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#### 1 Quotients of Complex Analytic Spaces

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Part I

Automorphic Analysis
Chapter 0

Introduction

0.1 Motivation: Discrete groups in complex analysis and mathematical physics

0.1.1 Universal covering of Kähler manifolds

A complex Kähler manifold $M$ (not necessarily compact) has a universal covering manifold $D = \tilde{M}$ such that

$$M = D/\Gamma$$

where $\Gamma = \pi_1(M)$ is a discrete group of holomorphic deck transformations acting on $D$. Thus complex analysis on $M$ is related to ‘automorphic’ analysis on $D$. We call $\Gamma$ co-compact if the quotient space $M = D/\Gamma$ is compact. More generally, we call $\Gamma$ of finite covolume if $D/\Gamma$ has finite volume under the volume form induced by the Kähler metric. For ‘hyperbolic’ Kähler manifolds, $D$ can often be realized as a bounded domain in $\mathbb{C}^d$, and in important cases as a bounded symmetric domain

$$D = K \setminus G,$$

where $G$ is a semi-simple real Lie group and $K$ is a maximal compact subgroup. In this case we have a discrete subgroup

$$\Gamma \subset G$$

which is called a lattice if it is co-compact. In the 1-dimensional case a compact Riemann surface $M$ is hyperbolic iff it has genus $> 1$. By the uniformization theorem, $D = \tilde{M}$ is the unit disk (or upper half-plane). Therefore

$$\Gamma \subset G = \text{SL}_2(\mathbb{R})$$

becomes a discrete group of Möbius transformations. One (i.e., Poincaré) calls $\Gamma$ of Klein type if it is co-compact and of Fuchs type if it has finite co-volume. Thus the
all-important modular group

\[
SL_2(\mathbb{Z}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2} : ad - bc = 1 \}
\]

is Fuchsian, but not Kleinian. In higher dimensions, there exist differential geometric criteria for \( M \) to ensure that \( D = \tilde{M} \) is the unit ball in \( \mathbb{C}^d \), or more generally a bounded symmetric domain. These criteria involve important geometric invariants of the underlying Kähler manifold.

### 0.1.2 Teichmüller space

For a compact Riemann surface \( X \) the Teichmüller space

\[
\mathcal{T}(X) = \text{Conf}(X)/\text{Diff}^0(X)
\]

consists of all conformal structures on \( X \) modulo equivalence by diffeomorphisms which are isotopic to the identity. Via Beltrami differentials, \( \mathcal{T}(X) \) can be realized as a convex bounded domain. However, the 'true' moduli space

\[
\mathcal{M}(X) = \text{Conf}(X)/\text{Diff}(X)
\]

consists of all conformal structures modulo equivalence by the full diffeomorphism group. Thus

\[
\mathcal{M}(X) = \mathcal{T}(X)/\Gamma
\]

where

\[
\Gamma = \text{Diff}(X)/\text{Diff}^0(X)
\]

is the discrete group of components of \( \text{Diff}(X) \), also called the mapping class group. The Teichmüller space becomes the universal covering

\[
\mathcal{T}(X) = \widetilde{\mathcal{M}(X)}.
\]

As an important step, this moduli space has to be compactified to a projective algebraic variety (or stack) by adding points at infinity, so \( \mathcal{M}(X) \) becomes a Zariski-open (dense) subset of the compactification. Actually, the important case is where \( X \) is not compact, but arises from a compact Riemann surface by removing finitely many punctures. Then the compactification \( \overline{\mathcal{M}}_{g,n} \) is the Mumford-Deligne moduli space.

### 0.1.3 String theory duality groups

In the preceding two examples, \( \Gamma \) is a discrete group of holomorphic transformations. These give rise to automorphic forms, which are better regarded as sections of a holomorphic line bundle (therefore 'forms' instead of 'functions'). On the other hand, in
number theory (Langlands program) and mathematical physics one encounters discrete subgroups of 'real' Lie groups which give rise to real automorphic (or better, invariant) functions. For example, for a metric $g$ on a pseudo-Riemannian manifold of Minkowski signature, the Einstein field equation

$$\text{Ric}(g) = 0$$

arises from a variational principle under the Einstein-Hilbert action

$$\mathcal{L}(g) = \int_X d\text{Vol}_g \text{Scal}(g).$$

Extending this concept to super-gravity in 10 dimensions, the corresponding solutions, when compactified on tori $T^n$ of dimension $0 \leq n \leq 10$ have scalar moduli which transform under the super-gravity duality groups

$$A_n(\mathbb{R}), \ D_n(\mathbb{R}), \ E_n(\mathbb{R}),$$

the real forms of algebraic groups of ADZ-type. Now super-gravity is regarded as the low-energy limit of string theory. Passing to string theory, which is a quantum field theory, one expects again that the corresponding solutions have scalar moduli which transform under the super-string duality groups

$$A_n(\mathbb{Z}), \ D_n(\mathbb{Z}), \ E_n(\mathbb{Z}),$$

which form a lattice within the real duality groups.

### 0.1.4 Free group von Neumann algebras

For any free group $\Gamma$ in $\ell$-generators (more generally, every group with only infinite conjugacy classes) the group von Neumann algebra $W^*(\Gamma) = \Gamma''$ (bicommutant) is a von Neumann factor of type $II_1$. We will show that this arises in the Berezin quantization on weighted Bergman spaces $H^2_{\nu}(D)$ over the unit disk (or upper half-plane), where $\nu$ becomes the number of generators.

### 0.2 Basic concepts

#### 0.2.1 Holomorphic automorphism groups

For a complex manifold $D$ (or even more general, a complex analytic space) we let $\text{Aut}(D)$ denote the 'automorphism' group of all biholomorphic transformations of $D$, acting from the right: $(z, g) \mapsto z \cdot g$. It is known that for a bijective holomorphic map $g : D \to D$ the inverse map $g^{-1} : D \to D$ is also holomorphic.
If $D$ is a locally compact and locally connected topological space, then Arens has shown that the homeomorphism group $\text{Top}(X)$, endowed with the so-called compact-open topology, is a topological group and the evaluation map $D \times G \to D$ is jointly continuous. In particular, for a domain $D \subset \mathbb{C}^d$ we consider the identity component

$$G = \text{Aut}(D)^0$$

of the holomorphic automorphism group $\text{Aut}(D) \subset \text{Top}(D)$. By Arens’ result this is a connected topological group. In general, it is not a Lie group. For example, $\text{Aut}(\mathbb{C}^2)$ has infinite dimension, since for every entire function $f : \mathbb{C} \to \mathbb{C}$ the mapping

$$\Phi_f(z, w) := (z, w + f(z))$$

is an automorphism of $\mathbb{C}^2$, with inverse $\Phi_f^{-1} = \Phi_{-f}$. On the other hand, if $D \subset \mathbb{C}^d$ is a bounded domain, then $G$ is a (finite-dimensional) Lie group by a deep theorem of H. Cartan. The first step in the proof is the following:

**Lemma 1.** Let $A, B \subset D$ be compact subsets of $D$. Then the set

$$G_{A,B} := \{g \in G : A \cdot g \cap B \neq \emptyset\}$$

is compact.

**Proof.** By separability, it is enough to show that $G_{A,B}$ is sequentially compact. Consider a sequence $g_n \in G_{A,B}$. Then there exist sequences $a_n \in A$, $b_n \in B$ such that $a_n \cdot g_n = b_n$. Since $g_n^\pm$ are bounded holomorphic maps on $D$ we may choose by Montel’s theorem convergent subsequences satisfying $g_n^\pm \to g^\pm : D \to \overline{D}$. Since $A, B$ are compact, taking further subsequences we may assume in addition that $a_n \to a \in A$, $b_n \to b \in B$. Then $a \cdot g_+ = b$, $b \cdot g_- = a$ since the evaluation map $D \times G \to D$ is jointly continuous. Choose open sets $a \in U \subset W \subset D$, $b \in V \subset D$ satisfying $U \cdot g_+ \subset V$ and $V \cdot g_- \subset W$. Then joint continuity implies $g_+ \circ g_- |_V = \text{id}$, $g_- \circ g_+ |_U = \text{id}$. A similar argument shows that $D \cdot g_\pm \subset D$. 

