The Berger-Shaw theorem for a pair of commuting operators

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Bangalore

International Conference on Analysis, Algebra, Combinatorics and their applications

January 20, 2020
An operator $T$ on a Hilbert space $\mathcal{H}$ is said to be hyponormal if the commutator $[T^*, T] := T^*T - TT^*$ is positive.

The Berger-Shaw theorem says that if $T$ is an $m$-cyclic hyponormal operator, then the commutator $[T^*, T]$ is trace class and

$$\text{tr}[T^*, T] \leq \frac{m}{\pi} A(\sigma(T))$$

There has been some attempt to show that if a commuting $n$-tuple of bounded linear operators $T$ is hyponormal and cyclic, then the cross commutators must be trace class. The first of these is due to Athavale and the other is due to Douglas and Yan.

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\[
\{p(T_1, \ldots, T_n)\zeta : \zeta \in \zeta\{m\}, p \in \mathbb{C}[z]\}.
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is all of \( \mathcal{H} \).

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[[T^*, T]] := \left(\left[[T_j^*, T_i]\right]\right)_{i,j=1}^n : \bigoplus_n \mathcal{H} \rightarrow \bigoplus_n \mathcal{H}
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is positive, that is, for each \( x \in \bigoplus_n \mathcal{H} \), \( \langle [[T^*, T]] x, x \rangle \geq 0 \), and it is said to be weakly hyponormal if for each vector \( (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \), the sum \( \sum_{i=1}^n \alpha_i T_i \) is a hyponormal operator on \( \mathcal{H} \).
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Question: If the \( n \)-tuple \( T \) is strongly hyponormal and cyclic, then does it follow that the commutators \([T_j^*, T_i]\), \(1 \leq i, j \leq n\) is necessarily trace class?

It is easy to verify that the answer is "no", in general. Take for instance, the example of the Hardy space \( H^2(\mathbb{D}^2) \) and the pair of operators to be the multiplication by the coordinate functions \((M_1, M_2)\). Here the operators \(M_j^*M_i - M_iM_j^* = 0\), \(j \neq i\). However, the commutators \(M_j^*M_j - M_jM_j^*\) are of infinite multiplicity and they are not even compact.

What might be a possible generalization of the Berger-Shaw theorem in the case of commuting tuples of operators?

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Instead of asking for the trace of the commutators to be finite, we only ask that the trace of a “certain” determinant (or, in the language of Helton and Howe, the generalized commutator) is finite.

One may argue that it is not asking for much. But then to arrive at this conclusion, we don’t assume much either.

As in the Berger-Shaw theorem, we assume finite multiplicity but instead of either strong or weak hyponormality, we only assume that the determinant is positive. In many ways, it is a mild condition and this gives us the finiteness of the trace, what is more, we can even get an explicit bound.
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what is the determinant

Let \( B := (B_{ij}) \) be an \( n \times n \) block matrix with entries from \( \mathcal{L}(\mathcal{H}) \). The determinant of \( B \) is the operator

\[
\text{Det}(B) := \sum_{\sigma, \tau} \text{sgn}(\sigma) B_{\tau(1), \sigma(\tau(1))} B_{\tau(2), \sigma(\tau(2))} \cdots B_{\tau(n), \sigma(\tau(n))}.
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The map \( \text{Det} : \mathcal{L}(\mathcal{H})^n \times \ldots \times \mathcal{L}(\mathcal{H})^n \mapsto \mathcal{L}(\mathcal{H}) \) is clearly an alternating multi-linear map.

Let \( T = (T_1, T_2, \ldots, T_n) \) be a \( n \) -tuple of commuting operators. Let us say that the determinant of the \( n \) -tuple \( T \) is the operator \( \text{Det}([T^*, T]) \). For operators of the form \([T^*, T]\), Helton and Howe define the generalized commutator of \( T = (T_1, T_2, \ldots, T_n) \): Let \( A_1 = T_1^*, A_2 = T_1, \ldots, A_{2n-1} = T_n^*, A_{2n} = T_n \). The generalized commutator of the \( n \) -tuple \( T \) is the operator

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Thanks to Cherian Varughese, we see that $\text{Det}(T)$ and $\text{GC}(T)$ are equal, which is perhaps implicit in the paper of Helton and Howe.

