^\# \otimes \varrho_0^0(\tilde{\mathcal{K}}(z, w))) P^\#.$$

When $\lambda \in \mathcal{W}_c(\alpha)$, we have $c_1(\lambda) > 0$.

Proof. Using the definition of $\tilde{\mathcal{K}}$, Lemma 1.7 and Lemma 2.1 we immediately get

$$(P\iota D^{(w)})^\# \varrho_0^0(\tilde{\mathcal{K}}(z, w)) = (P\iota D^{(w)}) \varrho_0^0(\tilde{\mathcal{K}}(w, z)) = -\overline{c_1(\lambda)} \varrho_0^0(\tilde{\mathcal{K}}(z, w)) \tilde{\varrho}_1(Y_{wz})^\#.$$

We set $\phi(z) = \tilde{\varrho}_1(\bar{z})^\#$, using (3.10), the right hand side can be written

$$-\overline{c_1(\lambda)} \varrho_0^0(\tilde{k}(\exp -\bar{w}, z)^{-1})(\phi(\exp -\bar{w}) \cdot z).$$

We have to apply $P\iota D^{(z)}$ to this. We do it by applying Lemma 2.2, which is certainly applicable to $F(z) = \phi(z)v$ with any $v \in V_0$, therefore also to $\phi(z)$ by linearity in v . The first term Lemma 2.2 gives is exactly $|c_1(\lambda)|^2$ times the first expression for $A(z, w)$ in (3.15) (which is equal to the second expression by (3.13)). The second term Lemma 2.2 gives is

$$-\overline{c_1(\lambda)} \varrho_0^0(\tilde{\mathcal{K}}(z, w))(P\iota D\phi)(\exp -\bar{w}) \cdot z.$$

Now $P\iota D\phi$ is constant since ϕ is linear in z . More exactly, for any $X \in \mathfrak{p}^+$, by (3.13) we have

$$\iota D_X \phi(z) = \iota(B(\cdot, X) \otimes I_{V_0}) P^\# = (X \otimes I_{V_0}) P^\#,$$

i.e.

$$\iota D\phi(z) = (I_{\mathfrak{p}^-} \otimes I_{V_0}) P^\#$$

and

$$P\iota D\phi(z) = P(I_{\mathfrak{p}^-} \otimes I_{V_0}) P^\# = I_{V_1}$$

finishing the proof of (3.14). To prove the last statement: Now $\varrho_0^0(\mathcal{K}(z, w))$ is a positive definite kernel, hence so is the whole left hand side of (3.14). Therefore, for $z = 0 = w$ it is a positive operator. The right hand side of (3.14) shows this to be equal to $\overline{c_1(\lambda)} I_{V_1}$. Hence $c_1(\lambda) \geq 0$. But $c_1(\lambda) = 0$ is impossible since it would imply that the left hand side is identically zero, which is not the case since $\varrho_0^0(\tilde{\mathcal{K}}(z, w))$ is the reproducing kernel of a space containing all polynomials. \square

Corollary 3.8. *If in the notation of Theorem 2.5, $E^e = E^y$ is regularly unitarizable, then the corresponding λ is regular.*

Proof. By Lemma 3.5, E^0 is regularly unitarizable, hence $\lambda < \lambda_\alpha + j$ for each $\alpha \in A_j$. This implies that each $c_\ell^\alpha(\lambda)$ occurring in (2.25) is positive. \square

Corollary 3.9. *If $\lambda \in \mathcal{W}_c(\alpha)$, then the following statements are equivalent:*

- (1) $A(z, w) \prec C \varrho_1^0(\mathcal{K}(z, w))$ for some $C > 0$.
- (2) $\lambda - 1 \in \mathcal{W}_c(\beta)$ and $P\iota D$ is a bounded linear operator from $\mathcal{H}^{(\alpha, \lambda)}$ to $\mathcal{H}^{(\beta, \lambda-1)}$.

Proof. This follows from Proposition 3.2. \square

The following is a partial reduction of the boundedness question to the “scalar case”, i.e. the case where $\alpha = \mathbf{1}$ is the trivial representation, so $V_0 = \mathbb{C}$. The corresponding vector Hhvb-s are the line bundles L_λ already occurring in Theorem 1.6.

