

Role of the curvature in Operator theory

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the Cowen-Douglas class

A class of operators which was introduced by Cowen and Douglas in the late seventies consists of those bounded commuting d -tuples of operators $\mathbf{T} = (T_1, \dots, T_d)$ on a complex separable Hilbert space \mathcal{H} which

- possess an open set $\Omega \subset \mathbb{C}^d$ of joint eigenvalues of constant multiplicity, say n , and
- admit a holomorphic choice of eigenvectors: $s_1(w), \dots, s_n(w)$, $w \in \Omega$, that is,

$$T_i s_j(w) = w_i s_j(w), \quad w \in \Omega, \quad 1 \leq i \leq d, \quad 1 \leq j \leq n.$$

Adjoint of the multiplication by the co-ordinate functions on a Hilbert space of holomorphic functions possessing a reproducing kernel are typical examples of operators in the Cowen-Douglas class.





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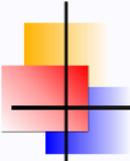
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the Cowen-Douglas theorem

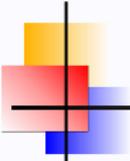
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- There is a one to one correspondence between the unitary equivalence class of the operators T and the equivalence classes of the holomorphic Hermitian vector bundles E_T determined by them.
- Furthermore, they find a set of complete invariants, not very tractable unless $n = 1$, for this equivalence. For $n = 1$, as is well-known, the curvature

$$K(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \|s(w)\|^2 dw \wedge d\bar{w}$$

of the line bundle L_T is a complete invariant of L_T , or equivalently, that of the operator T .





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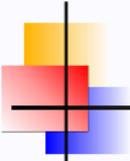


proof that the curvature is a complete invariant

Pick a holomorphic frame $s_i(w)$ for the line bundle E_i and let $\Gamma_i(w) = \langle s_i(w), s_i(w) \rangle$ be the Hermitian metric, $i = 1, 2$. Suppose that the two curvatures K_E and K_F are equal on some open (simply connected) subset $\Omega_0 \subseteq \Omega$. It then follows that $u = \log(\Gamma_1/\Gamma_2)$ is harmonic ensuring the existence of a harmonic conjugate v of u on Ω_0 . Define $\tilde{s}_2(w) = e^{(u(w)+iv(w))/2}s_2(w)$. Then clearly, $\tilde{s}_2(w)$ is a new holomorphic frame for F . Consequently, we have

$$\begin{aligned}\tilde{\Gamma}_2(w) &= \langle \tilde{s}_2(w), \tilde{s}_2(w) \rangle \\ &= \langle e^{(u(w)+iv(w))/2}s_2(w), e^{(u(w)+iv(w))/2}s_2(w) \rangle \\ &= e^{u(w)} \langle s_2(w), s_2(w) \rangle \\ &= \Gamma_1(w).\end{aligned}$$





kernel function

- The **kernel function** K is a complex valued function defined on $\Omega^* \times \Omega^*$ which is holomorphic in the first variable and anti-holomorphic in the second. Therefore, the map $w \rightarrow K(\cdot, w), w \in \Omega^*$, is holomorphic on $\Omega^* := \{\bar{w} : w \in \Omega\}$.
- It is Hermitian, $K(z, w) = \overline{K(w, z)}$, and positive definite, that is, $((K(w^i, w^j)))_{i,j=1}^n$ is positive definite for every subset $\{w^1, \dots, w^n\}$ of Ω^* , $n \in \mathbb{N}$.
- The kernel K reproduces the value of functions in \mathcal{H} , that is, for any fixed $w \in \Omega^*$, the holomorphic function $K(\cdot, w)$ belongs to \mathcal{H} and

$$f(w) = \langle f, K(\cdot, w) \rangle, f \in \mathcal{H}, w \in \Omega^*.$$

- The reproducing property of K ensures that $M_i^* K(\cdot, w) = \bar{w}_i K(\cdot, w)$. Therefore, we have a natural holomorphic frame $\gamma(w) := K(\cdot, w)$ on Ω^* for the commuting tuple M_1^*, \dots, M_d^* .





