The Grothendieck inequality

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max Cut

- A cut in a undirected graph $G = (V, E)$ is defined as partition of the vertices of $G$ into two sets; and the weight of a cut is the number of edges that has an end point in each set, that is, the edges that connect vertices of one set to the vertices of the other.
- The max-cut is the problem of finding a cut in $G$ with maximum weight.
- As an example, we note that the bipartite graph has maxcut exactly equal to the number of its edges.
- This is the MAX-2COLORING problem, namely, that of finding the maximum number of edges in a graph $G$ which can be colored by using only two colors.
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Max Cut:

A cut in a graph $G = (V, E)$ is a pair $(S, V \setminus S)$.

The edge set of the cut is the set of all edges:

$$E(S, V \setminus S) = \{ e \in E \mid |e \cap S| = |e \cap V \setminus S| = 1 \}$$
the edge set with labels
the edge set with crossings marked
Cut Norm

maximum, over all $I \subseteq R, J \subseteq S$,

$$\left| \sum_{i \in I, j \in J} a_{ij} \right|$$
Cut Norm

maximum, over all $I \subseteq R, J \subseteq S$, 

$$\left| \sum_{i \in I, j \in J} a_{ij} \right|$$

Claim: The cut norm (of the matrix on the right) is equal to the size of the max cut (of the graph on the left).
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Claim: The cut norm is at least the size of the max cut.
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\[
\left| \sum_{i \in I, j \in J} a_{ij} \right|
\]

We have shown:
The cut norm is at least the size of the max cut.
Cut Norm

maximum, over all $I \subseteq R, J \subseteq S$, 

$$|\sum_{i \in I, j \in J} a_{ij}|$$

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The cut norm is at most the size of the max cut.
the Cut norm

- The cut-norm $\|A\|_C$ of a real matrix $A = \left( (a_{ij}) \right)_{i \in R, j \in S}$ is the maximum, over all $I \subseteq R, J \subseteq S$, of the quantity $|\sum_{i \in I, j \in J} a_{ij}|$.
- It is not difficult to show that the norm $\| \cdot \|_C$ is equivalent to the norm $\|A\|_{\infty \to 1}$, that is, for any $n \times n$ matrix $A$, we have

$$4\|A\|_C \geq \|A\|_{\infty \to 1} \geq \|A\|_C,$$

where

$$\|A\|_{\infty \to 1} := \sup \left\{ \left| \sum_{j,k=1}^{n} a_{jk} s_j t_k \right| : |s_j|, |t_k| = 1, 1 \leq j, k \leq n \right\},$$

$s_j, t_k \in \mathbb{R}$ (resp. in $\mathbb{C}$).
proof

For any $x_i, y_j \in \{-1, 1\}$,

$$\sum_{i,j} a_{i,j} x_i y_j = \sum_{i:x_i=1, j:y_j=1} a_{i,j} - \sum_{i:x_i=1, j:y_j=-1} a_{i,j}$$

$$- \sum_{i:x_i=-1, j:y_j=1} a_{i,j} + \sum_{i:x_i=-1, j:y_j=-1} a_{i,j}.$$ 

The absolute value of each of the four terms in the right hand side is at most $\|A\|_C$, implying, by the triangle inequality, that

$$\|A\|_{\infty \rightarrow 1} \leq 4\|A\|_C.$$
proof (contd.)

Suppose, now, that \( \|A\|_C = \sum_{i \in I, j \in J} a_{i,j} \) (the computation in case it is \( -\sum_{i \in I, j \in J} a_{i,j} \) is essentially the same). Define \( x_i = 1 \) for \( i \in I \) and \( x_i = -1 \) otherwise, and similarly, \( y_j = 1 \) if \( j \in J \) and \( y_j = -1 \) otherwise. Then

\[
\|A\|_C = \sum_{i,j} a_{i,j} \frac{1+x_i}{2} \frac{1+y_j}{2} =
\]

\[
\frac{1}{4} \left( \sum_{i,j} a_{i,j} + \sum_{i,j} a_{i,j} x_i \cdot 1 + \sum_{i,j} a_{i,j} 1 \cdot y_j + \sum_{i,j} a_{i,j} x_i y_j \right).
\]

The absolute value of each of the four terms in the right hand side is at most \( \|A\|_{\infty \to 1}/4 \), implying, by the triangle inequality, that

\[
\|A\|_{\infty \to 1} \geq \|A\|_C.
\]
Finding the norm $\|A\|_{\infty \rightarrow 1}$ is called an integer linear program since

$$\|A\|_{\infty \rightarrow 1} := \sup \left\{ \left| \sum_{j,k=1}^{n} a_{jk}s_j t_k \right| : s_j, t_k \in \{-1,1\}, 1 \leq j, k \leq n \right\},$$

at least in the real case.