Recall the example of the pair of multiplication operators on the Hardy space, $H^2(\mathbb{D}^2)$. In this case,

$$\begin{bmatrix} M^* & M \end{bmatrix} = \begin{bmatrix} (M_z \otimes I)^* & (M_z \otimes I) \\ (I \otimes M_z)^* & (I \otimes M_z) \end{bmatrix} = \begin{bmatrix} P \otimes I & 0 \\ 0 & I \otimes P \end{bmatrix} \geq 0.$$

It now follows that $\text{Det}(\begin{bmatrix} M^* & M \end{bmatrix}) = 2(P \otimes P)$.

Thus $\text{Det}(\begin{bmatrix} M^* & M \end{bmatrix})$ is positive and trace class. Indeed, $\text{tr}(\text{Det}(\begin{bmatrix} M^* & M \end{bmatrix})) = 2$. 
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[[M^*, M]] = \begin{pmatrix}
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*Det and GC are the same*
Let \( T = (T_1, T_2) \) be a pair of commuting operators on a Hilbert space \( \mathcal{H} \) such that \( T \) is \( m \)-cyclic. Let \( \zeta \{ m \} \) be the minimal set of generating vectors for the pair \( (T_1, T_2) \). Set

\[
\mathcal{H}_N = \vee \{ T_1^{i_1} T_2^{i_2} v \mid v \in \zeta \{ m \} \text{ and } 0 \leq i_1 + i_2 \leq N \}
\]

and let \( P_N \) be the projection onto \( \mathcal{H}_N \).

Clearly, \( P_N \uparrow_{SOT} I \).

A pair of commuting operators \( T \) is said to be in the class \( BS_m(\mathcal{H}) \) if \( T \) is \( m \)-cyclic and for every \( N \in \mathbb{N} \), we have

\[
\left\| P_N (T_1^* T_1 T_2^* - T_2^* T_1 T_1^*) P_N \right\| \leq \frac{1}{N + 1} \left\| T_1 \right\|^2 \left\| T_2 \right\|.
\]  

(1)

and

\[
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Let $\mathbf{T} = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space $\mathcal{H}$ such that $\mathbf{T}$ is $m$-cyclic. Let $\zeta\{m\}$ be the minimal set of generating vectors for the pair $(T_1, T_2)$. Set

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$$\| P_N(T_1^*T_1T_2^* - T_2^*T_1T_1^*)P_N^\perp \| \leq \frac{1}{N+1} \| T_1 \|^2 \| T_2 \|$$

(1)

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\]
Theorem
Let $T = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space $\mathcal{H}$ in the class $BS_m(\mathcal{H})$. If the determinant operator $D\left(\begin{bmatrix} T^* & T \end{bmatrix}\right) (= GC(T))$ is non negative definite then it is in trace-class and

$$trace(D\left(\begin{bmatrix} T^* & T \end{bmatrix}\right)) \leq \frac{2m}{\pi^2} \nu(\sigma(T)),$$

where $\nu$ is the Lebesgue measure and $\sigma(T)$ is the Taylor-joint spectrum of the $n$-tuple $T$. 
**Lemma**

Let \( T = (T_1, T_2) \) be a pair of commuting operators on a Hilbert space \( \mathcal{H} \) such that \( T \) is \( m \)-cyclic. Furthermore assume that \( T \) is in the class \( BS_m(\mathcal{H}) \). If the determinant operator \( D([ [T^*, T] ]) \) is positive then it is in trace-class and

\[
\text{trace}(D([ [T^*, T] ])) \leq 2m \| T_1 \|^2 \| T_2 \|^2.
\]

**Outline of the proof:**

\[
\text{Det}( [ [T^*, T] ] ) = [T_1^* T_1 T_2^*, T_2] - [T_1^* T_2 T_2^*, T_1] + [T_2^* T_2 T_1^*, T_1] - [T_2^* T_1 T_1^*, T_2]
\]