Corollary 3.10. *Suppose $\lambda_0 \in \mathcal{W}_c(\mathbf{1})$, $\lambda_0 - 1 \in \mathcal{W}_c(\text{Ad}'_{\mathfrak{p}^-})$ and ιD is bounded from $\mathcal{H}^{(\mathbf{1}, \lambda_0)}$ to $\mathcal{H}^{(\text{Ad}'_{\mathfrak{p}^-}, \lambda_0-1)}$. Then for any irreducible α and any $\lambda \in \mathcal{W}_c(\alpha)$, we have $\lambda + \lambda_0 \in \mathcal{W}_c(\alpha)$, $\lambda + \lambda_0 - 1 \in \mathcal{W}_c(\beta)$, and $P\iota D$ is bounded from $\mathcal{H}^{(\alpha, \lambda+\lambda_0)}$ to $\mathcal{H}^{(\beta, \lambda+\lambda_0-1)}$.*

Proof. By Corollary 3.9, the hypothesis implies

$$\chi_{\lambda_0}(\tilde{\mathcal{K}}(z, w))Y_{zw}Y_{wz}^\# \prec C\chi_{\lambda_0-1}(\tilde{\mathcal{K}}(z, w))\text{Ad}'_{\mathfrak{p}-}(\tilde{\mathcal{K}}(z, w))$$

for some C . This relation remains true after tensoring with the positive definite kernel $(\chi_\lambda \otimes \alpha)(\tilde{\mathcal{K}}(z, w))$ and then multiplying by P on the left and $P^\#$ on the right. Hence

$$\chi_{\lambda+\lambda_0}(\tilde{\mathcal{K}}(z, w))P(Y_{zw}Y_{wz}^\# \otimes \alpha(\tilde{\mathcal{K}}(z, w)))P^\# \prec C\chi_{\lambda+\lambda_0-1}(\tilde{\mathcal{K}}(z, w))P(\text{Ad}'_{\mathfrak{p}-} \otimes \alpha)(\tilde{\mathcal{K}}(z, w))P^\#.$$

The left hand side is just $A(z, w)$ corresponding to $\chi_{\lambda+\lambda_0} \otimes \alpha$, and the right hand side is $\chi_{\lambda+\lambda_0-1} \otimes \beta(\tilde{\mathcal{K}}(z, w))$. Now Corollary 3.9 implies our statement. \square

The last corollary shows the particular importance of the scalar case. A number of things are known about this case (cf. [8, 15]):

$$(3.16) \quad h(z, w) = \chi_{\frac{n}{p}}(\tilde{\mathcal{K}}(z, w))$$

is a polynomial, holomorphic in z , anti-holomorphic in w , of bidegree (r, r) , where r is the rank of \mathcal{D} ; it can be characterized in several equivalent ways. It has an expansion

$$(3.17) \quad h(z, w) = 1 - \frac{1}{2p}\langle z, w \rangle + \cdots,$$

the other terms homogeneous of bidegree at least $(2, 2)$. ($\frac{1}{2p}$ is the factor normalizing the inner product so that the inscribed sphere of \mathcal{D} has radius 1. It was computed in [15].) Furthermore, it is known that

$$(3.18) \quad \mathcal{W}_c(\mathbf{1}) = \{\lambda < -\frac{n}{p}(r-1)\frac{a}{2}\},$$

i.e. $\lambda_1 = -\frac{n}{p}(r-1)\frac{a}{2}$.

Proposition 3.11. *For any irreducible \mathcal{D} ,*

$$(3.19) \quad \text{Ad}_{\mathfrak{p}-}(\tilde{\mathcal{K}}(z, w)) = -2p\iota D^{(z)}(\iota D^{(w)})^\# \log h(z, w).$$

Proof. First we prove the quasi-invariance of the right hand side. We use the abbreviation $H(z, w) = \log h(z, w)$. Since $h(z, w)$ is quasi-invariant with holomorphic multiplier, we have

$$(3.20) \quad (\iota D_1)(\iota D_2)^\# H(z, w) = \iota D^{(z)}(\iota D^{(w)})^\# \{H(g.z, g.w)\}.$$

To compute the right hand side, we use a ‘‘chain rule’’ for ιD :

From $D^{(z)}\{f(gz)\} = (DF)(gz)\text{Ad}_{\mathfrak{p}+}(\tilde{k}(g, z))$ (cf. Remark 1.8) we obtain, by (2.6),

$$(3.21) \quad \iota D^{(z)}\{f(gz)\} = \text{Ad}_{\mathfrak{p}-}(\tilde{k}(g, z)^{-1})(\iota Df)(gz).$$