Thus one may wish to simply compute the $\|A\|_{\infty \rightarrow 1}$ instead of the CUT norm. However, this is not easy either.

Let us see if we can give ourselves a little more room and compute a norm, namely, the 2-summing norm, related to the cut norm and the norm $\|A\|_{\infty \rightarrow 1}$ that we have already seen.
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The 2 - summing norm $\gamma(A)$ is defined as follows:

$$
\gamma(A) := \sup \left\{ \left| \sum_{j,k=1}^{n} a_{jk} \langle x_j, y_k \rangle \right| : x_j, y_k \in (\ell_2)_1, 1 \leq j, k \leq n \right\}.
$$

Finding $\gamma(A)$, the 2 - summing norm, is called a semi-definite program.

Define the numerical constant, the Grothendieck constant:

$$
K_G(n) \overset{\text{def}}{=} \sup \{ \gamma(A) : A = A_{n \times n}, \|A\|_{\infty \to 1} \leq 1 \}.
$$

The constant $K_G(n)$ depends on the ground field.
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what we know about the Grothendieck constant

- The fact that $K_G(n)$ remains finite, say $K_G$, as $n \to \infty$ was established by Grothendieck and is known as the Grothendieck constant, that is,

$$\sup\{ \frac{\gamma(A)}{\|A\|_{\infty \to 1}} : A \in \mathbb{C}^{n \times n}, n \in \mathbb{N} \} < \infty.$$ 

- The Grothendieck inequality says that the two norms $\|A\|_{\infty \to 1}$ and $\gamma(A)$ can differ only by a constant factor.
- The exact value of $K_G$ is not known. However, $K_G^\mathbb{C}(1) = K_G^\mathbb{C}(2) = 1$ and $K_G^\mathbb{R}(2) = \sqrt{2} = K_G^\mathbb{R}(3)$.
- Although, not entirely trivial, it is known that $K_G > 1$.
- Kirvine’s proof gives $\frac{\pi}{2 \ln(1+\sqrt{2})} = 1.782 \ldots$.
- Krivine conjectured that his bound is actually the exact value of $K_G$. Recently, this conjecture has been shown to be false.
Grothendieck constant for graphs

• Let $G$ be a graph with $n$ vertices denoted by $\{1, \ldots, n\}$ and $E \subseteq \{1, \ldots, n\}^2$ be the set of its edges.
• Following Noga Alon, Assaf Naor and many others, define the Grothendieck constant of the graph $G$, denoted by $K(G)$, to be the smallest constant $K$ such that

$$\sup \left\{ \left| \sum_{\{i,j\} \in E} a_{ij} \langle x_i, y_j \rangle : \|x_i\| = 1 = \|y_j\| \right| \right\} \leq K \sup \left\{ \sum_{\{i,j\} \in E} a_{ij} s_i t_j : |s_i| = 1 = |t_j| \right\}$$

holds true for any real matrix $A = ((a_{ij}))$. 
The original Grothendieck inequality is the particular case that corresponds to the bipartite graphs (i.e. of chromatic number 2) and, as a consequence,

\[ K_G = \sup_{n \in \mathbb{N}} \{ K(G) : G \text{ is a bipartite graph on } n \text{ vertices} \}. \]

Additionally, if \( C_n \) stands for the complete graph with \( n \) vertices, the corresponding Grothendieck constant is of order \( \log(n) \). The Grothendieck constant of a graph \( G \) is clearly related to the combinatorics of \( G \).
On the other hand, the expression on the right hand side of the Grothendieck inequality for graphs is relevant statistical physics: if $G$ weighted by the matrix $A$ represents the possible interaction of $n$ particles affected by a spin $i = \pm 1$, then the total energy generated by these particles in the system in the Ising model of the spin glass is

$$\mathcal{E} = -\left( \sum_{\{i,j\}\in E} a_{ij} \varepsilon_i \varepsilon_j \right).$$

A configuration of the spins $(\varepsilon_i) \in \{-1,1\}^n$ represents its ground state if it minimizes the energy.
Kirvine’s proof of the Grothendieck inequality

Let $S \subseteq \mathbb{C}^k$ be the Euclidean sphere of radius 1.