For \( j = 1, 2 \), note that \( P_N T_j P_N^\perp = 0 \), \( \text{rank}(P_N^\perp T_j P_N) \leq (N + 1)m \).
Lemma
Let $T = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space $\mathcal{H}$ such that $T$ is $m$-cyclic. Furthermore assume that $T$ is in the class $BS_m(\mathcal{H})$. If the determinant operator $D([[T^*, T]])$ is positive then it is in trace-class and

$$\text{trace}(D([[T^*, T]])) \leq 2m\|T_1\|^2\|T_2\|^2.$$

Outline of the proof:

$$\text{Det}([[T^*, T]])$$

$$= [T_1^*T_2^*, T_2] - [T_1^*T_2^*, T_1] + [T_2^*T_1^*, T_1] - [T_2^*T_1^*, T_2]$$

For $j = 1, 2$, note that $P_N T_j P_N^\perp = 0$, $\text{rank}(P_N^\perp T_j P_N) \leq (N + 1)m$. 

the proof
Lemma

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$$\text{trace}(D\left(\begin{bmatrix} T^* & T \end{bmatrix}\right)) \leq 2m \|T_1\|^2 \|T_2\|^2.$$ 

Outline of the proof:

\[
\text{Det}\left(\begin{bmatrix} T^* & T \end{bmatrix}\right) = [T_1^*T_1T_2^*, T_2] - [T_1^*T_2T_2^*, T_1] + [T_2^*T_2T_1^*, T_1] - [T_2^*T_1T_1^*, T_2]
\]

For $j = 1, 2$, note that $P_N T_j P_N^\perp = 0$, $\text{rank}(P_N^\perp T_j P_N) \leq (N + 1)m$. 
For $i \neq j$, therefore we have

$$P_N[T_i^* T_i T_j^* , T_j] P_N = P_N(T_j T_i^* T_i T_j^* - T_i^* T_i T_j^* T_j) P_N$$

$$= P_N T_j (P_N + P_N^\perp) T_i^* T_i T_j^* P_N - P_N T_i^* T_i T_j^* (P_N + P_N^\perp) T_j P_N$$

$$= [P_N T_i^* T_i T_j^* P_N , P_N T_j P_N] - P_N T_i^* T_i T_j^* P_N^\perp T_j P_N.$$ 

If $A, B$ are in trace-class, then $\text{trace}(AB) = \text{trace}(BA)$ and it follows that $\text{trace}([P_N T_i^* T_i T_j^* P_N , P_N T_j P_N]) = 0$. Hence

$$\text{trace}(P_N D([T^*, T]) P_N) = \text{trace}(P_N (T_2^* T_1 T_1^* - T_1^* T_1 T_2^*) P_N^\perp T_2 P_N) + \text{trace}(P_N (T_1^* T_2 T_2^* - T_2^* T_2 T_1^*) P_N^\perp T_1 P_N)$$
proof contd.

For \( i \neq j \), therefore we have

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P_N [T_i^* T_i T_j^* , T_j] P_N = P_N (T_j T_i^* T_i T_j^* - T_i^* T_i T_j^* T_j) P_N
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\]

\[
= [P_N T_i^* T_i T_j^* P_N, P_N T_j P_N] - P_N T_i^* T_i T_j^* P_N^\perp T_j P_N.
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\[
\text{trace}(P_N D([[[T^*, T]]] ) P_N) = \text{trace}(P_N (T_2^* T_1 T_1^* - T_1^* T_1 T_2^*) P_N^\perp T_2 P_N) + \\
\text{trace}(P_N (T_1^* T_2 T_2^* - T_2^* T_2 T_1^*) P_N^\perp T_1 P_N)
\]
Continuing, we have

\[
\left| \text{trace}(P_N D(\begin{bmatrix} T^* & T \end{bmatrix}) P_N) \right| \leq \left\| P_N(T_1^* T_1 T_2^* - T_2^* T_1 T_1^*) P_N \right\| \left\| P_N^\perp T_2 P_N \right\|_1 + \left\| P_N(T_2^* T_2 T_1^* - T_1^* T_2 T_2^*) P_N \right\| \left\| P_N^\perp T_1 P_N \right\|_1,
\]

where \( \| \cdot \|_1 \) is the trace norm.

