Research Statement

The development of multi-variate operator theory has been somewhat similar to that of function theory in several complex variables except that the non-commutating co-ordinates pose an entirely new set of challenges. The introduction of methods of commutative algebra for studying problems in this area by viewing a pair $(\mathscr{H}, \boldsymbol{T})$, where $\boldsymbol{T} := (T_1, \ldots, T_n)$ is a commuting n-tuple of operators on the Hilbert space $\mathscr{H}$, as a Hilbert module via the multiplication $(p, h) \mapsto p(\boldsymbol{T})h,\ p \in \mathbb{C}[z_1, \ldots, z_n],\ h \in \mathscr{H}$, was the beginning of a systematic new development. What makes this very useful is the possibility of using a vast array of tools from commutative algebra. On the other hand, these techniques don't apply directly because of the continuity assumption of the module multiplication either in just the first variable, or in both the variables. A choice is made depending on the problem at hand.

A second ingredient has been the action of a group G on the module $(\mathscr{H}, \boldsymbol{T})$. This occurs naturally in several instances. For example, if the joint spectrum $X := \sigma(\boldsymbol{T})$ of the d-tuple $\boldsymbol{T}$ is a G space, that is, G acts on X by biholomorphic automorphisms, then defining $g \cdot \boldsymbol{T}$, the classification of imprimitivities $\big(X, G, (\mathscr{H}, \boldsymbol{T})\big) := \big\{\boldsymbol{T} : U_g^* p(\boldsymbol{T}) U_g = (g \cdot p)(\boldsymbol{T})\}$, U_g is a unitary representation of the group G and $g \cdot p = p \circ g^{-1}$, is an important problem. One may study such imprimitivities by choosing a group G acting on X. This choice might vary from a transitive action to an action, where the orbits of G in X are quite big. For a typical example, consider $X = \bar{\mathbb{D}}^n$, the n-fold product of the unit closed disc, and the group G to be either the bi-holomorphic automorphism group of $\bar{\mathbb{D}}^n$ or simply the permutation group $\mathfrak{S}_n$ acting on $\bar{\mathbb{D}}^n$.

The study of these problems are greatly facilitated by the analysis of a class holomorphic curves in the Grassmannian $\mathrm{Gr}(\mathscr{H}, n)$ of rank n in some separable complex Hilbert space $\mathscr{H}$. These holomorphic curves arise from a class of operators acting on some Hilbert space $\mathscr{H}$ introduced in the paper of M. J. Cowen and R. G. Douglas, "Operator Theory and Complex Geometry", Acta Math., **141** (1978), 187 - 261. The operators in this class possess an open set $\Omega \subseteq \mathbb{C}$ of eigenvalues of (constant) multiplicity k and characterized by the existence of a holomorphic map $\gamma : \Omega \to \mathscr{H}$ such that $\boldsymbol{\gamma}(w) := (\gamma_1(w), \ldots, \gamma_k(w))$, $T\gamma_i(w) = w\gamma_i(w)$, $1 \le i \le k$, $w \in \Omega$. For $k = 1$, one of the main features of the operator T in this class is that the curvature $\mathscr{K}_T(w) := -\partial\bar{\partial} \log \|\gamma_T(w)\|^2$ of the holomorphic Hermitian line bundle E_T determined by the holomorphic map γ_T equipped with the Hermitian structure $\|\gamma_T(w)\|^2$ is a complete unitary invariant for the operator T.

