Geometric Quantization in Complex and Harmonic Analysis

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These are informal notes, subject to continuous changes and corrections

Contents

Chapter 0

Overview

0.1 d^+-Quantization, $d \geq 0$

border: d-dimensional manifold S, closed (compact) but possibly disconnected (many-particle system)
bordism: $d+1$ manifold Σ, connected but non-closed, with boundary $\partial\Sigma = S$
border symplectic manifold M
border complex manifold: family of Kähler manifolds M_τ
border complex quantization: family of Hilbert spaces $H^2(M_\tau)$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$
classical bordism: flow of symplectomorphisms
quantum bordism: flow of unitary operators

0.2 0^+-Quantization, Quantum Mechanics

border: point $S = \mathbf{S}^0$ or finite number of points
bordism: interval $[0,t]$, 1-manifold with boundary

Example 0.2.1. Q configuration space
border symplectic manifold T^*Q
border complex manifold: family of Kähler manifolds M_τ
border complex quantization: family of Hilbert spaces $H^2(M_\tau)$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $L^2(Q)$
classical bordism: geodesic flow
quantum bordism: time evolution e^{tH}

Example 0.2.2. G compact Lie group, T maximal torus
border symplectic orbit G/T
border complex orbit: family of Kähler manifolds $G^{\mathbf{C}}/G^{\mathbf{C}}_\tau$
border complex quantization: family of highest weight Hilbert spaces $G^{\mathbf{C}}/G^{\mathbf{C}}_\tau$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces
quantum bordism: no time evolution $H = 0$

0.3 1^+-Quantization, Conformal Field Theory

border: circle $S = \mathbf{S}^1$, or disjoint union of circles=compact 1-manifold without boundary
bordism: cylinder $[0,1] \times \mathbf{S}^1$ or connected Riemann surface Σ with boundary

Example 0.3.1. G compact Lie group
border symplectic quotient

$$\mathcal{C}^\infty(\mathbf{S}^1, G)/G$$

border complex quotient: family of Kähler manifolds

$$\mathcal{O}(\mathbf{D}, G^{\mathbf{C}})$$

border complex quantization: positive energy representations of loop group (G. Segal)
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$

Example 0.3.2. 1+1 gravity, $G = SL_2(\mathbf{R})$ non-compact

Example 0.3.3. Restricted Grassmannian, 2d QCD (Rajeev-Turgut)

0.4 2^+-Quantization,Topological Quantum Field Theory

border: non-connected compact oriented surface S without boundary
bordism: connected non-compact 3-manifold Σ with boundary $\partial\Sigma = S$

Example 0.4.1. Chern-Simons theory: G=compact Lie group
border symplectic quotient (compact)

$$H^1(S, G) = Hom(\pi_1(S), G)$$

border complex quotient: family of Kähler manifolds

$$H^1(S_\tau, G^{\mathbf{C}})$$

border complex quantization: family of Hilbert spaces $H^2(M_\tau)$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$

Example 0.4.2. 2+1 gravity=Chern-Simons theory for non-compact Lie group $SL_2(\mathbf{R})$ (Verlinde)

0.5 $3 \leq d \leq 8$, Higher gauge theory and special holonomy

- gauge theory in 4 dimensions, $SU(2)$-holonomy
- Calabi-Yau manifolds in 6 dimensions, $SU(3)$-holonomy
- G_2-manifolds in 7-dimensions
- $Spin(7)$-manifolds in 8 dimensions

Since spacetime is supposed to have dimension ≤ 11 (M-theory) or ≤ 12 (F-theory), Kaluza-Klein compactification to 4-dimensional Minkowski space yields 'border' manifolds of dimension ≤ 8.

Chapter 1

Manifolds, Connexions and Curvature

1.1 Manifolds

Consider smooth manifolds over $\mathbf{R}$ and complex manifolds over $\mathbf{C}$. We use the term $\mathbf{K}$-manifold for $\mathbf{K} = \mathbf{R}, \mathbf{C}$. If not specified otherwise, maps, functions, sections etc. will be smooth for $\mathbf{K} = \mathbf{R}$ or holomorphic for $\mathbf{K} = \mathbf{C}$.

• Jordan manifolds

Jordan manifolds are symmetric manifolds of arbitrary rank, associated with Jordan algebras and Jordan triples. The basic example is projective space (rank 1)

$$\mathbf{P}^s = \{E \subset \mathbf{K}^{1+s} : \dim E = 1\}.$$

Let Z be a $\mathbf{K}$-vector space, endowed with a ternary composition $Z \times Z \times Z \to Z$, denoted by

$$(x, y, z) \mapsto \{x; y; z\},$$

which is bilinear symmetric in (x, z) and anti-linear in the inner variable. Define

$$D(x, y)z := \{x; y; z\}.$$

Then Z is called a **Jordan triple** if the **Jordan triple identity**

$$\Big[D(x, y), D(u, v)\Big] = D(\{x; y; u\}, v) - D(u, \{v; x; y\})$$

holds. Z is called **hermitian** (over $\mathbf{K}$) if the sesqui-linear form

$$(x, y) \mapsto \text{tr } D(x, y)$$

is non-degenerate and hermitian

$$\overline{\text{tr } D(x, y)} = \text{tr } D(y, x).$$

A hermitian Jordan triple is called ^{0}**hermitian**, if the trace form (**??**) is positive definite. If there are q negative eigenvalues, then Z is called ^{q}**hermitian**. We will mostly be concerned with complex ^{0}hermitian Jordan triples.

The basic example is $Z = \mathbf{K}^{r\times s}$ with the ternary composition

$$\{u;v;w\} := uv^*w + wv^*u$$

which makes sense for rectangular matrices. More generally, the full **classification** of irreducible complex ^{0}hermitian Jordan triples is

- **matrix triple** $Z = \mathbf{C}^{r\times s},\ \{x;y;z\} = xy^*z + zy^*x,\ rank = r \le s,\ a = 2$ (complex case), $b = s - r$
- $r = 1,\ Z = \mathbf{C}^{1\times s} = \mathbf{C}^s,\ \{x;y;z\} = (x|y)z + (z|y)x$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- **spin factor** $Z = \mathbf{C}^{a+2},\ \{x;y;z\} = (x\cdot\overline{y})\ z + (z\cdot\overline{y})\ x + (x\cdot z)\overline{y},\ r = 2,\ b = 0$
- **exceptional Jordan triples** of dimension 16 ($r = 2$) and 27 ($r = 3$), $a = 8$ (octonion case)

For $(u,v) \in Z^2 := Z \times Z$ the endomorphism

$$B_{u,v}z := z - \{u;v;z\} + \frac{1}{4}\{u;\{v;z;v\};u\}$$

of Z is called the **Bergman operator**. For matrices it becomes

$$B_{u,v}z = (I_r - uv^*)z(I_s - v^*u)$$

which again makes sense for rectangular matrices.

A pair $(x,y) \in Z^2$ is called **quasi-invertible** if $B_{x,y}$ is invertible. In this case the element

$$x^y := B_{x,y}^{-1}(x - \{x;y;x\})$$

in Z is called the **quasi-inverse**. For rectangular matrices the quasi-inverse is given by

$$x^y := (I_r - xy^*)^{-1}x = x(I_s - y^*x)^{-1}$$

which is again a rectangular matrix.