Applying this and using (3.6) twice we obtain

$$\begin{aligned} (\iota D^{(w)})^\# \{H(gz, gw)\} &= (\iota D^{(w)})\{H(gw, gz)\}^\# \\ &= \text{Ad}_{\mathfrak{p}-}(\tilde{k}(g, z)^{-1})(\iota D_1 H)(gw, gz) \\ &= ((\iota D_2)^\# H)(gz, gw)(\text{Ad}_{\mathfrak{p}-}(\tilde{k}(g, z)^{-1}))^\#. \end{aligned}$$

When we apply $\iota D^{(z)}$ to this, we get exactly the quasi-invariance of $\iota D_1(\iota D_2)^\# H$ with respect to $\text{Ad}_{\mathfrak{p}-}(\tilde{k}(g, z)^{-1})$.

By transitivity of \tilde{G} , this proves (3.19) if we know that the two sides are equal for $z = 0 = w$. So, we evaluate at $z = w = 0$. On the left of (3.20) we have $I_{\mathfrak{p}-}$ since $\tilde{\mathcal{K}}(0, 0) = e$. On the right hand side, we use

$$(3.22) \quad -\iota D_1(\iota D_2)^\# (\log h) = h^{-2}(\iota Dh)((\iota D)^\# h) - h^{-1}\iota D(\iota D)^\# h.$$

Evaluating this at $(0, 0)$ with the aid of (3.17), the first term gives 0 and the second term gives $\frac{1}{2p}I_{\mathfrak{p}^-}$ by the easily checked identity

$$(3.23) \quad \iota D^{(z)}(\iota D^{(w)})\# \langle z, w \rangle = I_{\mathfrak{p}^-}$$

for all z, w in \mathfrak{p}^+ . This completes the proof. \square

Theorem 3.12. *Suppose \mathcal{D} is the Euclidean unit ball in some \mathbb{C}^n . If α is any irreducible representation of $\mathfrak{k}_{\mathbb{S}\mathbb{S}}^{\mathbb{C}}$, β an irreducible component of $\text{Ad}'_{\mathfrak{p}^-} \otimes \alpha$, and $\lambda \in \mathcal{W}_c(\alpha)$, then $\lambda - 1 \in \mathcal{W}_c(\beta)$ and $P\iota D$ is a bounded operator from $\mathcal{H}^{(\alpha, \lambda)}$ to $\mathcal{H}^{(\beta, \lambda-1)}$.*

Proof. When \mathcal{D} is the Euclidean ball, we have $r = 1$. We first prove the Theorem in the special case of $\alpha = \mathbf{1}$. For a more convenient parameter, we write $\ell = -\frac{p}{n}\lambda$. By Proposition 3.2, we have to prove only that if $\lambda \in \mathcal{W}_c(\mathbf{1})$, i.e. by (3.18) if $\lambda < 0$, i.e. if $\ell > 0$, then

$$(3.24) \quad \iota D^{(z)}(\iota D^{(w)})\#\{h(z, w)^{-\ell}\} \prec Ch(z, w)^{-\ell} \text{Ad}_{\mathfrak{p}^-}(\tilde{\mathcal{K}}(z, w))$$

for some C . The left hand side here equals

$$(3.25) \quad \ell(\ell + 1)h^{-\ell-2}(\iota Dh)((\iota D)\#h) - \ell h^{-\ell-1}(\iota D(\iota D)\#h),$$

and we have a similar expression for the right hand side from (3.19) and (3.22). It follows that choosing $C = \frac{\ell(\ell+1)}{p}$ we have

$$(3.26) \quad Ch(z, w)^{-\ell} \text{Ad}_{\mathfrak{p}^-}(\tilde{\mathcal{K}}(z, w)) - \iota D^{(z)}(\iota D^{(w)})\#\{h(z, w)^{-\ell}\} = -\ell^2 h(z, w)^{-\ell-1} \iota D(\iota D)\#h(z, w).$$

Since $r = 1$, the expansion (3.17) ends with the term of bidegree $(1, 1)$. So, because of (3.23), the right hand side of (3.26) is $\frac{1}{2p}\ell^2 h(z, w)^{-\ell-1}I_{\mathfrak{p}^-}$, which is positive definite. This proves (3.24) and the case $\alpha = \mathbf{1}$ of the Theorem.