It is easy to see that if T is a contraction in the Cowen-Douglas class of the unit disc $\mathbb{D}$, then $\mathscr{K}_T(w) \le \mathscr{K}_{S^*}(w)$, where S^* is the backward unilateral shift acting on ℓ^2. Choosing a holomorphic frame γ_{S^*}, say $\gamma_{S^*}(w) = (1, w, w^2, \ldots)$, it follows that $\|\gamma_{S^*}(w)\|^2 = (1 - |w|^2)^{-1}$ and that $\mathscr{K}_{S^*}(w) = -(1 - |w|^2)^{-2}$, $w \in \mathbb{D}$. Thus the the operator S^* is an extremal operator in the class of all contractive Cowen-Douglas operator. R. G. Douglas asked if the curvature $\mathscr{K}_T$ of a contraction T achieves equality in this inequality even at just one point, then does it follow that T must be unitarily equivalent to S^*? It is easy to see that the answer is "no", in general. However, if T is homogeneous, namely, $U_\varphi^* T U_\varphi = \varphi(T)$ for each bi-holomorphic automorphism φ of the unit disc and some unitary U_φ, then the answer is "yes". Of course, it is then natural to ask what are all the homogeneous operators. These are the holomorphic imprimitivities.

The question of classifying the holomorphic imprimitivities on a bounded symmetric domain Ω amounts to classification of commuting tuples of homogeneous of operators in the

Cowen-Douglas class of Ω. There is a one to one correspondence between these and the holomomorphic homogeneous vector bundles on the bounded symmetric domain Ω. The homogeneous bundles can be obtained by holomorphic induction from representations of a certain parabolic Lie algebra on finite dimensional inner product spaces. The representations, and the induced bundles, have composition series with irreducible factors. In joint work with A. Korányi [48, 61][1], our first main result is the construction of an explicit differential operator intertwining the bundle with the direct sum of its factors. Next, we study Hilbert spaces of sections of these bundles. We use this to get, in particular, a full description and a similarity theorem for homogeneous n-tuples of operators in the Cowen-Douglas class of the Euclidean unit ball in $\mathbb{C}^m$. A different approach is in [54]. The initial study of these questions restricted to the case of homogeneous holomorphic line bundles is in [18].

A little more can be said beyond holomorphic imprimitivities in the case of a single operator T acting on a Hilbert space $\mathscr{H}$. Assume that there is a projective unitary representation σ of Möb such that $\varphi(T) = \sigma(\varphi)^\star T\sigma(\varphi)$ for all φ in Möb. If this is the case, the operator T is homogeneous and we say that $\sigma(\varphi)$ is associated with the operator T. A Möbius equivariant version of the Sz.-Nagy–Foias model theory for completely non-unitary (cnu) contractions is developed in [67]. As an application, we prove that if T is a cnu contraction with associated (projective unitary) representation σ, then there is a unique projective unitary representation $\hat{\sigma}$, extending σ, associated with the minimal unitary dilation of T. The representation $\hat{\sigma}$ is given in terms of σ by the formula

$$\hat{\sigma} = (\pi \otimes D_1^+) \oplus \sigma \oplus (\pi_\star \otimes D_1^-),$$

where $D_1^\pm$ are the two Discrete series representations (one holomorphic and the other antiholomorphic) living on the Hardy space $H^2(\mathbb{D})$, and $\pi, \pi_\star$ are representations of Möb living on the two defect spaces of T and defined explicitly in terms of σ and T. Moreover, a cnu contraction T has an associated representation if and only if its Sz.-Nagy–Foias characteristic function θ_T has the product form $\theta_T(z) = \pi_\star(\varphi_z)^*\theta_T(0)\pi(\varphi_z)$, $z \in \mathbb{D}$, where φ_z is the involution in Möb mapping z to 0. We obtain a concrete realization of this product formula for a large subclass of homogeneous cnu contractions from the Cowen-Douglas class. These are the holomorphic imprimitivities among the homogeneous contractions.

Typically, contractive homomorphisms induced by an operator (or, even a commuting tuple of operators) in the Cowen-Douglas class give rise to curvature inequalities. However, given the curvature inequality, it is clear that it can only provide information about the second order jets of the holomorphic curve. Strengthening the curvature inequality to obtain additional information about the holomorphic curve remains an intriguing problem. Some partial answers to this question are in [51].