By [**?**,], we have the **addition formulas**

$$B_{x,y+z} = B_{x,y}\,B_{x^y,z}$$

and

$$x^{y+z} = (x^y)^z.$$

This implies that

$$[x,a] = [y,b] \ \Leftrightarrow\ (x, a-b) \text{ quasi-invertible and } y = x^{a-b}$$

defines an equivalence relation on Z^2. Informally, $[x,a] = [x^{a-b}, b]$. The compact quotient manifold

$$\hat{Z} = Z^2/R = \{[m,a] :\ z, a \in Z\}$$

is a compact symmetric space called the **conformal hull** of Z. Its non-compact dual is the connected 0-component

$$\check{Z} := \{m \in Z :\ B_{z,z} \text{ invertible}\}^0,$$

which is a bounded symmetric domain in its circular and convex Harish-Chandra realization.

Example 1.1.1. For the matrix triple $Z = \mathbf{K}^{r\times s}$, $\hat{Z}$ can be identified with the **Grassmannian**

$$\mathbf{G}_r(\mathbf{K}^{r+s}) = \{E \subset \mathbf{K}^{r+s} : \ \dim E = r\}.$$

The embedding $\sigma^0 : Z \subset \hat{Z}$ is given by mapping $m \in \mathbf{K}^{r\times s}$ to its graph

$$\sigma^0_m := \{(\xi, \xi z) : \ \xi \in \mathbf{K}^{1\times r}\} \subset \mathbf{K}^{1\times r} \times \mathbf{K}^{1\times s} = \mathbf{K}^{1\times(r+s)}.$$

of $m \in \mathbf{K}^{r\times s}$. Via this embedding, we have

$$\check{Z} = \{m \in Z : \ I_r - zz^* > 0\} = \{m \in Z : \ I_s - z^*z > 0\}.$$

For $r = 1$, $\hat{Z}$ becomes projective space $\mathbf{P}^s$ and $\check{Z}$ is the unit ball $\mathbf{B}^s$.

A basic theorem of M. Koecher characterizes hermitian symmetric spaces in terms of Jordan triples:

Theorem 1.1.2. *In the complex setting, for every $^+$hermitian Jordan triple Z the conformal hull $\hat{Z}$ is a compact hermitian symmetric space, and every such space arises this way. Similarly, every hermitian bounded symmetric domain can be realized as the spectral unit ball $\check{Z}$ of a hermitian Jordan triple Z.*

Thus there is a 1-1 correspondence between ^{0}hermitian Jordan triples and ^{0}hermitian symmetric spaces of compact/non-compact type. Via this correspondence the two exceptional symmetric spaces can be treated on an equal footing with the classical types. For real Jordan triples and symmetric spaces, the above 1-1 correspondence is 'almost' true (some exceptional symmetric spaces are missing).

*Peirce manifolds
*Jordan-Kepler manifolds
*Jordan-Schubert varieties

- **Restricted Grassmannian**

We now describe an infinite-dimensional example.

Example 1.1.3. Let A be an associative unital Banach algebra. The set

$$\mathcal{S} := \{s \in A : \ s^2 = 1\}$$

of all **symmetries** in A is a Banach manifold, with tangent space

$$T_s\mathcal{S} = \{\dot{s} \in A : \ s\dot{s} + \dot{s}s = 0\}.$$

The set

$$\mathcal{P} := \{p \in A : \ p^2 = p\}$$

of all **idempotents** in A is a manifold, with tangent space

$$T_p\mathcal{P} = \{\dot{p} \in A : \ p\dot{p} + \dot{p}p = \dot{p}\}.$$

If A is a $*$-algebra, one obtains real manifolds by restricting to **self-adjoint** symmetries or projections, resp.

Lemma 1.1.4. *There is a 1-1 correspondence between (self-adjoint) symmetries $s \in \mathcal{S}$ and idempotents $p \in \mathcal{P}$ given by $p = \frac{s+1}{2}$ and $s = 2p - 1$, respectively.*

Proof. We have

$$\Big(\frac{s+1}{2}\Big)^2 = \frac{1}{4}(s^2 + 2s + 1) = \frac{1}{4}(1 + 2s + 1) = \frac{s+1}{2}$$

and

$$(2p-1)^2 = 4p + 1 - 4p = 1.$$

□

Identifying a subspace E with its orthogonal projection p_E or the corresponding symmetry $s_E = 2p_E - 1$, the complex Grassmannian becomes a connected component of the manifold of self-adjoint projections (resp. symmetries) for the block-matrix algebra

$$A = \mathbf{C}^{(r+s)\times(r+s)} = \begin{pmatrix} \mathbf{C}^{r\times r} & \mathbf{C}^{r\times s} \\ \mathbf{C}^{s\times r} & \mathbf{C}^{s\times s} \end{pmatrix}.$$

This is more precisely the **symplectic realization** of the complex Grassmannian.

*The infinite-dimensional **restricted Grassmannian** $\mathbf{G}_{res}$ arises by taking symmetries s in $A = \mathcal{L}(H)$, for a complex Hilbert space H, such that $s-1$ is of trace class. In the approach by Rajeev-Turgut, it plays a basic role in 2-dimensional QCD.

• **Loop groups**

Let G be a compact connected 1-connected simple Lie group. Let

$$(\xi|\eta) := -\mathrm{tr}(\mathrm{ad}_\xi \mathrm{ad}_\eta)$$

be the negative Killing form. Let $\mathbf{S} := \mathbf{S}^1$ be the circle and

$$\mathcal{C}^\infty_*(\mathbf{S}, G) = \{m : \mathbf{S} \to G : \; m(1) = e\}$$

be the based loop group. It has the tangent space

$$T_m\mathcal{C}^\infty_*(\mathbf{S}, G) = \mathcal{C}^\infty_*(\mathbf{S}, \mathfrak{g}) = \{u : \mathbf{S} \to \mathfrak{g} : \; u(1) = 0\}.$$

• **Conformal blocks**

Let S be a compact oriented surface. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then the set

$$\Omega^1(S, G)$$

of all connexions A on the trivial G-bundle $S \times G$ is an affine space of infinite dimension. It has the tangent space

$$T_A(\Omega^1(S, G)) = \Omega^1(S, \mathfrak{g})$$

at any $A \in \Omega^1(S, \mathfrak{g})$.

In the following, most manifolds will be constructed as **quotient manifolds** under an equivalence relation. Let N be a (not necessarily connected) manifold and $R \subset N \times N$ be a closed submanifold which defines an equivalence relation on N. Then $M := N/R$ is a manifold if the *Godement properties [**?**,] hold: The projections $R \to M$ must be submersions. For $u \in N$ let $[u]$ denote the equivalence class in the quotient manifold $M = N/R$.