**Corollary 2.** For every $a \in D$ the isotropy subgroup

$$G_a := G_{a,a} = \{g \in G : a \cdot g = a\}$$

is compact.

Let $D$ be a complex manifold, for example a bounded domain $D \subset \mathbb{C}^d$. A group $\Gamma \subset \text{Aut}(D)$ of holomorphic transformations of $D$ is called properly discontinuous if for all compact subsets $A, B \subset D$ the set

$$\Gamma_{A,B} := \{\gamma \in \Gamma : A \cdot \gamma \cap B \neq \emptyset\}$$

is finite. Note that in general this is only a subset of $\Gamma$. For $A = B$ we obtain a (finite) subgroup

$$\Gamma_A := \Gamma_{A,A} = \{\gamma \in \Gamma : A \cdot \gamma \cap A \neq \emptyset\}.$$
In particular, for each point \( a \in D \) the **isotropy subgroup**
\[
\Gamma_a := \Gamma_{a,a} = \{ \gamma \in \Gamma : a \cdot \gamma = a \}
\]
is finite. The same concepts apply to more general 'analytic spaces' which may have singularities.

**Proposition 3.** For a bounded domain \( D \), every discrete subgroup \( \Gamma \subset \text{Aut}(D) \) acts properly discontinuous on \( D \).

**Proof.** For all compact subsets \( A, B \subset D \) the set
\[
\Gamma_{A,B} = \Gamma \cap G_{A,B}
\]
is compact and discrete, hence finite. \( \square \)

### 0.2.2 Holomorphic automorphic forms

Consider first a connected complex manifold \( D \) and a properly discontinuous group \( \Gamma \subset \text{Aut}(D) \). An **automorphic cocycle** \( J : \Gamma \times D \to \mathbb{C} \) consists of holomorphic functions \( J_\gamma : D \to \mathbb{C} \) which satisfy the cocycle property
\[
J_{\gamma \gamma'}(z) = J_\gamma(\gamma'z) J_{\gamma'}(z).
\]
The standard example, for a domain \( D \), is given by the Jacobian
\[
J_g(z) := \det g'(z)
\]
where \( g'(z) \) is the holomorphic derivative of \( g \in \text{Aut}(D) \) at \( z \in D \). Relative to the cocycle \( J \), a holomorphic function \( f : D \to \mathbb{C} \) is called an **\( m \)-automorphic form** if
\[
J_\gamma(z)^m f(\gamma z) = f(z)
\]
for all \( \gamma \in \Gamma \) and all \( z \in D \). This means that \( f \) is a holomorphic section of the \( m \)-th power of a line bundle determined by \( J \). For \( m = 0 \) one would say invariant function, but typically automorphic forms exist for large \( m \). Let \( \mathcal{O}_m^\Gamma(D, \mathbb{C}) \) denote the vector space of all \( m \)-automorphic forms. Then \( \mathcal{O}_m^\Gamma(D, \mathbb{C}) \cdot \mathcal{O}_n^\Gamma(D, \mathbb{C}) \subset \mathcal{O}_{m+n}^\Gamma(D, \mathbb{C}) \) and hence
\[
\mathcal{O}_m^\Gamma(D, \mathbb{C}) := \sum_{m \geq 0} \mathcal{O}_m^\Gamma(D, \mathbb{C}) \subset \mathcal{O}(D, \mathbb{C})
\]
is a graded subalgebra of holomorphic functions.
0.2.3 Holomorphic Eisenstein series on bounded domains

Let $D \subset \mathbb{C}^d$ be a bounded domain and $\Gamma \subset \text{Aut}(D)$ a discrete, hence properly discontinuous, subgroup. Let $f \in H^\infty(D)$ be a holomorphic function. For $m \geq 2$ define the Poincaré-Eisenstein series

$$f_m^\Gamma(z) := \sum_{\gamma \in \Gamma} J(z, \gamma)^m f(z \cdot \gamma).$$

Note that $\Gamma$ is acting from the right.

**Proposition 4.** For $m \geq 2$ the series

$$1_m^\Gamma(z) := \sum_{\gamma \in \Gamma} J(z, \gamma)^m$$

is compactly $| \cdot |$-convergent on $D$.

**Proof.** Let $A \subset \subset B \subset \subset D$ be compact subsets. Then for each $z \in A$ there exists an open polydisk (product of disks) $P_z \subset B$. If $P_z \cdot \sigma \cap P_z \cdot \tau \neq \emptyset$ then $B \cdot (\sigma \tau^{-1}) \cap B \supset P_z \cdot (\sigma \tau^{-1}) \cap P_z \neq \emptyset$ and hence $\sigma \tau^{-1} \in \Gamma_B$. Therefore the collection $(P_z \gamma)_{\gamma \in \Gamma}$ covers $D$ at most $|\Gamma_B|$ times. This implies for the volume $| \cdot |$

$$\sum_{\gamma \in \Gamma} |P_z \cdot \gamma| \leq |\Gamma_B| |D|.$$

The mean value theorem and integral transformation formula imply

$$|J(z, \gamma)|^2 \leq \frac{1}{|P_z|} \int_{P_z} dw |J(w, \gamma)|^2 = \frac{|P_z \cdot \gamma|}{|P_z|}$$

since $|J(w, \gamma)|^2$ is the real Jacobian determinant. It follows that

$$\sum_{\gamma \in \Gamma} |J(z, \gamma)|^2 \leq \sum_{\gamma \in \Gamma} \frac{|P_z \cdot \gamma|}{|P_z|} \leq \frac{1}{|P_z|} |\Gamma_B| |D|.$$

Since $A$ is covered by finitely many polydisks $P_z$, this proves uniform convergence on $A$ for $m = 2$. This in turn implies $\sup_{z \in A} |J(z, \gamma)| < 1$ for almost all $\gamma \in \Gamma$ and therefore $|J(z, \gamma)|^m \leq |J(z, \gamma)|^2$ of $m \geq 2$.

**Corollary 5.** If $f \in H^\infty(D)$ is a bounded holomorphic function, then for $m \geq 2$ the series

$$f_m^\Gamma(z) := \sum_{\gamma \in \Gamma} J(z, \gamma)^m f(z \cdot \gamma)$$

is compactly $| \cdot |$-convergent on $D$. 

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Proof.

\[ \sum_{\gamma \in \Gamma} |J(z, \gamma)|^m |f(z \cdot \gamma)| \leq \sup_D |f| \sum_{\gamma \in \Gamma} |J(z, \gamma)|^m \]

\qed

In case \( D \subset Z \) is a bounded domain, all polynomials \( f \in \mathcal{P}(Z) \) restricted to \( D \) are bounded.

### 0.2.4 Poincaré series on Lie groups

**Proposition 6.** Let \( G \) be a unimodular group and \( f \in L^1(G) \) be integrable (could be vector-valued). Then the Poincaré series

\[ f^\Gamma(g) := \sum_{\gamma \in \Gamma} f(g\gamma) \]

\( \| \cdot \| \)-converges compactly on \( G \) and is bounded.