A complete set of invariants, originally obtained by Cowen and Douglas, have been refined to provide a tractable set of invariants for large class of Cowen-Douglas operators [55,56]. In these papers, after imposing a mild condition on the Cowen-Douglas bundles, it is shown that the curvature together with the second fundamental form serves as a complete set of invariants.

In general, determining the moduli space for the isomorphism classes of sub-modules of a Hilbert module is a difficult problem. Thanks to Beurling's theorem, the moduli space is a singleton for the Hardy module $H^2(\mathbb{D})$ of the unit disc, that is all sub-modules of $H^2(\mathbb{D})$ are isomorphic. However, a rigidity phenomenon occurs in $H^2(\mathbb{D}^n)$, namely, no two sub-modules of $H^2(\mathbb{D}^n)$ are isomorphic barring a very few exceptions. This is typical of the multi-variable situation. The determination of the isomorphism classes of sub-modules of a Hilbert module for

[1]The numbers in square brackets refer to the list on the Publications page.

anlytic Hilbert modules seems intractable at the moment. Some partial results have been obtained by using the monoidal transform to resolve the singularities of the holomorphic curves corresponding to sub-modules of an analytic Hilbert module. More recently, the submodules of analytic Hilbert modules defined over certain algebraic varieties in bounded symmetric domains, the so-called Jordan-Kepler varieties V_ℓ of arbitrary rank ℓ have been studied. For $\ell > 1$, the singular set of V_ℓ is not a complete intersection. Hence the usual monoidal transformations do not suffice for the resolution of the singularities. Instead, we describe a new higher rank version of the blow-up process, defined in terms of Jordan algebraic determinants, and apply this resolution to obtain the rigidity of the sub-modules vanishing on the singular set [44, 47, 64]. The accompanying question of finding a model for the quotient modules might be thought of the model theory of Sz. - Nagy and Foias to the more general context of Hilbert modules over function algebras. Finding a complete set of invariants for the unitary equivalence classes of such canonical models is an important problem. Both of these questions have been addressed partially in [22, 37].

Given a pair of positive real numbers α, β and a sesqui-analytic function K on a bounded domain $\Omega \subseteq \mathbb{C}^m$, we investigate the properties of the real-analytic function

$$K^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{z}) := \Big(\big(\tfrac{\partial^2}{\partial \bar{z}_j \partial z_i} K^{(\alpha+\beta)} \log K(\boldsymbol{z},\boldsymbol{z})\big)\Big)_{1\le i,j\le m},\ \boldsymbol{z}\in\Omega,$$

taking values in $m \times m$ matrices. The kernel $K^{(\alpha,\beta)}$ is non-negative definite whenever K^α and K^β are non-negative definite. In this case, a realization of the Hilbert module determined by the kernel $K^{(\alpha,\beta)}$ is obtained. Let $\mathscr{M}_i$, $i = 1, 2$, be two Hilbert modules over the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$. The tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$ is clearly a module over the ring $\mathbb{C}[z_1,\ldots,z_{2m}]$. This module multiplication restricts to $\mathbb{C}[z_1,\ldots,z_m]$ via the diagonal map $\boldsymbol{z} \mapsto (\boldsymbol{z},\boldsymbol{z})$, $\boldsymbol{z}\in\Omega$. Now, a natural decomposition of the tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$ very similar to the Clebsch-Gordon decomposition of the tensor product of two irreducible unitary representations occurs relative to the multiplication restricted to $\mathbb{C}[z_1,\ldots,z_m]$. Two of the initial pieces in this decomposition have been identified in [65]. The first of these is simply the restriction of $\mathscr{M}_1 \otimes \mathscr{M}_2$ to the diagonal set $\triangle := \{(\boldsymbol{z},\boldsymbol{z}) : \boldsymbol{z}\in\Omega\}$, while the second piece in the decomposition is the module determined by the kernel $K^{(\alpha,\beta)}$. Moreover, if Ω is a bounded symmetric domain, then $K^{(\alpha,\beta)}$ is covariant whenever K is covariant under the action of the bi-holomorphic automorphism group of Ω.