1.1.1 Covered manifolds

A **covered manifold** is a $\mathbf{K}$-manifold M endowed with an open covering by local charts $\sigma^a : U_a \to M$, where U_a is a domain in a vector space $L \equiv \mathbf{K}^n$. Then the open sets $V_a := \sigma^a(U_a) \subset M$ cover M. Denote by

$$\sigma_a := (\sigma^a)^{-1} : V_a \to U_a \subset L$$

the inverse of σ^a. The charts are related by transition maps

$$\sigma^a_b = \sigma_b \circ \sigma^a$$

satisfying $\sigma^a = \sigma^b \circ \sigma_b^a, \quad \sigma_b^a \circ \sigma_a = \sigma_b$ and

$$\sigma_c^a = \sigma_c^b \circ \sigma_b^a.$$

We define two (closely related) equivalence relations for a covered manifold. First, consider the disjoint union

$$\mathcal{U} := \bigcup U_a \times \{a\}$$

endowed with the equivalence relation

$$(x,a) \approx (y,b) \Leftrightarrow \; x \in U_a, \; y \in U_b, \; \sigma_x^a = \sigma_y^b.$$

Equivalently, $y = \sigma_a^b(x)$. In the following, we often write argument variables, such as x, y, as a subscript, in order to save brackets. Now consider the disjoint union

$$\mathcal{V} := \bigcup V_a \times \{a\}$$

endowed with the equivalence relation

$$(m,a) \sim (m,b) \Leftrightarrow \; m \in V_a \cap V_b.$$

Then $M = \mathcal{U}/\approx = \mathcal{V}/\sim$.

• Jordan manifolds

Example 1.1.5. Consider the projective space $M = \mathbf{P}^s$. For $0 \le i \le s$ let $U_i = L = \mathbf{C}^s$ and define the charts

$$\sigma^i : \mathbf{C}^s \to \mathbf{P}^s, \quad \sigma^i(z^0, \ldots, \hat{1}^i, \ldots, z^s) := [z^0, \ldots, 1^i, \ldots, z^s].$$

Conversely, put

$$V_i := \{[\zeta] \in \mathbf{P}^s : \; \zeta^i \neq 0\}.$$

Then

$$\sigma_i : V_i \to \mathbf{C}^s, \quad \sigma_i[\zeta] = \Big(\frac{\zeta^0}{\zeta^i}, \vdots, \hat{1}^i, \vdots, \frac{\zeta^s}{\zeta^i}\Big).$$

The transition maps (for $i < j$) are given by

$$\sigma_j^i(z^0, \ldots, \hat{1}^i, \ldots, z^j, \ldots, z^s) = \Big(\frac{z^0}{z^j}, \ldots, \frac{1^i}{z^j}, \ldots, \hat{1}^j, \ldots, \frac{z^s}{z^j}\Big).$$

In this way $\mathbf{P}^s = \mathcal{U}/\approx$ becomes a covered manifold. In the special case $s = 1$ (Riemann sphere) we obtain

$$\sigma^0(z^1) := [1, z^1], \; \sigma_0[\zeta] := \frac{\zeta^1}{\zeta^0}$$

$$\sigma^1(z^0) := [m^0, 1], \; \sigma_1[\zeta] := \frac{\zeta^0}{\zeta^1}$$

$$\sigma_1^0(z^1) = \frac{1}{z^1}, \quad \sigma_0^1(z^0) = \frac{1}{z^0}.$$

*finite charts for Grassmannian

For the conformal hull $\hat{Z}$ of a hermitian Jordan triple Z, instead of a finite covering we have a 'continuous' covering by local charts

$$\sigma^a : Z \to \hat{Z}, \quad z \mapsto \sigma_z^a := [z, a]$$

for any $a \in Z$. Thus $\mathcal{U} = Z \times Z =: Z^2$ in this case, so that

$$\hat{Z} = Z^2 / \approx .$$

In the special case $a = 0$ we write $z^0 = z$ and obtain the affine embedding

$$\sigma^0 : Z \subset \hat{Z}.$$

If (z, a) is quasi-invertible, then $\sigma^a_z = [z, a] = [z^a, 0] = (z^a)^0 = z^a$. In view of the addition formula (**??**) the transition map between two local charts σ^a and σ^b is given by

$$\sigma^a_b(z) = z^{a-b}$$

on the open set $\{z \in Z : \ (z, a - b) \text{ quasi-invertible}\}$.

1.1.2 Homogeneous manifolds

Another basic type of quotient manifolds are the **homogeneous** manifolds. Let G be a Lie group with a closed subgroup $H \subset G$. Then the equivalence relation $R := \{(g, gh) : \ g \in G, \ h \in H\}$ on G is invariant under left G-translations and hence

$$M = G/H = G/R$$

becomes a quotient manifold with a left G-action.

• Jordan manifolds

projective space

$$\mathbf{P}^s = SU(1, s)/U(s)$$

Grassmannian: Let $Z = \mathbf{C}^{r\times s}$, endowed with the operator norm $\|z\| = \sup spec(zz^*)^{1/2}$. Then the **matrix unit ball**

$$\check{Z} = \{m \in \mathbf{C}^{r\times s} : \ \|z\| < 1\} = \{m \in \mathbf{C}^{r\times s} : \ I - zz^* > 0\}$$

is a symmetric domain under the **pseudo-unitary group**

$$\check{G} = U(r, s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(r + s) : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

which acts on $\check{Z}$ via **Moebius transformations**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (m) = (az + b)(cz + d)^{-1}.$$

Its maximal compact subgroup is

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : \ a \in U(r), \ d \in U(s) \right\}.$$

with the linear action $m \mapsto azd^*$. For the compact dual we have

$$\mathbf{G}_r(\mathbf{C}^{r+s}) = U(r + s)/U(r) \times U(s)$$

For a general hermitian Jordan triple, let $K = Aut(Z)$ denote the compact linear Lie group of all Jordan triple automorphisms of Z. The **structure group** $\mathring{K} \subset GL(Z)$ is generated by all invertible

Bergman operators $B_{a,b}$, where $(a,b) \in Z^2$ is quasi-invertible. It acts via linear transformations on $\hat{Z}$. On the other hand, the non-linear transformations of $\hat{Z}$ are the translations

$$\mathfrak{t}_a z := z + a$$

and the quasi-inverse maps

$$\mathfrak{t}_a^* z := z^{-a}$$

for $a \in Z$. The **conformal group** $\mathring{G}$ of Z is an algebraic Lie group generated by these three types of transformations. It acts transitively on $\hat{Z}$, giving a **conformal realization**

$$\hat{Z} = \mathring{G}/\mathring{G}_0$$

as a **flag manifold**. Here the **parabolic subgroup**

$$\mathring{G}_0 := \{g \in \mathring{G} : \ g(0) = 0\}$$

is generated by $\mathring{K}$ and the quasi-inverse maps (**??**). Putting $B_{a,b}^* = B_{b,a}$, one can show that $\mathring{K}$ carries an involution such that

$$K = \{k \in \mathring{K} : \ k^* = k^{-1}\}.$$

This can be extended to an involution of $\mathring{G}$ mapping $\mathfrak{t}_a$ to $\mathfrak{t}_a^*$. Then