Since \( \text{rank}(P_N^\perp T_j P_N) \leq (N + 1)m \), \( \left\| P_N^\perp T_j P_N \right\|_1 \leq (N + 1)m \| T_j \| \) and by definition \( \left\| P_N(T_i^* T_i T_j^* - T_j^* T_i T_i^*) P_N \right\| \leq \frac{1}{N+1} \| T_i \|^2 \| T_j \| \), we have

\[
\left| \text{trace}(P_N D(\begin{bmatrix} T^* & T \end{bmatrix}) P_N) \right| \leq 2m \| T_1 \|^2 \| T_2 \|^2.
\]

But \( D(\begin{bmatrix} T^* & T \end{bmatrix}) \) is non negative definite, therefore taking \( P_N \uparrow I \), we get

\[
\text{trace}(D(\begin{bmatrix} T^* & T \end{bmatrix})) \leq 2m \| T_1 \|^2 \| T_2 \|^2.
\]
Continuing, we have

$$\left|\text{trace}(P_N D(\left[\left[ T^*, T \right]\right]) P_N)\right| \leq \left\| P_N (T_1^* T_2^* - T_2^* T_1^*) P_N^\perp \right\|_1 \left\| P_N^\perp T_2 P_N \right\|_1$$

$$+ \left\| P_N (T_2^* T_1^* - T_1^* T_2^*) P_N^\perp \right\|_1 \left\| P_N^\perp T_1 P_N \right\|_1,$$

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Since $\text{rank}(P_N^\perp T_j P_N) \leq (N + 1)m$, $\left\| P_N^\perp T_j P_N \right\|_1 \leq (N + 1)m \| T_j \|$ and by definition $\left\| P_N (T_i^* T_j^* - T_j^* T_i^*) P_N^\perp \right\| \leq \frac{1}{N+1} \| T_i \|^2 \| T_j \|$, we have

$$\left|\text{trace}(P_N D(\left[\left[ T^*, T \right]\right]) P_N)\right| \leq 2m \| T_1 \|^2 \| T_2 \|^2.$$

But $D(\left[\left[ T^*, T \right]\right])$ is non negative definite, therefore taking $P_N \uparrow I$, we get

$$\text{trace}(D(\left[\left[ T^*, T \right]\right])) \leq 2m \| T_1 \|^2 \| T_2 \|^2.$$
Continuing, we have

\[ |\text{trace}(P_N D([[T^*, T]]) P_N)| \leq \|P_N (T_1^* T_2^* - T_2^* T_1^*) P_N\| \|P_N T_2 P_N\|_1 \]

\[ + \|P_N (T_2^* T_2^* T_1^* - T_1^* T_2 T_2^*) P_N\| \|P_N T_1 P_N\|_1, \]

where \(\| \cdot \|_1\) is the trace norm.

Since \(\text{rank}(P_N T_j P_N) \leq (N + 1)m\), \(\|P_N T_j P_N\|_1 \leq (N + 1)m\|T_j\|\) and by definition \(\|P_N (T_i^* T_j^* - T_j^* T_i^*) P_N\| \leq \frac{1}{N+1} \|T_i\|^2 \|T_j\|\), we have

\[ |\text{trace}(P_N D([[T^*, T]]) P_N)| \leq 2m \|T_1\|^2 \|T_2\|^2. \]

But \(D([[T^*, T]])\) is non negative definite, therefore taking \(P_N \uparrow I\), we get

\[ \text{trace}(D([[T^*, T]])) \leq 2m \|T_1\|^2 \|T_2\|^2. \]
For the second half of the theorem, we need two preparatory lemmas. The first one says that if $T_i$ is $m_i$-multicyclic, $i = 1, 2$, and $\sigma(T_1) \cap \sigma(T_2)$ is empty, then $T_1 \oplus T_2$ is $m$-multi-cyclic, where $m = \max\{m_1, m_2\}$.

The other one is essentially the Vitali covering lemma.

A Vitali covering of a finite measure space $(E, m)$ is a collection of closed balls $B$ such that for each $x \in E$ and any $\varepsilon > 0$, there is a $B \in B$ with the property: $x \in B$ and $m(B) < \varepsilon$.

The Vitali covering Lemma says that if $(E, m)$ is a finite measure space and $B$ is a “Vitali covering” of $E$, then given any $\delta > 0$, we can find finitely many disjoint balls $B_1, \ldots, B_N$ in $B$ such that

$$\sum_{i=1}^{N} m(B_i) \geq m(E) - \delta.$$
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$$\sum_{i=1}^{N} m(B_i) \geq m(E) - \delta.$$
Let \( R_i = \| T_i \| \), \( i = 1, 2 \), and put \( D_{12} = D_{\| T_1 \|} \times D_{\| T_2 \|} \).