**Proof.** Since \( \Gamma \) is discrete there exists a symmetric compact \( e \)-neighborhood \( P \subset G \) such that \( \Gamma \cap P^2 = \{e\} \). By a deep result of Harish-Chandra [Baily, Theorem 19, p. 154] there exists a 'Dirac' like function \( \delta \in \mathcal{C}^\infty(G) \) with compact support \( \text{supp}(\delta) \subset P \) (which is \( K \)-invariant \( \delta(k^{-1}gk) = \delta(k) \ \forall \ k \in K \)) and satisfies the convolution equation

\[ f \ast \delta = f. \]

Putting \( h' = h\gamma \), it follows that

\[ f(g\gamma) = (f \ast \delta)(g\gamma) = \int_G dh \ f(g\gamma h'^{-1}) \ \delta(h') \]

\[ = \int_G dh \ f(gh^{-1}) \ \delta(h\gamma) = \int_{P\gamma^{-1}} dh \ f(gh^{-1}) \ \delta(h\gamma) \]

Therefore

\[ \|f(g\gamma)\| \leq \int_{P\gamma^{-1}} dh \ |f(gh^{-1})| \ |\delta(h\gamma)| \leq \sup_{G} |\delta| \int_{P\gamma^{-1}} dh \ |f(gh^{-1})| \]

If \( \gamma_1, \gamma_2 \in \Gamma \) are distinct, then \( P_{\gamma_1^{-1}} \cap P_{\gamma_2^{-1}} = \emptyset \). Putting \( h'' = gh^{-1} \), it follows that

\[ \|f\|^\Gamma(g) := \sum_{\gamma \in \Gamma} \|f(g\gamma)\| \leq \sup_{G} |\delta| \sum_{\gamma \in \Gamma} \int_{P\gamma^{-1}} dh \ |f(gh^{-1})| \]

\[ \leq \sup_{G} |\delta| \int_{G} dh \ |f(gh^{-1})| = \sup_{G} |\delta| \int_{G} dh'' \ |f(h'')| = \sup_{G} |\delta| \|f\|_1 \]
using that $G$ is unimodular. This shows that the series converges normally on $G$. Since $f$ is integrable, for any $\epsilon > 0$ there exists a compact set $Q \subset G$ such that
\[
\int_{G \setminus Q} dh \| f(h) \| \leq \epsilon.
\]

For any compact subset $C \subset G$ the set
\[A := \{ \gamma \in \Gamma : C\gamma P \cap Q \neq \emptyset \}\]
is finite. For $\gamma \in \Gamma \setminus A$ the sets $g\gamma P$ are pairwise disjoint and contained in $G \setminus Q$. Therefore, for any $g \in C$
\[
\sum_{\gamma \in \Gamma \setminus A} \| f(g\gamma) \| \leq \sup_{g \in G} |\delta| \sum_{\gamma \in \Gamma \setminus A} \int_{g\gamma P} dh \| f(h) \| \leq \sup_{g \in G} |\delta| \int_{G \setminus Q} dh \| f(h) \| \leq \epsilon \sup_{g \in G} |\delta|.
\]
Hence the series converges uniformly on $C$.

In general, it is difficult to decide whether these Poincaré series do not vanish identically. This can be studied, e.g., by Fourier expansions to be considered later.
Chapter 1

Quotients of Complex Analytic Spaces

1.1 Overview

The quotient space $D/\Gamma$ of a complex manifold $D$ (e.g., a domain $D \subset \mathbb{C}^d$) by a properly discontinuous group $\Gamma$ is in general not a complex manifold, because of singularities arising at fixed points $a \in D$ where the (finite) isotropy group $\Gamma_a$ is not trivial. Nevertheless, it will be shown that $D/\Gamma$ is always a so-called analytic space. More precisely,

- The quotient $Z/\Gamma$ by a **finite linear group** $\Gamma \subset GL(Z) = GL_n(\mathbb{C})$ (not necessarily a reflection group) is a complex analytic space.

- As a consequence, the quotient $D/\Gamma$ of any complex analytic space $D$ by a **properly discontinuous group** $\Gamma \subset Aut(D)$ (not necessarily finite or linear) is again a complex analytic space.

- If $D$ is a **bounded domain** and $\Gamma \subset Aut(D)$ is a **co-compact discrete** subgroup, then $D/\Gamma$ is a projective algebraic variety. This deep result of H. Cartan was a primary motivation for Kodaira’s embedding theorem.

- If $D = K \backslash G$ is a **bounded symmetric domain** and $\Gamma \subset G$ is an ‘arithmetic’ discrete subgroup (of finite co-volume) then $D/\Gamma$ is a Zariski-dense open subset of a projective algebraic variety.

In this chapter we prove the first three assertions. The fourth assertion (Satake compactification) lies deeper and will be proved later.
1.2 Commutative algebra

1.2.1 Integral closure and Krull topology

We consider unital commutative rings $A$. For an integral domain $A$ let

$$A := \{ \frac{a}{b} : a, b \in A, b \neq 0 \}$$

denote its field of fractions. For a commutative ring extension $A \subset B$ let

$$A^B := \{ b \in B : A[b] = A\langle fin \rangle \}$$

denote the integral closure of $A$ in $B$. This shorthand notation means that the algebra $A[b]$ generated by $A$ and $b \in A$ (in short, the $A$-algebra generated by $b$) is a finitely generated $A$-module. One can show that

$$A \subset A^B \subset B$$

is a subring of $B$. We define the notion of integrally closed and integrally dense by looking at the extreme cases

$$A \overset{\text{int closed}}{\subset} B \iff A = A^B,$$

$$A \overset{\text{int dense}}{\subset} B \iff A^B = B.$$

An integral domain $A$ is called normal if

$$A = A^A \overset{\text{int closed}}{\subset} A^A$$

is integrally closed in its field of fractions. Consider a group $\Gamma \subset \text{Aut}(A)$ of ring automorphisms of $A$. Then

$$A^\Gamma := \{ a \in A : \gamma \cdot a = a \ \forall \ \gamma \in \Gamma \}$$

is a subring of $A$.

**Lemma 7.** Let $A$ be a normal ring. Then the subring

$$A^\Gamma := \{ a \in A : \gamma \cdot a = a \ \forall \ \gamma \in \Gamma \}$$

is also normal.

**Proof.** Since $A$ is an integral domain, its subring $A^\Gamma$ is also an integral domain. Now let $f = \frac{p}{q} \in A^\Gamma$, where $p, q \in A^\Gamma$ and $q \neq 0$. Then $f \in A^A = A$ and for all $\gamma \in \Gamma$ we have

$$\gamma \cdot f = \frac{\gamma \cdot p}{\gamma \cdot q} = \frac{p}{q} = f$$

Therefore $f \in A^\Gamma$ and hence $A^\Gamma = A^\Gamma A^\Gamma$  \qed
Lemma 8. Let $A$ be a noetherian ring, and $M = A\langle\text{fin}\rangle$ a finitely generated $A$-module. Then every $A$-submodule $N \subset M$ is also finitely generated, $N = A\langle\text{fin}\rangle$.

The following integrality criterion will often be used.

**Proposition 9.** Let $A \subset B$ be a commutative ring extension. Then $B = A\langle\text{fin}\rangle$ is a finitely generated $A$-module if and only if $B = A[\text{fin}]$ is a finitely generated $A$-algebra and $B = \overline{A}^B$. In short,
\[
B = A\langle\text{fin}\rangle \iff B = A[\text{fin}] = \overline{A}^B
\]

Now we study ring completions under the so-called Krull topology. For any ring $A$ and ideal $m \triangleleft A$ the $m$-closure of an ideal $a \triangleleft A$ is given by
\[
\overline{a} = \bigcap_{n \geq 0} (a + m^n A)
\]
More generally, a submodule $N \subset M$ of an $A$-module $M$ has the $m$-closure
\[
\overline{N} = \bigcap_{n \geq 0} (N + m^n M)
\]
The following closure criterion is proved in [Zariski-Samuel, p. 262, Theorem 9].