$$\hat{G} = \{g \in \mathring{G} : \ g^* = g^{-1}\}$$

is a compact subgroup of $\mathring{G}$ which still acts transitively on $\hat{Z}$ and satisfies

$$\hat{G} \cap \mathring{G}_0 = K.$$

This yields a **metric realization**

$$\hat{Z} = \hat{G}/K$$

of $\hat{Z}$ as a compact hermitian symmetric space. Let

$$s_0 z := -z$$

denote the **symmetry** at the origin $0 \in Z$. Then

$$\check{G} := \{g \in \mathring{G} : \ g^{-1} = s_0 g^* s_0\}$$

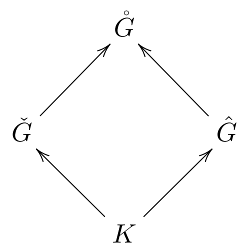
is a non-compact subgroup of $\mathring{G}$ which acts transitively on the spectral unit ball $\check{Z}$ and also satisfies

$$\hat{G} \cap \mathring{G}_0 = K.$$

This gives a **metric realization**

$$\check{Z} = \check{G}/K$$

of $\check{Z}$ as a non-compact hermitian symmetric space. In summary, we have a diagram of Lie groups



For $\mathbf{K} = \mathbf{C}$ the structure group $\mathring{K}$ is a complexification of K, and the conformal group $\mathring{G}$ is a complexification of $\hat{G}$ and of $\check{G}$. Moreover, $\hat{G}$ is the full biholomorphic automorphism group of $\hat{Z}$, and $\check{G}$ is the full biholomorphic automorphism group of $\check{Z}$. (These remarks hold more precisely for the connected components of the identity.)

In terms of the classification of complex hermitian Jordan triples we have

- $K = U(r) \times U(s): \ z \mapsto uzv, \ u \in U(r), \ v \in U(s), \quad Z = \mathbf{C}^{r \times s}$
- $K = U(s), \quad Z = \mathbf{C}^{1 \times s} = \mathbf{C}^s$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- $K = \mathbf{T} \cdot SO(a+2)$ **spin factor**$Z = \mathbf{C}^{a+2}$
- $K = ?, \quad Z = \mathbf{C}^{16}_{exc}$ and $K = \mathbf{T} \cdot E_6, \quad Z = \mathbf{C}^{27}_{exc}$.

1.2 Bundles

For any fibre bundle B over a manifold M, let $\Gamma(B)$ denote the set of all sections (smooth/holomorphic) $\Phi : M \to B$, satisfying $\pi \circ \Phi = I_M$.. For the trivial bundle $B = M \times F$ with fibre F we write

$$\Gamma(M \times F) = \Gamma(M, F).$$

Thus $\Gamma(M, F) = \mathcal{C}^\infty(M, F)$ for $\mathbf{K} = \mathbf{R}$ and $\Gamma(M, F) = \mathcal{O}(M, F)$ for $\mathbf{K} = \mathbf{C}$. Denote by TM the tangent bundle if $\mathbf{K} = \mathbf{R}$ and the holomorphic tangent bundle if $\mathbf{K} = \mathbf{C}$. Thus in the complex case we have the complexified tangent space

$$T^{\mathbf{C}}_m M := T_m M \oplus \overline{T_m M}$$

and the real tangent space $T^{\mathbf{R}}_m M$ is the real subspace

$$T^{\mathbf{R}}_m M = \{v + \overline{v} : \ v \in T_m M\}.$$

The complex structure $J_m : T^{\mathbf{R}}_m M \to T^{\mathbf{R}}_m M$ is given by

$$J_m(v + \overline{v}) = iv + \overline{iv}$$

for all $v \in T_m M$.

Example 1.2.1. For a domain $M \subset \mathbf{C}$, with coordinate $z = x+iy$, we have the holomorphic/antiholomorphic tangent vectors

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

satisfying

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \ i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}.$$

The complex structure is

$$J\frac{\partial}{\partial x} = J\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\Big) = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}} = -\frac{\partial}{\partial y}$$

$$J\frac{\partial}{\partial y} = -iJ\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big) = -i\Big(i\frac{\partial}{\partial z} + i\frac{\partial}{\partial \overline{z}}\Big) = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}.$$

Let P be a principal fibre bundle with Lie structure group H, called an H-bundle in the following, over $M = P/H$. Any H-module (E, π) (i.e., a finite dimensional vector space E endowed with a representation π of H), gives rise to an **associated vector bundle**

$$P \overset{\pi}{\underset{H}{\times}} E = \{[p, \phi] = [ph, h^{-\pi}\phi] : \ p \in P, \ h \in H, \ \phi \in E\}.$$

Let $[p] \in M$ denote the equivalence class of $p \in P$. Writing $\Phi_{[p]} = [p, \tilde{\Phi}_p]$, for the so-called **homogeneous lift** $\tilde{\Phi}$ of a section Φ, one obtains an isomorphism

$$\Gamma(P \overset{\pi}{\underset{H}{\times}} E) \equiv \{f \in \Gamma(P, E) : \; f_{ph} = h^{-\pi} \; f_p \; \forall p \in P, \; h \in H\}.$$

In the following, principal bundles and their associated vector bundles will be defined in terms of **cocycles.**

Proposition 1.2.2. *Let $M = N/\sim$ be a quotient manifold for an equivalence relation $R \subset N$. Let $\beta : R \to H$ be a smooth map into a Lie group H, denoted by $(u, v) \mapsto \beta_u^v$, which has the* **cocycle property**

$$\beta_u^v \; \beta_v^w = \beta_u^w$$

for all triples $u \sim v \sim w$ in N. Then

$$N \overset{\beta}{\underset{\sim}{\times}} H := \{[u, h] = [v, (\beta_u^v)^{-1} h] : \; (u, v) \in R, \; h \in H\}$$

becomes an H-bundle over $M = N/R$, with projection $N \times_\sim^\beta H \to M, \; [u, h] \mapsto [u]$.

As a consequence any H-module E gives rise to an **induced vector bundle**

$$N \overset{\beta,\pi}{\underset{\sim}{\times}} E := (N \overset{\beta}{\underset{\sim}{\times}} H) \overset{\pi}{\underset{H}{\times}} E = \{[u, \phi] = [v, (\beta_u^v)^{-\pi} \phi] : \; (u, v) \in R, \; \phi \in E\}$$

over M. Writing $\Phi_{[u]} = [u, \tilde{\Phi}_u]$ one obtains an isomorphism

$$\Gamma(N \overset{\beta,\pi}{\underset{\sim}{\times}} E) \equiv \{f \in \Gamma(N, E) : \; f_v = (\beta_u^v)^{-\pi} \; f_u \; \forall \; (u, v) \in R\}.$$

We often omit the reference to π if the context is clear.