Let \( \varepsilon > 0 \), by Vitali covering lemma, there exist \( B_1, \ldots, B_n \) pairwise disjoint balls in \( D_{12} \setminus \sigma(T) \) such that
\[
\nu(D_{12}) < \nu(\sigma(T)) + \sum_j \nu(B_j) + \varepsilon.
\]

If \( B_j = \mathbb{B}(a_j; r_j) \), where \( a_j \in \mathbb{C}^2 \) the above inequality gives
\[
\pi^2 \| T_1 \|^2 \| T_2 \|^2 - \frac{\pi^2}{2} \sum_j r_j^4 < \nu(\sigma(T)) + \varepsilon.
\]
Let $\mathbf{S}$ be the pair of shift operators on the Hardy space over ball $H^2(\mathbb{B}^2)$.

Define $L_j(Z) = (a_j + r_jZ)$.

Let $S_j$ be the $m$-fold direct-sum copy of the operator $L_j(\mathbf{S})$ on the Hilbert space $H_j = \bigoplus_{j=1}^m H^2(B_j)$.

The pair $A = T \oplus \bigoplus_{j=1}^n S_j$ is $m$-cyclic on the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \bigoplus_{j=1}^n H_j$ since the spectrum of the summands are pairwise disjoint and each $S_j$ is $m$-cyclic.

Clearly, $D\left(\left[[A^*,A]\right]\right) = D\left(\left[[T^*,T]\right]\right) \oplus \bigoplus D\left(\left[[S_j^*,S_j]\right]\right)$ is nonnegative definite and $\|A_i\| = \|T_i\|$, $i = 1, 2$. For $i \neq j$, we have

$$\left\|\tilde{P}_N(A_i^*A_iA_j^* - A_j^*A_iA_i^*)\tilde{P}_N^1\right\| \leq \frac{1}{N+1} \|A_i\|^2 \|A_j\|.$$
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Clearly, $D([[A^*, A]]) = D([[T^*, T]]) \oplus \bigoplus D([[S_j^*, S_j]])$ is non-negative definite and $\|A_i\| = \|T_i\|$, $i = 1, 2$. For $i \neq j$, we have

$$\|\tilde{P}_N(A_i^*A_jA_j^* - A_j^*A_iA_i^*)\tilde{P}_N^\perp\| \leq \frac{1}{N+1} \|A_i\|^2 \|A_j\|.$$
completing the proof

Thus all the hypothesis made for the pair $T$ also holds good for the pair $A$. Hence

$$\text{trace}(D([[A^*, A]])) \leq 2m\|A_1\|^2\|A_2\|^2.$$ 

Now, it follows that

$$\text{trace}(D([[T^*, T]])) = \text{trace}(D([[A^*, A]])) - \sum_j \text{trace}(D([[S_j^*, S_j]]))$$

$$\leq 2m\|A_1\|^2\|A_2\|^2 - m\sum_j r_j^A$$

$$= \frac{2m}{\pi^2} \left( \pi^2\|T_1\|^2\|T_2\|^2 - \sum_j \frac{\pi^2}{2} r_j^A \right)$$

$$\leq \frac{2m}{\pi^2} \left( \nu(\sigma(T)) + \varepsilon \right).$$

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Now, it follows that

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More examples

Let \( \{e_{k,l}\} \) be an orthonormal basis \( \ell^2(\mathbb{N} \times \mathbb{N}) \) and \( T = (T_1, T_2) \) be a pair of joint weighted shifts:

\[
T_1(e_{k,l}) = w^1_{k,l} e_{k+1,l}, \quad \text{where} \quad w^1_{k,l} = \delta_k \sqrt{\frac{k-l+1}{k+2}}
\]

\[
T_2(e_{k,l}) = w^2_{k,l} e_{k+1,l+1}, \quad \text{where} \quad w^2_{k,l} = \delta_k \sqrt{\frac{l+1}{k+2}}.
\]

A simple computation gives:

\[
D([[[T^*, T]]) e_{k,l} = \left( \frac{\delta^4_k}{k+2} - \frac{k\delta^4_{k-1}}{(k+1)^2} \right) e_{k,l}.
\]

It is then easy to verify that

\[
\text{trace}(D([[[T^*, T]]]) \leq r(T_\delta)^4,
\]

where \( r(T_\delta) \) denotes the spectral radius of the operator \( T_\delta \). This is in conformity with our inequality.
More examples

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Thank You!