**Proposition 10.** Consider a noetherian ring $A$ and an ideal $m \triangleleft A$ contained in every maximal ideal. An equivalent condition is that
\[
1 + m \subset \hat{A}
\]
is invertible. Then every ideal $a \triangleleft A$ is $m$-closed
\[
a = \bigcap_{n \geq 0} (a + m^n A)
\]
More generally, every submodule $N \subset M = A\langle\text{fin}\rangle$ of a finitely generated $A$-module $M$ is $m$-closed:
\[
N = \bigcap_{n \geq 0} (N + m^n M)
\]

An important special case is a (noetherian) local ring $A$ with a unique maximal ideal
\[
m = A \setminus \hat{A}.
\]
Here $\hat{A}$ denotes the group of units in $A$. 

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1.2.2 Power series and germs of analytic functions

For a field $K$ and indeterminates $x = (z_1, \ldots, z_n)$ we denote by

$$K[z] = K[z_1, \ldots, z_n]$$
$$K[z] = K[z_1, \ldots, z_n]$$
$$C\{z\} = C\{z_1, \ldots, z_n\}$$

the ring of polynomials/formal power series/convergent power series

$$f(z) = \sum_{\nu \in \mathbb{N}^n} f_{\nu} x^{\nu}$$

with coefficients $f_{\nu} \in K$. These rings are integral domains (no zero divisors). Putting

$$x = (z_1, z_n) = (z', z_n),$$

with $z' = (z_1, \ldots, z_{n-1})$, we have

$$K[z] = K[z'][z_n]$$
$$K[z] = K[z'][z_n]$$
$$C\{z\} = C\{z'\}\{z_n\}$$

The **Weierstrass division theorem** states

**Theorem 11.** Let $f, g \in K[z]$ such that $f(0, z_n) \neq 0$, i.e., $o(f(0', z_n)) = k < \infty$. Then there exist unique $q \in K[z]$ and $r \in K[z'][z_n]$ such that the order

$$o(g(0', z_n) - r(0', z_n)) \geq k$$

and

$$f = qg + r.$$ 

Similarly for convergent power series.

Thus the Taylor coefficients in the $z_n$-variable satisfy $g_i(0') = r_i(0')$ for $0 \leq i < k$.

**Corollary 12.** We have

$$f = qz^k + r$$

with $q(0) \neq 0$, i.e., $q$ is a unit.

**Proposition 13.** The rings $K|z|$ and $C\{z\}$ are noetherian.

**Proof.** Use induction on $n$ and, in the convergent setting, the Weierstrass theorem. □

**Proposition 14.** The rings $K[z], K|z|, C\{z\}$ are normal.
For any \( a \in \mathbb{C}^n \) let
\[
\mathcal{O}_a = \mathcal{O}_{a}^{\mathbb{C}^n} \approx \mathbb{C}\{z - a\}
\]
denote the local ring of \textbf{germs of analytic functions} at \( a \). Given an open subset \( U \subset \mathbb{C}^n \) a closed subset \( X \subset U \) is called \textbf{analytic} if for every \( a \in X \) there exist \( a \in U_a \subset_{\text{open}} U \) and holomorphic functions \( h_i \in \mathcal{O}(U_a) \), \( i \in I \) such that
\[
X \cap U_a = \{ z \in U_a : h_i(z) = 0 \ \forall \ i \in I \}.
\]
By the noetherian property, one may always choose \( I \) to be a finite set. For an analytic set \( X \) we denote by \( \mathcal{O}_X^a \) the ring of germs of analytic functions on \( X \).

There are \textbf{two basic ways} to construct local rings of analytic functions. Suppose first that \( \pi : D \to D/\Gamma \) is a quotient map. Endow \( D/\Gamma \) with the quotient topology, and for \( a \in D \), let \( \mathcal{O}_{\pi a}^{D/\Gamma} \) denote the ring of germs of continuous functions. Then we define
\[
\mathcal{O}_{\pi a}^{D/\Gamma} := \{ f \in \mathcal{O}_{\pi a}^{D/\Gamma} : f \circ \pi \in \mathcal{O}_a^D \} =: \pi_{\ast}(\mathcal{O}_a^D).
\]
On the other hand, for an analytic subset \( X \subset U \) with inclusion map \( \iota : X \to U \), and \( b \in X \) we define
\[
\mathcal{O}_b^X := \{ f|_X = f \circ \iota : f \in \mathcal{O}_b^U \} =: \iota_{\ast}\mathcal{O}_b^U.
\]
The maximal ideal \( m \) in the local power series ring \( K[z]/\mathbb{C}\{z\} \) are the power series \( f \) without constant term, i.e. \( f(0) = 0 \). Given power series \( f_i \in \mathbb{C}\{z\} \) without constant term we put
\[
f_{\ast} = (f_1, \ldots, f_m).
\]
Since \( f_{\ast}(0) = 0 \) we have the \textbf{substitution homomorphism}
\[
\mathbb{C}\{z\} \xrightarrow{\circ f_{\ast}} \mathbb{C}\{w\}, \quad g(w) \mapsto g(f_{\ast}z)
\]
for \( x \) near 0, inducing a commuting diagram
\[
\begin{array}{ccc}
\mathbb{C}\{z\} & \xrightarrow{\circ f_{\ast}} & \mathbb{C}\{w\} \\
\uparrow & & \downarrow
\\
\mathbb{C}\{f_{\ast}\} & \xrightarrow{\approx} & \mathbb{C}\{w\}/\ker(\circ f_{\ast})
\end{array}
\]
where the range
\[
\mathbb{C}\{f_{\ast}\} = \mathbb{C}\{f_1, \ldots, f_m\} := \mathbb{C}\{w\} \circ f_{\ast}
\]
is a subring of \( \mathbb{C}\{z\} \) and the kernel
\[
\ker(\circ f_{\ast}) = \{ g \in \mathbb{C}\{w\} : g \circ f_{\ast} = 0 \} \subset \mathbb{C}\{w\} 
\]
is called the **ideal of analytic relations** between \( f_1, \ldots, f_m \).

For a polynomial ideal \( I \triangleleft K[z] \) we denote by
\[
I^\triangledown := \{ z \in Z : p(z) = 0 \ \forall \ p \in I \}
\]
the **algebraic variety** in \( Z = \mathbb{C}^n \). If \( p_\ast = (p_1, \ldots, p_m) \) we also write
\[
p_\ast^\triangledown = K[z]\langle p_\ast \rangle^\triangledown = \{ z \in Z : p_i(z) = 0 \ \forall \ i \}
\]
by considering the ideal \( K[z]\langle p_\ast \rangle \) generated by the \( p_i \). For convergent power series \( f_i \in C\{z\} \) we have instead the **analytic variety** (germ)
\[
f_\ast^\triangledown := \{ z : f_1(z) = \ldots = f_m(z) = 0 \}
\]
near 0. Then \( 0 \in f_\ast^\triangledown \) and \( f_\ast \) defines an analytic mapping into \( \ker(\circ f_\ast) \).

The following **geometric integrality criterion** is proved in [Baily, Corollary, p. 19].

**Proposition 15.** 0 is isolated in \( f_\ast^\triangledown \) if and only if the ring \( C\{z\} \) is integral over its subring \( C\{f_\ast\} \), i.e.,
\[
C\{f_\ast\}^\text{int} \subset C\{z\} = C\{f_\ast\}^{C\{z\}}
\]

### 1.3 Quotient by a finite linear group

Let \( \Gamma \subset \text{GL}_d(C) \) be a finite group of linear transformations. More generally, let \( K \) be a field, not necessarily of characteristic 0 or algebraically closed. We often write \( Z = K^d \) (resp., \( Z = \mathbb{C}^d \)) since the coordinates play no distinguished role. Thus \( \Gamma \subset \text{GL}(Z) \).