1.2.1 Covered manifolds

For a covered manifold M consider maps $\beta_b^a : V_a \cap V_b \to H$ satisfying the cocycle property

$$\beta_b^a(m) \; \beta_c^b(m) = \beta_c^a(m)$$

for all $m \in V_a \cap V_b \cap V_c$. Then

$$\beta_{a,m}^{b,m} := \beta_a^b(m)$$

defines a cocycle $\beta : R \to H$ in the sense of (**??**). Hence

$$\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} H = \{[m, h]_a = [m, \beta_a^b(m) h]_b : \; m \in V_a \cap V_b, \; h \in H\}$$

becomes an H-bundle over the quotient manifold $M = \mathcal{V}/\sim$. Any H-module E gives rise to an **induced vector bundle**

$$\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} E := (\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} H) \underset{H}{\times} E = \{[m, \phi]_a = [m, \beta_a^b(m) \phi]_b : \; m \in V_a \cap V_b, \; \phi \in E\}$$

over M. Writing $\Phi_{[m]} = [m, \Phi^a(m)]_a$ one obtains an isomorphism

$$\Gamma(\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} E) \equiv \{(\Phi^a) \in \prod_a \Gamma(V_a, E) : \; \Phi^a(m) = \beta_b^a(m) \; \Phi^b(m) \; \forall \; m \in V_a \cap V_b\}.$$

By local triviality, every principal bundle and every vector bundle can be realized this way.

The **tangent bundle** arises as follows. For a covering family of charts σ^a of M and a map $f : V_a \to E$ we write

$$\frac{\partial f}{\partial \sigma_a}(m) := (f \circ \sigma^a)'(x) \in \mathrm{Hom}(L, E)$$

if $m = \sigma^a_x \in V_a$. Applying this notation to $E = L$ and $f = \sigma_b$, we obtain

$$\frac{\partial \sigma_b}{\partial \sigma_a}(m) = (\sigma_b \circ \sigma^a)'(x) = (\sigma^a_b)'(x) \in \mathcal{L}(L) \text{ endomorphisms}$$

for $m = \sigma^a_x \in V_a \cap V_b$. Now let $m = \sigma^a_x = \sigma^b_y$. Since $y = \sigma^a_b(x)$, the chain rule yields

$$\frac{\partial \sigma_c}{\partial \sigma_a}(m) = (\sigma^a_c)'(x) = (\sigma^b_c)'(y)(\sigma^a_b)'(x) = \frac{\partial \sigma_c}{\partial \sigma_b}(m)\frac{\partial \sigma_b}{\partial \sigma_a}(m)$$

for all x, with $y := \sigma^a_b(x)$. We sometimes write this relation in the opposite order

$$t \cdot d_x\sigma^a_c = (t \cdot d_x\sigma^a_b) \cdot d_y\sigma^b_c$$

for all $t \in L$. It follows that

$$\dot{\sigma}^b_a(m) := \frac{\partial \sigma_a}{\partial \sigma_b}(m)$$

defines a $GL(L)$-valued cocycle on R. Hence we obtain a $GL(L)$-bundle

$$\mathcal{V} \underset{\sim}{\overset{\dot{\sigma}}{\times}} GL(L) = \{[m, h]_a = [m, h\frac{\partial \sigma_b}{\partial \sigma_a}(m)]_b : \ m \in V_a \cap V_b\}$$

over M, called the **bein bundle**. Via the defining representation of $GL(L)$, we obtain the associated vector bundle

$$\mathcal{V} \underset{\sim}{\overset{\dot{\sigma}}{\times}} L = \{[m, t]_a = [m, \frac{\partial \sigma_b}{\partial \sigma_a}(m)t]_b : \ m \in V_a \cap V_b, \ t \in L\}$$

which is isomorphic to the tangent bundle TM by identifying $[m, t]_a$ with $(T_x\sigma^a)t$ for $m = \sigma^a_x$. In fact, we have

$$(T_x\sigma^a)t = T_x(\sigma^b \circ \sigma^a_b)t = (T_y\sigma^b)(\sigma^a_b)'_x t = (T_y\sigma^b)\frac{\partial \sigma_b}{\partial \sigma_a}(m)t$$

for $m = \sigma^a_x = \sigma^b_y$. The corresponding sections (vector fields) are

$$\Gamma(\mathcal{V} \underset{\sim}{\overset{\dot{\sigma},\iota}{\times}} L) \equiv \{(T^a) \in \prod_a \Gamma(V_a, L) : \ T^b_m = \frac{\partial \sigma_b}{\partial \sigma_a}(m)T^a_m \ \forall m \in V_a \cap V_b\}.$$

Similarly, the cotangent bundle T^*M is isomorphic to the cocycle bundle

$$\mathcal{V} \underset{\sim}{\overset{\dot{\sigma}}{\times}} L^* = \{[m, \vartheta]_a = [m, \vartheta \circ \frac{\partial \sigma_a}{\partial \sigma_b}(m)]_b : \ m \in V_a \cap V_b, \ \vartheta \in L^*\}$$

by identifying $[m, \vartheta]_a$ with $\vartheta \circ (T_m\sigma_a)$ when $m = \sigma^a_x$. In fact, we have

$$\vartheta \circ (T_m\sigma_a) = \vartheta \circ T_m(\sigma^b_a \circ \sigma_b) = \vartheta \circ (\sigma^b_a)'(x)(T_m\sigma_b) = \vartheta \circ \frac{\partial \sigma_a}{\partial \sigma_b}(m)(T_m\sigma_b)$$

for $m = \sigma^a_x = \sigma^b_y$. The corresponding sections (1-forms) are

$$\Gamma(\mathcal{V} \underset{\sim}{\overset{\dot{\sigma},\iota}{\times}} L^*) \equiv \{(\Theta^a) \in \prod_a \Gamma(V_a, L^*) : \ \Theta^b_m = \Theta^a_m \circ \frac{\partial \sigma_a}{\partial \sigma_b}(m) \ \forall m \in V_a \cap V_b\}.$$

These bundles can also be described in the setting $M = \mathcal{U}/\approx$. The formulas are

$$\mathcal{U} \underset{\sim}{\overset{\dot{\sigma}}{\times}} GL(L) = \{[x, h]_a = [y, h(\sigma^a_b)'_x]_b : \ x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y\}$$

$$\mathcal{U}\overset{\dot\sigma}{\underset{\sim}{\times}}L=\{[x,t]_a=[y,(\sigma^a_b)'_x t]_b:\ x\in U_a,\ y\in U_b,\ \sigma^a_x=\sigma^b_y,\ t\in L\}$$

$$\Gamma(\mathcal{U}\overset{\dot\sigma}{\underset{\sim}{\times}}L)\equiv\{(T^a)\in\prod_a\Gamma(U_a,L):\ T^b_y=(\sigma^a_b)'_x T^a_x\ \forall x\in U_a,\ y\in U_b,\ \sigma^a_x=\sigma^b_y\},$$

$$\mathcal{U}\overset{\dot\sigma}{\underset{\sim}{\times}}L^*=\{[x,\vartheta]_a=[y,\vartheta\circ(\sigma^b_a)'_y]_b:\ x\in U_a,\ y\in U_b,\ \sigma^a_x=\sigma^b_y,\ \vartheta\in L^*\}$$

$$\Gamma(\mathcal{U}\overset{\dot\sigma}{\underset{\sim}{\times}}L^*)\equiv\{(\Theta^a)\in\prod_a\Gamma(U_a,L^*):\ \Theta^b_y=\Theta^a_x\circ(\sigma^b_a)'_y\ x\in U_a,\ y\in U_b,\ \sigma^a_x=\sigma^b_y\}.$$

In case M carries an H-**structure**, for a closed subgroup $H\subset GL(L)$, the transition maps σ^a_b can be chosen such that $\frac{\partial\sigma_b}{\partial\sigma_a}(m)=(\sigma^a_b)'_x\in H$, and we obtain H-bundles instead.