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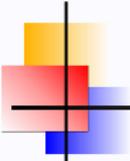
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An alternative description of the Cowen-Douglas class

A Hilbert module over the polynomial ring $\mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_m]$ is a Hilbert space \mathcal{H} which is a $\mathbb{C}[z]$ -module if for some $C_p > 0$,

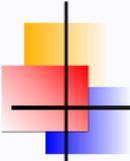
$$\|p \cdot f\| \leq C_p \|f\|, f \in \mathcal{H}, p \in \mathbb{C}[z].$$

The multiplication M_j by the coordinate functions z_j , $M_j f := z_j \cdot f$, $1 \leq j \leq m$, then defines a commutative tuple $\mathbf{M} = (M_1, \dots, M_m)$ of linear bounded operators acting on \mathcal{H} and vice-versa.

A Hilbert module \mathcal{H} over the polynomial ring $\mathbb{C}[z]$ is said to be in the Cowen-Douglas class $B_n(\Omega)$, $n \in \mathbb{N}$, if

- $\dim \mathcal{H} / \mathfrak{m}_w \mathcal{H} = n < \infty$ for all $w \in \Omega$, where \mathfrak{m}_w is the maximal ideal in $\mathbb{C}[z]$ at w and
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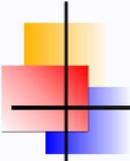
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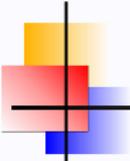
tensor product

Let \mathcal{M}_1 and \mathcal{M}_2 be Hilbert spaces of holomorphic functions on Ω so that they possess reproducing kernels K_1 and K_2 , respectively. Assume that the natural action of $\mathbb{C}[z]$ on the Hilbert space \mathcal{M}_1 is continuous, that is, the map $(p, h) \rightarrow ph$ defines a bounded operator on M_p for $p \in \mathbb{C}[z]$. (We make no such assumption about the Hilbert space \mathcal{M}_2 .) Now, $\mathbb{C}[z]$ acts naturally on the Hilbert space tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ via the map

$$(p, (h \otimes k)) \rightarrow ph \otimes k, p \in \mathbb{C}[z], h \in \mathcal{M}_1, k \in \mathcal{M}_2.$$

The map $h \otimes k \rightarrow hk$ identifies the Hilbert space $\mathcal{M}_1 \otimes \mathcal{M}_2$ as a reproducing kernel Hilbert space of holomorphic functions on $\Omega \times \Omega$. The module action is then the point-wise multiplication $(p, hk) \rightarrow (ph)k$, where $((ph)k)(z_1, z_2) = p(z_1)h(z_1)k(z_2)$, $z_1, z_2 \in \Omega$.





a new kernel

Let \mathcal{H} be the Hilbert module $\mathcal{M}_1 \otimes \mathcal{M}_2$ over $\mathbb{C}[z]$. Let $\Delta \subseteq \Omega \times \Omega$ be the diagonal subset $\{(z, z) : z \in \Omega\}$ of $\Omega \times \Omega$. Let \mathcal{I} be the maximal submodule of functions in $\mathcal{M}_1 \otimes \mathcal{M}_2$ which vanish on Δ . Thus

$$0 \rightarrow \mathcal{I} \xrightarrow{X} \mathcal{M}_1 \otimes \mathcal{M}_2 \xrightarrow{Y} \mathcal{Q} \rightarrow 0$$

is a short exact sequence, where $\mathcal{Q} = (\mathcal{M}_1 \otimes \mathcal{M}_2) / \mathcal{I}$, X is the inclusion map and Y is the natural quotient map. One can appeal to an extension of an earlier result of Aronszajn to analyze the quotient module \mathcal{Q} when the given modules are reproducing kernel Hilbert spaces. The reproducing kernel of \mathcal{H} is then the pointwise product $K_1(z, w)K_2(u, v)$ for z, w, u, v in Ω . Set $\mathcal{H}_{\text{res}} = \{f|_{\Delta} : f \in \mathcal{H}\}$ and $\|f\|_{\Delta} = \inf\{\|g\| : g \in \mathcal{H}, g|_{\Delta} \equiv f|_{\Delta}\}$.