Via substitution
\[
(\gamma \cdot p)(z) := p(z \cdot \gamma)
\]
the group \( \Gamma \) acts by ring automorphisms on the polynomials \( K[z] \). Consider the invariant subalgebra
\[
K[z]^\Gamma := \{ p \in K[z] : \gamma \cdot p = p \ \forall \ \gamma \in \Gamma \}.
\]
Since the \( \Gamma \)-action preserves degrees, the homogeneous terms of a \( \Gamma \)-invariant polynomial are also \( \Gamma \)-invariant. It follows that \( K[z]^\Gamma \) is a graded \( K \)-algebra.

**Lemma 16.** The ring extension \( K[z]^\Gamma \subset K[z] \) is integral, i.e.
\[
K[z]^\Gamma \subset K[z]\text{int} = \overline{K[z]^\Gamma K[z]}^{K[z]}
\]

**Proof.** For \( p \in K[z] \) the monic polynomial
\[
\hat{p}(t) = \prod_{\gamma \in \Gamma} (t - \gamma \cdot p) = (t - p) \prod_{1 \neq \gamma \in \Gamma} (t - \gamma \cdot p) \in K[z]^\Gamma[t]
\]
satisfies \( \hat{p}(p) = 0 \). Therefore \( p \in \overline{K[z]^\Gamma K[z]} \). \( \square \)
Lemma 17.  

\[ K[z] = K[z]^{\Gamma} \langle \text{fin} \rangle \]

is a finitely generated \( K[z]^{\Gamma} \)-module.

Proof. Since  

\[ K[z] = K[\text{fin}] = K[z]^{\Gamma}[\text{fin}] = \overline{K[z]^{\Gamma}}^K[z], \]

the assertion follows from the 'integrality criterion'. \( \square \)

The 'polynomial' finite generation theorem is

Theorem 18. There exist finitely many homogeneous polynomials \( p_1, \ldots, p_m \) such that  

\[ K[z]^{\Gamma} = K[p_1, \ldots, p_m] = K[p_*] \]

is a finitely generated \( K \)-algebra. Here we write \( p_* := (p_1, \ldots, p_m) \). In other words, the substitution homomorphism  

\[ K[z]^{\Gamma} \xleftarrow{op_*} K[w] \]

is surjective.

Proof. Applying (??) to the coordinate functions \( z_i \) we obtain  

\[ \hat{z}_i(t) = \sum_{n \geq 0} t^n z_{i,n} \in K[z]^{\Gamma}[t] \]

where \( z_{i,n} \in K[z]^{\Gamma} \). Define the unital \( K \)-algebra  

\[ A := K[z_{i,n}] \subset K[z]^{\Gamma}. \]

For each \( i \) we have \( z_i \in \overline{A}^{K[z]} \) since \( \hat{z}_i(t) \in A[t] \) by definition of \( A \). Since the integral closure \( \overline{A}^{K[z]} \) is a subring and even a \( K \)-subalgebra, it follows that  

\[ K[z] = \overline{A}^{K[z]} . \] (1.3.1)

Therefore  

\[ K[z] = K[\text{fin}] = A[\text{fin}] = \overline{A}^{K[z]} \]

and the 'integrality criterion' implies that  

\[ K[z] = A \langle \text{fin} \rangle \] (1.3.2)

is a finitely generated \( A \)-module. Now \( A \) is a homomorphic image of a polynomial ring, hence noetherian. By (1.3.3) it follows that \( K[z] \) is a noetherian \( A \)-module. Hence the \( A \)-submodule \( K[z]^{\Gamma} \subset K[z] \) is also noetherian. Now the Lemma implies  

\[ K[z]^{\Gamma} = A \langle \text{fin} \rangle = K[\text{fin}] \langle \text{fin} \rangle = K[\text{fin}] \]

This yields finitely many algebra-generators \( p_1, \ldots, p_m \), which may be assumed homogeneous, since \( K[z]^{\Gamma} \) is a graded algebra. \( \square \)
For $1 \leq 1 \leq m$ define
\[ d_j = \deg p_j > 0. \]
For any multi-index $\mu = (\mu_1, \ldots, \mu_m)$ of length $m$ put
\[ d \cdot \mu := \sum_j d_j \cdot \mu_j. \]
A polynomial of the form
\[ \phi(w) = \sum_{d \cdot \mu = k} \phi_\mu w^\mu \]
for some integer $k$ is called $k$-isobaric. Let
\[ (j) := (0, \ldots, 0, 1_j, 0, \ldots, 0). \]

**Lemma 19.** Let $\phi \in K[w]$ be $k$-isobaric. If there exists $1 \leq j \leq m$ with coefficient $\phi(j) \neq 0$, then
\[ \phi - \phi(j) w_j \in K[w_1, \ldots, \hat{w}_j, \ldots, w_m] \]

**Proof.** If $\phi(j) \neq 0$, then $d_j = k$. Now
\[ \phi - \phi(0) = \sum_{|\mu| > 1} \phi_\mu w^\mu. \]
Let $|\mu| > 1$ satisfy $\mu_j > 0$. If $\mu_j > 1$ then $d \cdot \mu \geq d_j \mu_j > k$. Therefore $\phi_\mu = 0$. If $\mu_j = 1$ there is another index $i \neq j$ such that $\mu_i > 0$. Then $d \cdot \mu \geq d_j + d_i \mu_i > d_j = k$. Therefore $\phi_\mu = 0$. \qed

We say that a set of generators $p_\ast = (p_1, \ldots, p_m)$ of $K[z]^G$ is reduced if every isobaric polynomial $\phi \in K[w]$ satisfying $\phi \circ p_\ast = 0$ has a vanishing linear term $\phi'(0) = 0$.

**Lemma 20.** Every homogeneous set $p_1, \ldots, p_m$ of generators of $K[z]^G$ contains a reduced set of generators.

**Proof.** If $p_1, \ldots, p_m$ is not reduced, there exists an isobaric polynomial $\phi \in K[w]$, satisfying $\phi \circ p_\ast = 0$, with non-vanishing linear term $\phi'(0) \neq 0$. Hence $\phi(j) \neq 0$ for some $j$. By the Lemma we have
\[ 0 = \phi(p_\ast) = (\phi - \phi(j) w_j)(p_1, \ldots, \hat{p}_j, \ldots, p_m) + \phi(j) p_j. \]
Since $\phi(j) \neq 0$ it follows that $p_j \in K[p_1, \ldots, \hat{p}_j, \ldots, p_m]$. Therefore $p_1, \ldots, \hat{p}_j, \ldots, p_m$ is a smaller set of generators. Repeating this argument, we obtain a reduced set of generators. \qed

From now on we assume that the generators $p_\ast$ are homogeneous and reduced. Next we obtain the 'power series’ finite generation theorem.
Theorem 21. For formal/convergent power series we have

$$K|z|^\Gamma = K|p_*| / C\{z\}^\Gamma = C\{p_*\}.$$

Note that $p_*(0) = 0$ is needed to define these rings. In other words, the substitution homomorphisms

$$K|z|^\Gamma \xleftarrow{op_*} K|w|,$$

$$C\{z\}^\Gamma \xleftarrow{op_*} C\{w\}$$

are surjective: for every $f(z) \in K|z|^\Gamma / C\{z\}^\Gamma$ there exists $\hat{f} \in K|w| / C\{w\}$ such that

$$f(z) = \hat{f}(p_*z)$$

Proof. The assertion for formal power series follows from the expansion into homogeneous terms. In the convergent setting $C\{p_*\} \subset C\{z\}^\Gamma$, Taylor expansion into homogeneous terms shows that