• Jordan manifolds

Example 1.2.3. For the projective space $M=\mathbf{P}^s$ taking derivatives of (**??**), we obtain

$$e_k\cdot(\partial_m\frac{z^i}{z^j})=\frac{\partial}{\partial z^k}\partial_m\frac{z^i}{z^j}=\frac{\delta^i_k\ z^j-z^i\ \delta^j_k}{(z^j)^2}$$

for the standard base e_k of $\mathbf{C}^s$. For any other $u=u^k\ e_k\in\mathbf{C}^s$ we obtain

$$u\cdot(\partial_m\frac{z^i}{z^j})=u^k\ e_k\cdot(\partial_m\frac{z^i}{z^j})=u^k\frac{\delta^i_k\ z^j-z^i\ \delta^j_k}{(z^j)^2}=\frac{u^i\ z^j-z^i\ u^j}{(z^j)^2}$$

Proposition 1.2.4. *For a hermitian Jordan triple Z the conformal hull carries a $\mathring{K}$-structure. More precisely, the map $\beta:Z^2\to\mathring{K}$ defined by*

$$\beta^{w,b}_{z,a}:=B_{z,a-b}$$

is a cocycle, and the induced $\mathring{K}$-bundle $Z^2\times^\beta_\sim\mathring{K}$ over $\hat{Z}$ is the bein (tangent frame) bundle.

Proof. The cocycle property follows from the addition formula (**??**). The well-known identity

$$\partial_z\mathfrak{t}^*_a=B^{-1}_{z,-a}$$

implies that the transition map $\sigma^a_b=\mathfrak{t}^*_{b-a}$ has the derivative

$$\partial_z\sigma^a_b=\partial_z\mathfrak{t}^*_{b-a}=B^{-1}_{z,a-b}.$$

This gives the bein bundle. □

As a consequence, any $\mathring{K}$-module E yields an induced vector bundle

$$Z^2\overset{\beta}{\underset{\sim}{\times}}E:=(Z^2\overset{\beta}{\underset{\sim}{\times}}\mathring{K})\overset{\pi}{\underset{\mathring{K}}{\times}}E=\{[z,\phi]_a=[z^{a-b},B^{-\pi}_{z,a-b}\phi]_b:\ (z,a-b)\text{ quasi-invertible}\}$$

over $\hat{Z}=Z^2/\approx$. Writing $\Phi_{[z,a]}=[z,\Phi^a_z]_a$ the sections Φ are described by

$$\Gamma(Z^2\overset{\beta}{\underset{\sim}{\times}}E)\equiv\{(\Phi^a)\in\Pi_a\Gamma(Z,E):\ \Phi^b_{z^{a-b}}=B^{-\pi}_{z,a-b}\Phi^a_z\}.$$

Since $Z\subset\hat{Z}$ is a dense open subset via the embedding $z\mapsto z^0=[z,0]$, a section Φ is uniquely determined by its trivialization $\underline{\Phi}:=\Phi^0$. Thus the mapping $\Phi\mapsto\underline{\Phi}$ identifies $\Gamma(Z^2\times^\beta_\sim E)$ with a vector space of maps from Z to E. For the defining representation $\mathring{K}\subset GL(Z)$ we obtain the **tangent bundle**

$$Z^2\overset{\beta}{\underset{\sim}{\times}}Z:=(Z^2\overset{\beta}{\underset{\sim}{\times}}\mathring{K})\underset{\mathring{K}}{\times}Z=\{[z,t]_a=[z^{a-b},B^{-1}_{z,a-b}t]_b:\ (z,a-b)\text{ quasi-invertible},\ t\in Z\}\equiv T\hat{Z}$$

and the cotangent bundle

$$Z^2 \underset{\sim}{\overset{\beta}{\times}} Z^* = (\hat{Z} \underset{\sim}{\overset{\beta}{\times}} \mathring{K}) \underset{\mathring{K}}{\times} Z^* = \{[z,\vartheta]_a = [z^{a-b}, \vartheta \circ B_{z,a-b}]_b : \ (z, a-b) \text{ quasi-invertible}\} \equiv T^*\hat{Z}.$$

Example 1.2.5. Riemann sphere

1.2.2 Homogeneous manifolds

For a Lie group G with a closed subgroup $H \subset G$, we may regard

$$G = G \underset{H}{\times} H$$

as an H-bundle over $M := G/H$. The **homogeneous vector bundle** associated to an H-module E of H is given by

$$G \underset{H}{\overset{\pi}{\times}} E := \{[g,\phi] = [gh, h^{-\pi}\phi] : \ g \in G, \ h \in H, \ \phi \in E\}.$$

It is G-equivariant under the action

$$g_{g'H}[g',\phi] := [gg',\phi].$$

• Jordan manifolds

The derivative $\partial_0 q$ of $q \in \mathring{G}_0$ belongs to $\mathring{K}$, and

$$\partial_0 : \mathring{G}_0 \to \mathring{K}, \quad q \mapsto \partial_0 q.$$

is a homomorphism.

Proposition 1.2.6. *The mapping*

$$[z,h]_a \mapsto [\mathsf{t}^*_{-a}\mathsf{t}_z, h]$$

induces an isomorphism

$$Z^2 \underset{\sim}{\overset{\beta}{\times}} \mathring{K} \equiv \mathring{G} \underset{\mathring{G}_0}{\overset{\partial_0}{\times}} \mathring{K}$$

of $\mathring{K}$-bundles over $\hat{Z}$.

Proof. The transformation $g := \mathsf{t}^*_{-a}\mathsf{t}_z \in \mathring{G}$ has the derivative

$$\partial_0 g = (\partial_z \mathsf{t}^*_{-a})(\partial_0 \mathsf{t}_z) = \partial_z \mathsf{t}^*_{-a} = B^{-1}_{z,a}.$$

Now let $[z,\phi]_a = [w, B^{-\pi}_{z,a-b}\phi]_b$. Then $\mathsf{t}^*_{-a}\mathsf{t}_z(0) = \mathsf{t}^*_{-a}(z) = z^a = w^b = \mathsf{t}^*_{-b}(w) = \mathsf{t}^*_{-b}\mathsf{t}_w(0)$. Hence there exists $q \in \mathring{G}_0$ such that $\mathsf{t}^*_{-b}\mathsf{t}_w = \mathsf{t}^*_{-a}\mathsf{t}_z q$. Then

$$B^{-1}_{w,b} = \partial_0(\mathsf{t}^*_{-b}\mathsf{t}_w) = \partial_0(\mathsf{t}^*_{-a}\mathsf{t}_z q) = \partial_0(\mathsf{t}^*_{-a}\mathsf{t}_z)\ \partial_0 q = B^{-1}_{z,a}\ \partial_0 q.$$