To prove the general case, suppose $\lambda \in \mathcal{W}_c(\alpha)$, i.e. $\lambda < \lambda_\alpha$. We choose λ' such that $\lambda < \lambda' < \lambda_\alpha$. Then $\lambda = \lambda' + \lambda_0$, with $\lambda_0 < 0$, i.e. $\lambda_0 \in \mathcal{W}_c(\mathbf{1})$. We now apply Corollary 3.10 with λ' in place of λ , and get the general statement of our theorem. \square

4. HOMOGENEOUS COWEN-DOUGLAS TUPLES

We will be mostly concerned with the modified Cowen-Douglas class $\hat{B}_k(\mathcal{D})$ which has all the basic geometric properties of the original Cowen-Douglas class but is easier to handle (see [6, Remark p. 5]). To recall the definitions, let $\mathcal{D} \subseteq \mathbb{C}^m$ be an arbitrary domain, and let $\mathcal{H} \subset \text{Hol}(\mathcal{D}, \mathbb{C}^k)$ be a Hilbert space containing all the \mathbb{C}^k -valued polynomials as a dense set and having a reproducing kernel $K = K(z, w)$. Suppose also that the operators M_j , defined by $(M_j)f(z) = z_j f(z)$ preserve \mathcal{H} and are bounded on it. An n -tuple (T_1, \dots, T_m) of commuting bounded operators on any Hilbert space H is said to belong to $\hat{B}_k(\mathcal{D})$ if there is a unitary isomorphism of H onto \mathcal{H} which carries T_j to M_j^* for each $j = 1, \dots, m$. From now on we write V in place of \mathbb{C}^k , this is more convenient for what follows. We keep assuming that V has an inner product $\langle \cdot, \cdot \rangle$ (corresponding to the standard inner product in \mathbb{C}^k).

The original Cowen-Douglas class $B_k(\mathcal{D})$ (see [5, 4]) can be characterized in a similar way, with the requirement of density of polynomials replaced by the condition that the range of $\bigoplus_{j=1}^m (M_j^* - \bar{w}_j)$ mapping \mathcal{H} into $\mathcal{H} \oplus \dots \oplus \mathcal{H}$ is closed for all $w \in \mathcal{D}$. For the precise relationship between these classes, see [4] and [1].

The essential fact about $\hat{B}_k(\mathcal{D})$ (and about $B_k(\mathcal{D})$ as well) is that the joint eigenspace F_z of the operators M_j^* for eigenvalue \bar{z}_j is, for all $z \in \mathcal{D}$, k dimensional and equal to $\{K_z v : v \in V\}$.

The spaces F_z with their inner product inherited from \mathcal{H} form the fibres of a Hermitian anti-holomorphic vector bundle F over \mathcal{D} . In a natural way, the space \mathcal{H} is the space of sections of the complex antidual E of F , which is a Hermitian holomorphic vector bundle. In the trivialization the fibre E_z becomes V with the inner product $\langle K(z, z)^{-1} \cdot, \cdot \rangle$.

It is a fundamental result ([5, 4, 1]) that the unitary equivalence class of elements of $\hat{B}_k(\mathcal{D})$ (and also of $B_k(\mathcal{D})$) and the corresponding isomorphism class of holomorphic Hermitian vector bundles mutually determine each other.

When \mathcal{D} is a bounded symmetric domain and H any Hilbert space, one calls an n -tuple $T = (T_1, \dots, T_n)$ of commuting bounded operators homogeneous (cf. [20, 3]) if their joint Taylor spectrum is contained in $\bar{\mathcal{D}}$ and for every holomorphic automorphism g of \mathcal{D} , there exists a unitary operator U_g such that

$$g(T_1, \dots, T_n) = (U_g^{-1}T_1U_g, \dots, U_g^{-1}T_nU_g),$$

or more briefly

$$(4.1) \quad g(T)_i = U_g^{-1}T_iU_g \quad (1 \leq i \leq n).$$

A description of all homogeneous n -tuples in $B_1(\mathcal{D})$, when \mathcal{D} is a domain of classical type is in [3, 20], for arbitrary \mathcal{D} it is in [2]. When \mathcal{D} is the unit disc in \mathbb{C} , a complete description of all homogeneous operators in $B_k(\mathcal{D})$ is in [17]. It is easily seen that the answer is the same for $\hat{B}_k(\mathcal{D})$. For a large subclass of $B_k(\mathcal{D})$ for arbitrary \mathcal{D} , there are precise results in [21].

Here we prove some simple results about the most general case, then specialize to the case of the unit ball in \mathbb{C}^n and prove the main results of this section.