$$C[p_*] = C[z]^\Gamma \overset{m}{\subset} \text{dense } C\{z\}^\Gamma$$

in the topology induced by the powers of the maximal ideal

$$m = \{ f \in C\{z\} : f(0) = 0 \} \triangle C\{z\}.$$

A fortiori, we obtain

$$C\{p_*\} \overset{m}{\subset} \text{dense } C\{z\}^\Gamma.$$

We will now show that $C\{p_*\}$ is also $m$-closed in $C\{z\}^\Gamma$. For any $f \in C\{z\}^\Gamma$ consider the algebra $C\{p_*\}[f]$. Then (??) implies

$$C\{p_*\} \overset{m}{\subset} \text{dense } C\{p_*\}[f].$$

Since $C\{z\}$ is a noetherian ring, its homomorphic image $C\{p_*\}$ is also noetherian. Since 0 is isolated in $V(p_*)$, the ‘geometric integrality criterion’ implies

$$C\{p_*\} \overset{\text{int}}{\subset} \text{dense } C\{z\} = \overline{C\{p_*\}^{C\{z\}}}$$

is integrally dense in $C\{z\}$. A fortiori,

$$C\{p_*\} \overset{\text{int}}{\subset} \text{dense } C\{p_*\}[f] = \overline{C\{p_*\}^{C\{p_*\}[f]}}$$

is also integrally dense in the subring $C\{p_*\}[f] \subset C\{z\}$. Thus

$$C\{p_*\}[f] = C\{p_*\}[fin] = \overline{C\{p_*\}^{C\{p_*\}[f]}}$$

and the integrality criterion implies that

$$C\{p_*\}[f] = C\{p_*\}\langle fin \rangle$$
is a finitely generated \( C\{p_*\}\)-module. Applying the 'closure criterion' to the noetherian ring \( C\{p_*\} \) and its maximal ideal \( m \cap C\{z\} \) it follows that the \( C\{p_*\}\)-submodule

\[ C\{p_*\} \overset{m}{\subseteq} \text{closed} C\{p_*\}[f] \]

is \( m \)-closed. Since it is also \( m \)-dense, we obtain \( C\{p_*\} = C\{p_*\}[f] \). Therefore \( f \in C\{p_*\} \).

Since \( f \in C\{z\}\) is arbitrary, it follows that \( C\{p_*\} = C\{z\}^\Gamma \).

\[ \square \]

Proposition 22. Consider power series \( q_1, \ldots, q_m \in K[z]^\Gamma / C\{z\}^\Gamma \) which satisfy

\[ o(q_j - p_j) > d_j. \]

Then there exist power series \( \Lambda_j(w) \in K|w| / C\{w\} \) such that

\[ \Lambda_* \circ p_* = q_*, \quad \Lambda_j(p_*z) = q_j(z) \]

and the linear term \( \Lambda_i(0) \) is invertible. Here we put

\[ q_*(z) := (q_1(z), \ldots, q_m(z)), \quad \Lambda_*(w) := (\Lambda_1(w), \ldots, \Lambda_m(w)). \]

Proof. We may assume that \( d_1 \leq \ldots \leq d_m \). Write

\[ q_i(z) = \sum_{n \geq 0} q_i^{(n)}(z) \]

where \( q_i^{(n)} \in K[z]^\Gamma \) is \( n \)-homogeneous. Then \( q_i^{(n)} \in K[z]^\Gamma \) can be (non-uniquely) written as

\[ q_i^{(n)} = \sum_{d\mu = n} a_{i, \mu}^n p_*^{\mu} \]

with coefficients \( a_{i, \mu}^n \in K \). Define formal power series

\[ \Lambda_i(w) := \sum_{\mu} a_{i, \mu}^{d_* \mu} w^\mu \in K|w|. \]

Then

\[ \Lambda_i(p_*z) = \sum_{\mu} a_{i, \mu}^{d_* \mu} (p_*z)^\mu = \sum_{n} q_i^{(n)}(z) = q_i(z). \]

For each fixed \( i \) the assumption

\[ o(\Lambda_i \circ p_* - p_i) = o(q_i - p_i) > d_i \]

implies that the isobaric polynomial

\[ \phi(w) := \begin{cases} \sum_{d\mu = n} a_{i, \mu}^n w^\mu & n < d_i \\ \sum_{d\mu = d_i} a_{i, \mu}^{d_i} w^\mu - w_i & \end{cases} \]
satisfy $\phi \circ p_* = 0$. By reducedness, the linear term vanishes:

$$0 = \phi'(0)y = \begin{cases} \sum_{d_j=n} a^n_{i,(j)}w_j & n < d_i \\ \sum_{d_j=d_i} a^d_{i,(j)}w_j - w_i & \end{cases}.$$ 

Therefore

$$\begin{cases} 
 a^n_{i,(j)} = 0 & \forall d_j = n \quad n < d_i \\
 d^d_{i,(i)} = 1 & a^d_{i,(j)} = 0 \quad \forall j \neq i, d_j = d_i .
\end{cases} \tag{1.3.3}$$

We claim that the linear terms

$$\Lambda'_i(0)y = \sum_j a^d_{i,(j)}w_j$$

form a unipotent upper triangular matrix. On the diagonal we have $d_{i,(i)} = 1$ by (1.3.3). Now let $j < i$. Then $d_j \leq d_i$. If $d_j < d_i$, then $a^d_{i,(j)} = 0$ by (1.3.3). If $d_j = d_i$, then $a^d_{i,(j)} = a^d_{d,i,(i)} = 0$ by (1.3.3). \hfill \Box

Note that this argument needs coordinates $w_1, \ldots, w_m$ (instead of just a complex vector space of dimension $m$) in order to define upper triangular matrices. The deeper reason is that the degrees $d_j$ of the generators $p_j$ will in general be distinct.

**Corollary 23.** For every $f \in K[z]/C\{z\}$ there exists $\tilde{f} \in K[w]/C\{w\}$ such that

$$\tilde{f} \circ q_* = f, \quad \tilde{f}(q_*z) = f(z)$$

In other words, the substitution homomorphisms

$$K[z]/C\{z\} \xleftarrow{\circ q_*} K[w], \quad C\{z\} \xrightarrow{\circ q_*} C\{w\}$$

are surjective.

**Proof.** Since the power series map

$$y \mapsto \Lambda_*(w) = (\Lambda_1(w), \ldots, \Lambda_m(w))$$

satisfies $\Lambda_*(0) = 0$ and $\Lambda'_*(0)$ is invertible, the inverse mapping theorem for formal/convergent power series implies that the (composition) inverse $\Lambda_*^{-1}(w)$ exists as a formal/convergent power series near 0. Now the 'power series’ finite generation theorem implies

$$f = \tilde{f} \circ p_* = f \circ (\Lambda_*^{-1} \circ q_*) = (f \circ \Lambda_*^{-1}) \circ q_* = \tilde{f} \circ q_*$$

with $\tilde{f} = \tilde{f} \circ \Lambda_*^{-1} \in K[z]/C\{z\}$. \hfill \Box
1.3.1 \(Z/\Gamma\) as an algebraic variety

Consider, as before, the ideal

\[
\ker(\circ p_*) = \{f \in K[w] : f \circ p_* = 0\} \subset K[w].
\]

**Proposition 24.** The ring \(K[z]^\Gamma = K[w]/\ker(\circ p_*)\) is normal.

**Proof.** The substitution homomorphism

\[
K[z]^\Gamma \xleftarrow{\circ p_*} K[w], \quad f(w) \mapsto f \circ p_*
\]

is surjective. Hence there is a commuting diagram

\[
\begin{array}{ccc}
\ker(\circ p_*) & \rightarrow & K[w] \\
0 & \Downarrow{\circ p_*} & \Downarrow{} \\
K[z]^\Gamma & \rightarrow & K[w]/\ker(\circ p_*)
\end{array}
\]

\[\square\]