- The quotient module is isomorphic to the module \mathcal{H}_{res} whose reproducing kernel is the pointwise product $K_1(z, w)K_2(z, w)$, $z, w \in \Omega$.



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another kernel!

Suppose $\Omega \subseteq \mathbb{C}^d$ is open connected and bounded. Let $K : \Omega \times \Omega$ be a non-negative definite kernel. Then \tilde{K} defined by

$$\tilde{K}(z, w) = ((K^2 \partial_i \bar{\partial}_j \log K(z, w)))_{1 \leq i, j \leq d}$$

is a $\mathbb{C}^{d \times d}$ valued non-negative definite kernel.

- We point out that $\sum_{i,j} \partial_i \bar{\partial}_j \log K(w, w) dw_i \wedge d\bar{w}_j$ is the curvature of the metric $K(w, w)$.

To see that \tilde{K} defines a positive definite kernel on Ω , set

$$\phi_i(w) := K_w \otimes \bar{\partial}_i K_w - \bar{\partial}_i K_w \otimes K_w, 1 \leq i \leq m$$

and note that each $\phi : \Omega \rightarrow \mathcal{H}$ is holomorphic. A simple calculation then shows that

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what is the Hilbert module?

How to describe the Hilbert space, or more importantly, the Hilbert module $\mathcal{H}(\tilde{K})$? May be, it is a quotient of the Hilbert module $\mathcal{H} \otimes \mathcal{H}$? If so, How do we identify the corresponding submodule?

Let \mathcal{H}_0 be the subspace of $\mathcal{H}(K) \otimes \mathcal{H}(K)$ given by $\overline{\bigvee \{ \phi_i(w) : w \in \Omega, 1 \leq i \leq m \}}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

Let \mathcal{H}_1 and \mathcal{H}_2 be the submodules defined by

$$\mathcal{H}_1 = \{ f \in \mathcal{H}(K) \otimes \mathcal{H}(K) : f|_{\Delta} = 0 \}$$

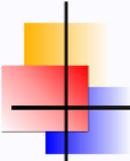
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- $\mathcal{H}_{11} = \mathcal{H}_2^{\perp} \ominus \mathcal{H}_1^{\perp}$





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- $\mathcal{H}_{11} = \mathcal{H}_2^{\perp} \ominus \mathcal{H}_1^{\perp}$





what is the Hilbert module?

How to describe the Hilbert space, or more importantly, the Hilbert module $\mathcal{H}(\tilde{K})$? May be, it is a quotient of the Hilbert module $\mathcal{H} \otimes \mathcal{H}$? If so, How do we identify the corresponding submodule?

Let \mathcal{H}_0 be the subspace of $\mathcal{H}(K) \otimes \mathcal{H}(K)$ given by $\overline{\{\phi_i(w) : w \in \Omega, 1 \leq i \leq m\}}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

Let \mathcal{H}_1 and \mathcal{H}_2 be the submodules defined by

$$\mathcal{H}_1 = \{f \in \mathcal{H}(K) \otimes \mathcal{H}(K) : f|_{\Delta} = 0\}$$

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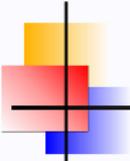
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a limit computation

The point of what we have said so far is that we can explicitly describe the Hilbert modules \mathcal{H}_2^\perp and \mathcal{H}_1^\perp , upto an isomorphism of modules. Using the jet construction followed by the restriction map, one may also describe the direct sum $\mathcal{H}_2^\perp \oplus \mathcal{H}_1^\perp$, again upto an isomorphism.

But what is the module \mathcal{H}_{11} as a submodule of \mathcal{H} ? To answer this question, one must find the kernel function for \mathcal{H}_{11} . Set K_1 to be the kernel function of the module \mathcal{H}_1 . Assuming $d = 1$, we have

$$\left(\frac{K_1(z, u, v, w)}{(z - u)(\bar{w} - \bar{v})} \right) \Big|_{\substack{z=u, z \neq u \\ w=v, w \neq v}} = \frac{1}{2} K(z, w)^2 \partial \bar{\partial} \log K(z, w).$$





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Thank You!