**Lemma**

$$\sup \left\{ \left| \sum_{i,j=1}^{n} a_{ij} \sin^{-1} \langle u_i, v_j \rangle \right| : \|A\|_{1 \rightarrow \infty} \leq 1; u_i, v_j \in S \right\} \leq \frac{\pi}{2}. $$

**Proof.** Let $\mu$ be the unique probability measure on $S$ which is rotation invariant. First, show that

$$ I := \int_S \text{sign} \langle x, u \rangle \text{sign} \langle y, u \rangle \, d\mu(u) = 1 - \frac{2\psi}{\pi}, \psi = \cos^{-1} \langle x, y \rangle, x, y \in S. $$

- The verification consists of finding an unitary $U : \ell_2(k) \to \ell_2(k)$ with

  $$ Ux = (1, 0, \ldots, 0), \; Uy = (\cos \psi, \sin \psi, 0, \ldots, 0), $$

  where $\psi = \cos^{-1} \langle x, y \rangle$, $0 \leq \psi \leq \pi$ and $\sin^{-1} \langle x, y \rangle = \frac{\pi}{2} - \psi$. 

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Kirvine’s proof

- If $x$ and $y$ are linearly dependent, namely $x = -y$, then $Ux = (1, 0, \ldots, 0)$, $Uy = (-1, 0, \ldots, 0)$ and $\psi = \pi$. Similarly, if $x = y$, then choose $Ux = (1, 0, \ldots, 0)$, $Uy = (1, 0, \ldots, 0)$ and $\psi = 0$. Now, extend this map linearly to all of $\ell_2(k)$ to an unitary.

- If $x$ and $y$ be linearly independent, then applying Gram-Schmidt, obtain a pair of orthonormal vectors $\alpha_1, \alpha_2$ and define a linear map $U$ on the span of these two vectors:

$$U\alpha_1 := (1, 0, \ldots, 0), \ U\alpha_2 := (0, 1, 0, \ldots, 0)$$

and extend it, as before, to an unitary on all of $\ell_2(k)$. 
Kirvine’s proof

• If $x$ and $y$ are linearly dependent, namely $x = -y$, then $Ux = (1, 0, \ldots, 0)$, $Uy = (-1, 0, \ldots, 0)$ and $\psi = \pi$. Similarly, if $x = y$, then choose $Ux = (1, 0, \ldots, 0)$, $Uy = (1, 0, \ldots, 0)$ and $\psi = 0$. Now, extend this map linearly to all of $\ell_2(k)$ to an unitary.

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and extend it, as before, to an unitary on all of $\ell_2(k)$. 
an integral

• A simple calculation gives $U_x = (1, 0, \ldots, 0)$, $U_y = (\cos \psi, \sin \psi, 0, \ldots, 0)$. Therefore, in computing $\langle U_x, U_u \rangle$ and $\langle U_y, U_u \rangle$, we assume without loss of generality: $U_u = (\cos \theta, \sin \theta, 0 \ldots, 0)$.

• The integral $I$ is $U$ invariant, we have

\[
I = \int_S \text{sign} \langle U_x, U_u \rangle \text{sign} \langle U_y, U_u \rangle d\mu(U_u) \\
= \int_S \text{sign} u_1 \text{sign}(\cos \psi u_1 + \sin \psi u_2) d\mu(U_u) \\
= \frac{1}{2\pi} \int_0^{2\pi} \text{sign}(\cos \theta) \text{sign}(\cos(\theta - \psi)) d\theta \\
= 1 - \frac{2\psi}{\pi} \\
= \frac{2}{\pi} \sin^{-1} \langle x, y \rangle.
\]
evaluation of the integral
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \text{sign}(\cos \theta) \text{sign}(\cos(\theta - x)) d\theta \]

Integrand is +1 when \( \theta \) lies in the red regions and -1 when \( \theta \) lies in the green regions.
The hypothesis on $A$ implies that

$$-1 \leq \sum_{i,j=1}^{n} a_{ij} \text{sign}\langle u_i, x \rangle \text{sign}\langle v_j, x \rangle \leq 1,$$

for any choice of vectors $\|u_i\|_2 = 1 = \|v_j\|_2$. The proof is then completed by integrating with respect to $x$.