Therefore $\partial_0 q = B_{z,a}\ B^{-1}_{w,b} = B_{z,a-b}$ by the addition formula (**??**). This implies

$$[\mathsf{t}^*_{-a}\mathsf{t}_z, h] = [\mathsf{t}^*_{-b}\mathsf{t}_w q^{-1}, h] = [\mathsf{t}^*_{-b}\mathsf{t}_w, (\partial_0 q)^{-1} h] = [\mathsf{t}^*_{-b}\mathsf{t}_w, B^{-1}_{z,a-b} h].$$

Hence the assignment (**??**) is a well-defined map $Z^2 \times^\beta_\sim \mathring{K} \to \mathring{G} \times^{\partial_0}_{\mathring{G}_0} \mathring{K}$, which is a bijection. □

Thus for any $\mathring{G}_0$-module E the mapping $[z,\phi]_a \mapsto [\mathsf{t}^*_{-a}\mathsf{t}_z, \phi]$ induces a vector bundle isomorphism

$$Z^2 \underset{\sim}{\overset{\beta}{\times}} E \equiv \mathring{G} \underset{\mathring{G}_0}{\overset{\partial_0}{\times}} E.$$

As a consequence, the vector bundle $Z^2 \times^\beta_\sim E$ carries a $\mathring{G}$-action. This is not obvious in the coordinate chart picture.

1.3 0-Geometry: Hermitian Metrics

A **0-geometry** on a vector bundle is a **hermitian metric**. We can also allow pseudo-metrics of indefinite signature and speak generally of **metric** vector spaces and vector bundles. The positive definite case will be called ^{0}metric (0 negative eigenvalues).

Let P be an H-bundle over $M = P/H$. Then any metric H-module E defines a metric

$$([p, \phi]|[p, \eta] := (\xi|\eta)$$

on the associated vector bundle $P \times_H^\pi E$. This is well-defined since

$$([ph, h^{-\pi}\phi]|[ph, h^{-\pi}\phi]) = (h^{-\pi}\xi|h^{-\pi}\xi) = (\xi|\eta).$$

1.3.1 Covered manifolds

Consider an H-valued cocycle β_a^b and a metric H-module E.

Lemma 1.3.1. *Let E be a metric vector space, with inner product $(\xi|\eta)$. A family of smooth maps $h^a : V_a \to \mathcal{H}^\times(E)$ (self-adjoint invertible), satisfying the compatibility condition*

$$h_m^a = \beta_a^b(m)^* \ h_m^b \ \beta_a^b(m)$$

for all $m \in V_a \cap V_b$ defines a metric on $\mathcal{V} \times_\sim^\beta E$ via

$$([m, \phi]_a|[m, \eta]_a)_m := (\xi|h_m^a\eta).$$

For $E = \mathbf{C}$, a family of smooth functions $\boldsymbol{h}^a : V_a \to \mathbf{R}^>$, satisfying the compatibility condition

$$\boldsymbol{h}_m^a = |\beta_a^b(m)|^2 \ \boldsymbol{h}_m^b$$

for all $m \in V_a \cap V_b$ defines a ^{0}metric on the line bundle $\mathcal{V} \times_\sim^\beta \mathbf{C}$ via

$$([m, \phi]_a|[m, \eta]_a)_m := (\xi|\boldsymbol{h}_m^a\eta).$$

Proof. The identification $[m, \phi]_a = [m, \beta_a^b(m)\phi]_b$ yields

$$([m, \beta_a^b(m)\phi]_b|[m, \beta_a^b(m)\eta]_b) = (\beta_a^b(m)\xi|h_m^b\beta_a^b(m)\eta)$$

$$= (\xi|\beta_a^b(m)^* h_m^b \beta_a^b(m)\eta) = (\xi|h_m^a\eta) = ([m, \phi]_a|[m, \eta]_a).$$

□

Proposition 1.3.2. *Let E be a ^{0}metric vector space. Let (χ_a) be a partition of unity subordinate to $\mathcal{V}$. Then the family*

$$h^a := \sum_c \ \beta_a^{c*} \ \chi_c \ \beta_a^c, \quad h_m^a = \sum_c \ \beta_a^c(m)^* \ \chi_c(m) \ \beta_a^c(m)$$

defines a ^{0}metric on $\mathcal{V} \times_\sim^\beta E$. For $E?\mathbf{C}$, the family

$$\boldsymbol{h}^a := \sum_c \ |\beta_a^c|^2 \ \chi_c, \quad \boldsymbol{h}_m^a = \sum_c \ |\beta_a^c(m)|^2 \ \chi_c(m)$$

defines a ^{0}metric on the line bundle $\mathcal{V} \times_\sim^\beta \mathbf{C}$.

Proof. Since the sum (**??**) is locally finite and the $\chi_c(m)$ add up to 1, (**??**) defines a smooth map from V_a to the positive definite matrices. For $m \in V_a \cap V_b$ the cocycle property (**??**) implies

$$\beta_a^b(m)^* \ h_m^b \ \beta_a^b(m) = \beta_a^b(m)^* \sum_c \beta_b^c(m)^* \ \chi_c(m) \ \beta_b^c(m) \ \beta_a^b(m)$$

$$= \sum_c (\beta_b^c(m)\beta_a^b(m))^* \ \chi_c(m) \ \beta_b^c(m)\beta_a^b(m) = \sum_c \beta_a^c(m)^* \ \chi_c(m) \ \beta_a^c(m) = h_m^a$$

□

Proposition 1.3.3. *Let* (h^a) *be a* ^{0}*metric on* $\mathcal{V} \times_{\sim}^{\beta} E$. *Then*

$$\kappa_b^a(m) := (h_m^a)^{1/2} \ \beta_b^a(m) \ (h_m^b)^{-1/2}$$

defines a unitary cocycle, i.e. $\kappa_g^a(m) \in U(E)$ *for all* $m \in V_a \cap V_b$.

Proof. The cocycle property follows from

$$\gamma_b^a \gamma_c^b = (h^a)^{1/2} \ \beta_b^a \ (h^b)^{-1/2} \ (h^b)^{1/2} \ \beta_c^b \ (h^c)^{-1/2} = (h^a)^{1/2} \ \beta_b^a \ \beta_c^b \ (h^c)^{-1/2} = (h^a)^{1/2} \ \beta_c^a \ (h^c)^{-1/2} = \gamma_c^a.$$

Moreover, we have

$$\kappa_b^{a*} \ \kappa_b^a = ((h^a)^{1/2} \ \beta_b^a \ (h^b)^{-1/2})^* \ (h^a)^{1/2} \ \beta_b^a \ (h^b)^{-1/2} = (h^b)^{-1/2} \ \beta_b^{a*} \ h^a \ \beta_b^a \ (h^b)^{-1/2} = (h^b)^{-1/2} \ h^b \ (h^b)^{-1/2} = I.$$

□

As a consequence we may form the ^{0}metric vector bundle

$$\mathcal{V} \underset{\sim}{\overset{\gamma}{\times}} E = \{\langle m, \xi\rangle_a = \langle m, \gamma_b^a(m)\xi\rangle_b : \ m \in V_a \cap V_b\}.$$

It carries the ^{0}metric

$$(\langle m, \xi\rangle_a | \langle m, \eta\rangle_a) = (\xi|\eta).$$

since the condition (**??**) is trivially satisfied by $\boldsymbol{h}_m^a = I_E$.