Let \(\pi : Z = K^d \rightarrow Z/\Gamma\) be the canonical projection. The map \(p_* := (p_1, \ldots, p_m) : Z \rightarrow W := K^m\) is \(\Gamma\)-invariant and therefore has a factorization

\[
\begin{array}{ccc}
Z & \xrightarrow{p_*} & W \\
\pi & \downarrow & \ \downarrow{p_*} \\
Z/\Gamma & & 
\end{array}
\]

Consider the associated affine algebraic variety

\[
\ker(\circ p_*)^\sim := \{w \in W : \ker(\circ p_*)(w) = 0\}
\]

**Theorem 25.** Suppose that \(K\) is algebraically closed. Then the range

\[
\overline{p}_*(Z/\Gamma) = p_*(Z) = \ker(\circ p_*)^\sim.
\]

**Proof.** Let \(a \in Z\) and \(f \in \ker(\circ p_*)\). Then \(f \circ p_* = 0\) and therefore

\[
f(\overline{p}_*(a\Gamma)) = f(p_*(a)) = (f \circ p_*)(a) = 0.
\]

It follows that \(p_*(Z) \subset \ker(\circ p_*)^\sim\). Conversely, let \(b \in \ker(\circ p_*)^\sim\). Then \(\ker(\circ p_*)(b) = 0\). Hence there is a commuting diagram

\[
\begin{array}{ccc}
\ker(\circ p_*) & \rightarrow & K[w] \\
0 & \Downarrow{\epsilon_b} & \Downarrow{} \\
K & \rightarrow & K[w]/\ker(\circ p_*)
\end{array}
\]
for the evaluation maps $\epsilon$. Hence

$$\ker \bar{\epsilon}_b \triangleleft K[w]/\ker(\circ \rho_*)$$

which implies

$$\mathfrak{m} := \ker(\epsilon_b \circ \rho_*) = (\ker \bar{\epsilon}_b) \circ \rho_* \triangleleft K[z]^\Gamma$$

We claim that $K[z] \mathfrak{m} \triangleleft K[z]$ is a proper ideal. In fact, if $1 = \sum_i u_i a_i$ with $a_i \in K[z]$ and $u_i \in \mathfrak{m}$ then for each $\gamma \in \Gamma$ we have $1 = \gamma \cdot 1 = \sum_i u_i (\gamma \cdot a_i)$ and therefore

$$1 = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_i u_i (\gamma \cdot a_i) = \frac{1}{|\Gamma|} \sum_i u_i \sum_{\gamma \in \Gamma} \gamma \cdot a_i \in \mathfrak{m}$$

since $\sum_{\gamma \in \Gamma} \gamma \cdot a_i \in K[z]^\Gamma$. This contradiction shows $1 \notin K[z] \mathfrak{m}$. By Zorn’s Lemma there exists a maximal ideal

$$K[z] \mathfrak{m} \triangleleft \mathfrak{n} \triangleleft K[z].$$

Then $\mathfrak{m} \subset \mathfrak{n} \cap K[z]^\Gamma \triangleleft \neq K[z]^\Gamma$, since $1 \notin \mathfrak{n} \cap K[z]^\Gamma$. It follows that

$$(\ker \epsilon_b) \circ \rho_* = \mathfrak{m} = \mathfrak{n} \cap K[z]^\Gamma.$$}

Since $K$ is algebraically closed, Hilbert’s Nullstellensatz implies $\mathfrak{n} = \ker \epsilon_a$ for some $a \in Z$. For each $j$ the affine polynomial $\lambda(w) := w_j - b_j$ belongs to $\ker \epsilon_b$, showing that $\lambda \circ \rho_* \in \mathfrak{n} = \ker \epsilon_a$. Therefore $0 = (\lambda \circ \rho_*)(a) = p_j(a) - b_j$ for all $j$ showing that $p_*(a) = b$. \qed

**Proposition 26.** For $K = \mathbb{C}$ the map

$$Z/\Gamma \xrightarrow{\overline{\rho}_*} \ker(\circ \rho_*^\mu) \subseteq \mathbb{C}^m$$

is a homeomorphism for the quotient topology on $Z/\Gamma$ and the relative topology on $\ker(\circ \rho_*^\mu)$, respectively.

**Proof.** For $t > 0$ define homotheties

$$\rho_t(z) := tx, \quad \sigma_t(w_j)_{j=1}^m := (t^{d_j}w_j)_{j=1}^m.$$  

Then the diagram

$$\begin{array}{ccc}
Z/\Gamma & \xrightarrow{\overline{\rho}_*} & \ker(\circ \rho_*^\mu) \\
\rho_t \downarrow & & \sigma_t \downarrow \\
Z/\Gamma & \xrightarrow{\overline{\rho}_*} & \ker(\circ \rho_*^\mu)
\end{array}$$

commutes. Let $C \subset Z$ be a compact 0-neighborhood. Then there exists $r > 0$ such that $\overline{\rho}_*(\pi a) \neq 0$ for $\|a\| = r$ and

$$\{z \in Z : \|z\| \leq r\} \subset C.$$
Suppose there exists a sequence \( w_\ell \in p_*Z \setminus p_*C \) such that \( w_\ell \to 0 \). Then \( w_\ell = p_*z_\ell \) for some \( z_\ell \in Z \setminus C \). Hence \( \|z_\ell\| > r \) and \( r \frac{z_\ell}{\|z_\ell\|} \in C \) has a convergent subsequence

\[
\frac{r}{\|z_\ell\|} z_\ell \rightarrow a \in C
\]

with \( \|a\| = r > 0 \), hence \( \overline{p}_*(\pi a) \neq 0 \). On the other hand,

\[
\overline{p}_*(r \frac{z_\ell}{\|z_\ell\|}) = \overline{p}_*(\rho_r/\|z_\ell\| z_\ell \Gamma) = \sigma_r/\|z_\ell\| \overline{p}_*(z_\ell \Gamma) = \sigma_r/\|z_\ell\| w_\ell \to 0
\]

since \( \frac{r}{\|z_\ell\|} < 1 \). This contradiction shows that \( p_* C \) is a neighborhood of \( 0 \in \ker(o p_*) \).

Since \( p_* \) is bijective and continuous, it follows that \( \pi C \xrightarrow{p_*} p_* C \) is a homeomorphism. Using the homotheties again, we can reach any point in \( Z \), and the assertion follows. \( \square \)

### 1.3.2 \( Z/\Gamma \) as a ringed space

The preceding theorem shows that \( Z/\Gamma \) is isomorphic to a normal affine algebraic variety \( \ker(o p_*) \subset W = K^m \) as a set. We will now show that this isomorphism holds on the level of ringed analytic spaces, if \( K \) is an algebraically closed, non-discrete, complete valuation field, e.g. \( K = \mathbb{C} \). This more difficult part of Cartan’s theorem proceeds by investigating the isotropy subgroups

\[
\Gamma_a := \{ \gamma \in \Gamma : a \cdot \gamma = 0 \}
\]

at all points \( a \in Z \). Since \( \Gamma_a \) is also a finite linear group, the preceding results apply to \( \Gamma_a \) as well. Note that \( \Gamma_0 = \Gamma \).