Lemma

For each positive integer $k$, there is a mapping $w_k : l_2^n \to l_2^N$ such that for all $x, y$, $\langle w_k(x), w_k(y) \rangle = \langle x, y \rangle^k$.

For the proof, set $w_k(x)$ to be the $k$-fold tensor product of the vector $x$. 
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sine hyperbolic

Lemma

Given $c > 0$, there exists $u : \ell_2(n) \rightarrow \ell_2$ and $v : \ell_2(n) \rightarrow \ell_2$ such that

$$\langle u(x), v(y) \rangle = \sin c \langle x, y \rangle,$$

$$\|u(x)\|^2 = \sinh (c \|x\|^2) \quad \text{and} \quad \|v(y)\|^2 = \sinh (c \|y\|^2), \ x, y \in \ell_2(n).$$

Proof. From the Taylor series expansion

$$\sin c \langle x, y \rangle = \sum_{1}^{\infty} (-1)^{k-1} c_k \langle w_{2k-1}(x), w_{2k-1}(y) \rangle,$$

where $c_k = \frac{c^{2k-1}}{(2k-1)!}$, we see that we just have to set

$$u(x) := \sum_{1}^{\infty} \sqrt{c_k} w_{2k-1}(x),$$

$$v(y) := \sum_{1}^{\infty} (-1)^{k-1} \sqrt{c_k} w_{2k-1}(y).$$
completing the proof

- Let $c = \sinh^{-1}(1) = \ln(1 + \sqrt{2})$.
  Set $u_i = u(x_i)$, $v_j = v(y_j)$, $\|x_i\|_2 = 1 = \|y_j\|_2$, and note that $\|u_i\| = 1 = \|v_j\|$.

- However, we know that
  $$c \langle x_i, y_j \rangle = \sin^{-1} \langle u_i, v_j \rangle, \quad |c \langle x_i, y_j \rangle| \leq 1$$
  and
  $$\left| \sum_{i,j=1}^{n} a_{ij} \sin^{-1} \langle u_i, v_j \rangle \right| \leq \frac{\pi}{2}.$$

So
  $$\left| \sum_{i,j=1}^{n} a_{ij} \langle x_i, y_j \rangle \right| \leq \frac{\pi}{2c} = \frac{\pi}{2 \ln(1 + \sqrt{2})}.$$
completing the proof

• Let \( c = \sinh^{-1}(1) = \ln(1 + \sqrt{2}) \).
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\]

So

\[
\left| \sum_{i,j=1}^{n} a_{ij} \langle x_i, y_j \rangle \right| \leq \frac{\pi}{2c} = \frac{\pi}{2 \ln(1 + \sqrt{2})}.
\]
Theorem (Varopoulos inequality)

Suppose $K^C_G$ denote the complex Grothendieck constant. Then

$$K^C_G \leq \sup \| p(T_1, \ldots, T_n) \| \leq 2K^C_G$$

where supremum is over all $n \in \mathbb{N}$, tuples of commuting contractions $T = (T_1, \ldots, T_n)$ and polynomial $p$ of degree 2 with $\|p\|_\infty \leq 1$. 
sharpening the Varopolous inequality

• Thus Grothendieck constant had made an unexpected appearance in the early work of Varopoulos. Setting

\[ C_2(n) = \sup \left\{ \| p(T) \| : \| p \|_{\mathbb{D}^n, \infty} \leq 1, \| T \|_{\infty} \leq 1 \right\}, \]

where the supremum is taken over all complex polynomials \( p \) in \( n \) variables of degree at most 2 and commuting \( n \)-tuples \( T := (T_1, \ldots, T_n) \) of contractions, he shows that

\[ \lim_{n \to \infty} C_2(n) \leq 2K_G^C, \]

where \( K_G^C \) is the complex Grothendieck constant.

• Rajeev Gupta in his PhD thesis shows that

\[ \lim_{n \to \infty} C_2(n) \leq \frac{3\sqrt{3}}{4} K_G^C, \]

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Thank you