Theorem 4.1. *Let $\mathcal{D} \subseteq \mathbb{C}^n$ be an irreducible bounded symmetric domain. An irreducible Hermitian holomorphic vector bundle E over \mathcal{D} corresponds to a homogeneous n -tuple in $\hat{B}_k(\mathcal{D})$ for some k if and only if it is homogeneous under \tilde{G} and its Hermitian structure comes from a regular unitary structure \mathcal{H} such that each multiplication operator M_i , $1 \leq j \leq n$, preserves \mathcal{H} and is bounded.*

Proof. For the “if” part: By Theorem 3.6 E is an E^y and \mathcal{H} is an \mathcal{H}_μ^y with some y and μ . The polynomials are dense in \mathcal{H}_μ^y and it has the reproducing kernel K_μ^y . By hypothesis, (M_1^*, \dots, M_n^*) is a well-defined n -tuple in $\hat{B}_k(\mathcal{D})$. We have to prove that M^* is homogeneous; for this, it is enough to prove that M is homogeneous. (As is well-known, if (4.1) holds for T with U_g , then it holds also for T^* with $U_{\bar{g}}$, where \bar{g} is defined by $\bar{g}(z) = \overline{g(\bar{z})}$ and \bar{z} is the ordinary complex conjugation.) Now $U = U^y$ acts on \mathcal{H}_μ^y via a multiplier $m(g, z)$, and we have

$$\begin{aligned} (M_i U_g f)(z) &= z_i m(g^{-1}, z)^{-1} f(g^{-1}z) \\ (U_g g(M)_i f)(z) &= m(g^{-1}, z)^{-1} (g(\zeta)_i f(\zeta))_{\zeta=g^{-1}(z)}. \end{aligned}$$

The two expressions being equal, M is homogeneous.

In proving the converse, \mathcal{H} is given with reproducing kernel K , polynomials dense, and M^* homogeneous. As recalled above, the joint \bar{z} -eigenspaces F_z of M^* form a bundle F and E is the anti-dual of F . We must prove that E is homogeneous. By [17, Theorem 2.1], for this it is enough to prove that for every $g \in \text{Aut}(\mathcal{D})$, there exists an automorphism of E acting on \mathcal{D} as g (i.e. a bundle map $E \rightarrow E$ projecting to g). For this, in turn, it is enough to prove that F has a similar property.

For all g in $\text{Aut}(\mathcal{D})$ we have by hypothesis a unitary operator U_g on \mathcal{H} intertwining M^* and $g(M^*)$. We show that $U_{\bar{g}}$ maps each F_z (which is a subspace of \mathcal{H}) linearly onto $F_{g(z)}$. This will give the desired automorphism of F . So, let $f \in F_z$, i.e. $M_i^* f = \bar{z}_i f$, ($1 \leq i \leq n$). We have

$$M_i^* U_{\bar{g}} f = U_{\bar{g}} \bar{g}(M^*)_i f = U_{\bar{g}} \bar{g}(\bar{z})_i f = \overline{g(z)}_i U_{\bar{g}} f,$$

which shows $U_{\bar{g}} f \in F_{g(z)}$. Doing the same with g^{-1} , we see that $U_{\bar{g}}$ gives a vector space isomorphism $F_z \rightarrow F_{g(z)}$, hence an automorphism of F . \square

The following corollary is immediate from the last statement of Theorem 3.6.

Corollary 4.2. *For a regularly unitarizable irreducible Hhhvb the boundedness of M_i holds either for all or none of the regular unitary structures. If it holds, then the corresponding commuting tuples of multiplication operators are all similar.*

For general \mathcal{D} , the following proposition provides a sufficient condition.

Proposition 4.3. *Let $\mathcal{D} \subseteq \mathbb{C}^n$ be an irreducible bounded symmetric domain and let $E^y = E^{\alpha, \lambda}$ be an irreducible Hhhvb. We write*

$$\lambda_y = \min_{0 \leq j \leq m} \min_{\alpha \in A_j} \lambda_\alpha.$$

If $\lambda < \lambda_y - \frac{n}{p}(r-1)\frac{a}{2}$, then E^y is regularly unitarizable. Each one of the Hermitian structures on E^y obtained in this way corresponds to a homogeneous tuple in some $\hat{B}_k(\mathcal{D})$.