We make \( Z/\Gamma \) into a ringed topological space. For an open subset \( V \subset Z/\Gamma \) define

\[
O_{Z/\Gamma}^V := \{ f : V \to \mathbb{C} : f \circ \pi : \pi^{-1}(V) \to \mathbb{C} \text{ holomorphic} \}
\]

This yields a presheaf on \( Z/\Gamma \). The local ring \( O_{Z/\Gamma}^{\pi_a} \) consists of all germs

\[
\tilde{f}(\pi z) := \sum_{\gamma \in \Gamma_a \setminus \Gamma} f(z\gamma - a),
\]

where \( f \in \mathbb{C}\{z\}^{\Gamma_a} \), since

\[
(\tilde{f} \circ \pi)(z) = \tilde{f}(\pi z)
\]

is holomorphic near \( a \). This yields

\[
O_{Z/\Gamma}^{\pi_a} = \{ \tilde{f} : f \in \mathbb{C}\{z\}^{\Gamma_a} \}.
\]

The (affine) algebraic variety \( Y = \ker(o p_*) \) has the regular functions

\[
K[Y] = K[w]/\ker(o p_*).
\]
At any point $b \in Y$ we may form the localization

$$K[Y]_b := \{ \frac{\phi}{\psi} : \phi, \psi \in K[Y], \psi(b) \neq 0 \}$$

These local rings form a coherent sheaf over $Y$. Thus the algebraic variety $Y = \ker(\circ p_*) ^\perp \subset W = \mathbb{C}^m$ is a ringed space with local rings

$$\mathcal{O}^{\ker(\circ p_*)}_b := \{ (t_b \phi)|_Y : \phi \in \mathbb{C}\{w\} \},$$

where we define

$$t_b \phi(w) := \phi(w - b)$$

For each $a \in Z$ define the ring homomorphism $\Lambda_a$ by the commuting diagram

$$\begin{array}{c c c c c c c c}
\mathbb{C}\{z\}^{\Gamma_a} & \xrightarrow{\Lambda_a} & \mathbb{C}\{w\} = \mathcal{O}^W_0 \\
\uparrow t_a & \approx & \uparrow t_{p_* a} \\
\mathcal{O}^{\Gamma_a}_a & \xleftarrow{\circ p_*} & \mathcal{O}^W_{p_* a}
\end{array}$$

where $t$ denotes the translation actions. Similarly, for formal power series.

**Theorem 27.** For any $a \in Z$ the homomorphism $\Lambda_a$ is surjective: If $f \in K[z]^{\Gamma_a} / \mathbb{C}\{z\}^{\Gamma_a}$, there exists $\tilde{f} \in K[w] / \mathbb{C}\{w\}$ such that

$$f(z - a) = \tilde{f}(p_* z - p_* a)$$

In other words, we have

$$t_a f = (t_{p_* a} \tilde{f}) \circ p_*$$

**Proof.** Applying the ‘polynomial’ finite generation theorem to $\Gamma_a$ it follows that

$$K[z]^{\Gamma_a} = K[r_*]$$

for a finite reduced set of homogeneous polynomials $r_*(z) = (r_1(z), \ldots, r_m(z))$. By Lemma (?), there exist invariant polynomials $t_a s_k \in K[z]^{\Gamma}$ with

$$\circ (s_k - r_k) = \circ_a (t_a s_k - t_a r_k) > \deg r_k.$$ 

Then $s_k \in K[z]^{\Gamma_a} \subset K[z]^{\Gamma_a}$. It follows from Proposition (??) applied to $\Gamma_a$ that

$$r_k = h_k \circ s_*$$

for some power series $h_k \in K[z] / \mathbb{C}\{z\}$. Since $t_a s_k \in K[z]^{\Gamma}$ we can write

$$s_k(z - a) = (t_a s_k)(z) = g_k(p_* z - p_* a)$$

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for polynomials $g_k \in K[w]$. Note that $p_*z - p_*a$ (unlike $p_*(z - a)$) is still a set of (inhomogeneous) $\Gamma$-invariant generators. Then

$$g_k(0) = g_k(r_*a - r_*a) = s_k(0) = 0.$$  

Thus we may form the formal power series $\tilde{r}_k = h_k \circ g_*$ and obtain

$$r_k(z - a) = h_k(s_*(z - a)) = h_k(g_*(p_*z - p_*a)) = (h_k \circ g_*)(p_*z - p_*a) = \tilde{r}_k(p_*z - p_*a).$$

This proves the assertion for the generators $r_k$. Since $K[z]^{\Gamma_a} = K[r_*/ \mathbb{C}\{z\}^{\Gamma_a} = \mathbb{C}\{r_*\}$ by the 'power series' finite generation theorem applied to $\Gamma_a$, this suffices for the assertion in general.

**Corollary 28.** For each $a \in Z$ there is a ring isomorphism

$$\mathcal{O}^{\ker (op_*)}_{p_*a} \approx \mathcal{O}^\Gamma_{p_*a}.$$

**Proof.** Every germ $\psi \in \mathcal{O}^\ker (op_*)^k$ is of the form

$$\psi(w) = g_{p_*a}(w) := g(w - p_*a)$$

where $g \in \mathbb{C}\{w\}$. Then the convergent power series

$$f_a(z) := g(p_*z + p_*a) \in \mathbb{C}\{z\}^{\Gamma_a}$$

satisfies

$$\psi \circ p_*(z) = g_{p_*a}(p_*z) = g(p_*z - p_*a) = f_a(z)$$

**Proposition 29.** At any point $b = p_*a \in Y$ the power series completion

$$K[Y]_b := \widehat{K[Y]_b} \approx K[z]^{\Gamma_a}$$

is normal.

**Proof.** The formal power series ring $K[z]$ is normal. By Lemma, its subring $K[z]^{\Gamma_a}$ is also normal.

**Lemma 30.** Let $A \subset Z$ be a finite set of $\Gamma$-inequivalent elements. For each $a \in A$ let $\phi_a \in K[z]^{\Gamma_a}$ satisfy $0_a(\phi_a) > r$. Then there exists $\psi \in K[z]^{\Gamma}$ such that $0_a(\psi - \phi_a) > r$ for all $a \in A$.

**Proof.** For each $b \in A$, the finite set $A \cdot \Gamma \setminus b$ is $\Gamma_b$-invariant. There exists a polynomial $p_b$ such that

$$\varrho_b(p_b - 1) > r, \varrho_{A \cdot \Gamma \setminus b}(p_b) > r.$$
The polynomial
\[ q_b := \prod_{\gamma \in \Gamma} \gamma \cdot p_b \in K[z]^{\Gamma_b} \]
has the same vanishing properties, since
\[ p_1 \cdots p_n - 1 = (p_1 - 1)p_2 \cdots p_n + (p_2 - 1)p_3 \cdots p_n + \cdots + p_n - 1 \]
Define
\[ \psi_b := \sum_{\Gamma_b \setminus \Gamma} \gamma \cdot (\phi_b q_b) = \phi_b q_b + \sum_{b \gamma \neq b} \gamma \cdot (\phi_b q_b) \in K[z]^\Gamma. \]
Then
\[ \sum_b \psi_b - \phi_a = \psi_a - \phi_a + \sum_{b \neq a} \psi_b = \phi_a (q_a - 1) + \sum_{a \gamma \neq a} \gamma \cdot (\phi_a q_a) + \sum_{b \neq a} \sum_{\Gamma_b \setminus \Gamma} \gamma \cdot (\phi_b q_b). \]
For the first term we have
\[ o_a(\phi_a (q_a - 1)) \geq o_a(q_a - 1) > r. \]
For the second term, if \( a \gamma \neq a \) then \( a \gamma \in A \cdot \Gamma \setminus a \) and therefore
\[ o_a(\gamma \cdot (\phi_a q_a)) = o_{a, \gamma}(\phi_a q_a) \geq o_{a, \gamma}(q_a) \geq \min_{A \cdot \Gamma \setminus a} o_{A, \Gamma \setminus a}(q_a) > r. \]
For the third term, if \( b \neq a \) we have \( a \gamma \in A \cdot \Gamma \setminus b \) since \( a, b \) are \( \Gamma \)-inequivalent. Therefore
\[ o_a(\gamma \cdot (\phi_b q_b)) = o_{a, \gamma}(\phi_b q_b) \geq o_{a, \gamma}(q_b) \geq \min_{A \cdot \Gamma \setminus b} o_{A, \Gamma \setminus b}(q_b) > r. \]
In summary, \( o_a(\sum_{b \in A} \psi_b - \phi_a) > r. \)