Proof. We choose $\lambda_0 < -\frac{n}{p}(r-1)\frac{a}{2}$ such that $\lambda' = \lambda - \lambda_0 < \lambda_y$. So $E^{y, \lambda} = L_{\lambda_0} \otimes E^{y, \lambda'}$. Choosing some $\mu = \{\mu_{j\alpha}\}$, we have the regular unitary structure $\mathcal{H}_\mu^{(y, \lambda')}$ on $E^{y, \lambda'}$. By (3.18) we can also choose a regular unitary structure H on L_{λ_0} . Now $H \otimes \mathcal{H}_\mu^{(y, \lambda')}$ is a regular unitary structure on $E^{\lambda, y}$ with reproducing kernel $h(z, w) \frac{p}{n} \lambda_0 K_\mu^{(y, \lambda')}(z, w)$. As proved in [2], each M_i is bounded on H . So by our Proposition 3.2,

$$(c^2 - z_i \bar{w}_i) h(z, w) \frac{p}{n} \lambda_0 \succ 0$$

with some $c > 0$. It follows that

$$(c^2 - z_i \bar{w}_i) h(z, w) \frac{p}{n} \lambda_0 K_\mu^{(y, \lambda')}(z, w) \succ 0.$$

which shows that M_i is bounded on $H \otimes \mathcal{H}_\mu^{(y, \lambda')}$ (again by Proposition 3.2). But then the last statement of Theorem 3.6 implies that M_i is bounded on any of the regular structures of $E^{y, \lambda}$. \square

Corollary 4.4. *When \mathcal{D} is the Euclidean unit ball in \mathbb{C}^n , every Hermitian hhvb whose Hermitian metric comes from a regular unitary structure corresponds to a homogeneous tuple in $\hat{B}_k(\mathcal{D})$ for some k .*

Proof. The domain \mathcal{D} is the Euclidean unit ball if and only if $r = 1$. On the other hand by Theorem 3.6 we know that $E^{y, \lambda}$ is regularly unitarizable exactly when $\lambda < \lambda_y$. \square

In the case of a general \mathcal{D} , these arguments leave a gap, an interval of λ for which the question remains open.

We shall say that a homogeneous n -tuple in $\hat{B}_k(\mathcal{D})$ is *basic* if the corresponding Hermitian hhvb is of the form $E^{\alpha, \lambda}$, i.e., is induced by an irreducible representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$. If \mathcal{D} is the unit ball, then Corollary 4.4 gives a complete characterization of these. This result and the following theorem generalize the main results of [17, Theorem 4.2] to the case of the Euclidean unit ball in \mathbb{C}^n , ($n \geq 0$).

Theorem 4.5. *If \mathcal{D} is the Euclidean unit ball in \mathbb{C}^n , then every homogeneous n -tuple in $\hat{B}_k(\mathcal{D})$ is similar to the direct sum of basic homogeneous n -tuples.*

Proof. We know that the bundle for the homogeneous n -tuple in $\hat{B}_k(\mathcal{D})$ is an E with regular unitary structure. We may assume that E is irreducible. By Theorem 3.6 this means that $E = E^y$ with the corresponding E^0 a direct sum $\bigoplus_{j=0}^m \bigoplus_{\alpha \in A_j} E^{\alpha, \lambda-j}$ and $\mathcal{H}^y = \Gamma \mathcal{H}^0$, with $\mathcal{H}^0 = \bigoplus \mathcal{H}^{(\alpha, \lambda-j)}$. Each $\mathcal{H}^{(\alpha, \lambda-j)}$ and hence \mathcal{H}^0 is stable under M_j ($1 \leq j \leq n$) by Proposition 4.5(a). The essential point is that \mathcal{H}^0 and \mathcal{H}^y are the same as sets. This follows immediately from Theorem 3.12 and the definition of Γ .

Let I be the identity map regarded as a linear transformation from \mathcal{H}^0 to \mathcal{H}^y . Let $M_j^{(0)}$, respectively $M_j^{(y)}$, be the multiplication operators as before but regarded as operators on \mathcal{H}^0 and \mathcal{H}^y respectively. They are clearly intertwined by I , so we have

$$M_j^{(y)} = I M_j^{(0)} I^{-1}$$

Since $M^{(0)} = (M_1^{(0)}, \dots, M_n^{(0)})$ is the direct sum of basic n -tuples coming from the Hilbert spaces $\mathcal{H}_j = \bigoplus_{\alpha \in A_j} \mathcal{H}^{(\alpha, \lambda-j)}$, the theorem follows. \square

Remark 4.6. If the analogue of Theorem 3.12 can be proved for more general \mathcal{D} , then the present theorem will also hold, at least if λ is outside the gap mentioned after Corollary 4.4.

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