For the tangent bundle, a family of smooth maps $\boldsymbol{h}^a : V_a/U_a \to \mathcal{H}^\times(L)$, satisfying the compatibility condition

$$\boldsymbol{h}_m^a = \frac{\partial \sigma_a}{\partial \sigma_b}(m)^* \ \boldsymbol{h}_m^b \ \frac{\partial \sigma_a}{\partial \sigma_b}(m), \quad \boldsymbol{h}_x^a = (\sigma_a^b)'(y)^* \ \boldsymbol{h}_y^b \ (\sigma_a^b)'(y)$$

on $V_a \cap V_b / U_a \cap U_b$ defines a tangent metric on $\mathcal{V}/\mathcal{U} \times_{\sim}^{\dot\sigma} L \equiv TM$ via the assignment

$$([m, u]_a | [m, v]_a)_m := (u|\boldsymbol{h}_m^a v), \quad ([x, u]_a | [x, v]_a) := (u|\boldsymbol{h}_x^a v),$$

Similar for the cotangent bundle. In the positive case the family

$$\boldsymbol{h}_m^a := \sum_c \frac{\partial \sigma_a}{\partial \sigma_c}(m)^* \ \chi_c(m) \ \frac{\partial \sigma_a}{\partial \sigma_c}(m)$$

induces a tangent ^{0}metric on M. The associated unitary cocycle is

$$\kappa_b^a(m) := (\boldsymbol{h}_m^a)^{1/2} \ \frac{\partial \sigma_b}{\partial \sigma_a}(m) \ (\boldsymbol{h}_m^b)^{-1/2}$$

• **Jordan manifolds**

Example 1.3.4. For $\mathbf{K} = \mathbf{C}$ consider the holomorphic tangent bundle $T\mathbf{P}$ on the Riemann sphere $\mathbf{P}^1$, endowed with the tangent metric

$$\boldsymbol{h}_z^0 = (1 + z\overline{z})^{-2}.$$

The coordinate change $w := \frac{1}{z}$ yields

$$\boldsymbol{h}_w^1(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}) = (1 + w\overline{w})^{-2}.$$

*The metric is invariant under $SU(2)$.

1.3.2 Homogeneous manifolds

If $H \subset G$ is a closed subgroup and E is a metric H-module, then $G \times_H E$ becomes a G-equivariant metric vector bundle with respect to the fibre metric

$$([g, \phi]|[g, \psi])_{gH} := (\phi|\psi).$$

For line bundles, with $\phi, \psi \in \mathbf{C}$, the ^{0}metric is

$$([g, \phi]|[g, \psi])_{gH} := \overline{\phi}\psi.$$

• **Jordan manifolds**

Any unitary K-representation (E, π) has a holomorphic extension to $\mathring{K}$. Then the mapping $[z, \phi]_a \mapsto [\mathsf{t}_{-a}^* \mathsf{t}_z, \phi]$ induces an isomorphism

$$Z^2 \overset{\kappa}{\underset{\sim}{\times}} E \equiv \hat{G} \overset{\pi}{\underset{K}{\times}} E$$

of hermitian holomorphic vector bundles. As a consequence, the restricted $\hat{G}$-action on the vector bundle $Z^2 \times_{\sim}^{\beta} E$ is isometric.

1.4 1-Geometry: Connexions

The Lie algebra $\Gamma(TM)$ of vector fields on M is endowed with the commutator

$$[X, Y]_m = X_m \cdot d_m Y - Y_m \cdot d_m X.$$

The infinitesimal action of vector fields on maps $\Phi : G \to E$ is given by

$$(d_X \Phi)_g := X_g \cdot T_g \Phi.$$

Proposition 1.4.1.

$$d_X(d_Y \Phi) - d_Y(d_X \Phi) = d_{[X,Y]}\Phi.$$

For a real manifold M let

$$\Omega^r(M, \mathbf{R}) = \Gamma(T^r M)$$

denote the space of all smooth real r-forms over M. Thus

$$\Omega^0(M) = \Gamma(M \times \mathbf{R}) = \Gamma(M, \mathbf{R})$$

consists of all smooth functions on M. The exterior derivative

$$d : \Omega^r(M) \to \Omega^{r+1}(M)$$

is defined by the **Palais formula**: Given vector fields $\xi^0,\dots,\xi^p$ then

$$(d\omega)(X^0,\dots,X^p) = \sum_{i=0}^{p}(-1)^i\ X^i_\delta\ \omega(X^0,\dots,\hat{X}^i,\dots,X^p)$$

$$+\sum_{i<j}(-1)^{j-i}\omega([X^i,X^j],X^0,\dots,\hat{X}^i,\dots,\hat{X}^j,\dots,X^p).$$

This definition differs from [**?**, Proposition 3.11] by a factor of $\frac{1}{p+1}$, but makes sense in any characteristic. For a 2-form ω we obtain

$$(X,Y,Z)d\omega_m = X\cdot(Y,Z)\omega - Y\cdot(X,Z)\omega + Z\cdot(X,Y)\omega - ([X,Y],Z)\omega - ([Y,Z],X)\omega + ([X,Z],Y)\omega$$

If B is a smooth vector bundle over a real manifold M, one can still define differential forms $\Omega^r(B)$, but the exterior differential d makes sense only if $B = M\times E$ is trivial. In this case we write $\Omega^r(M\times E) = \Omega^r(M,E)$.

For a complex manifold M, the complexified tangent space splits into the holomorphic and anti-holomorphic tangent space. The complexified smooth differential forms have a splitting

$$\Omega^r(M,\mathbf{C}) = \sum_{p+q=r}\Omega^{p,q}(M)$$

into (p,q)-forms. Accordingly, the differential

$$d:\Omega^r(M,\mathbf{C})\to\Omega^{r+1}(M,\mathbf{C})$$

splits as $d=\partial+\overline{\partial}$, with

$$\partial:\Omega^{p,q}(M)\to\Omega^{p+1,q}(M),\quad \overline{\partial}:\Omega^{p,q}(M)\to\Omega^{p,q+1}(M).$$

If B is a holomorphic vector bundle over a complex manifold M, the anti-linear part $\overline{\partial}$ of the exterior differential is still well-defined.

In general, for a G-bundle P over $M=P/G$ let $P\times_G^{\mathrm{ad}}\mathfrak{g}$ denote the adjoint $\mathfrak{g}$-bundle of P. The space $\Omega^1(P)$ of all G-connexions on P over M is an affine space, with tangent space

$$T_A\Omega^1(P) = \Omega^1(P\overset{\mathrm{ad}}{\underset{G}{\times}}\mathfrak{g})$$

at any $A\in\Omega^1(P)$. We write $\Omega^1(M,G)$ for the space of connexions on the trivial G-bundle $M\times G$, with tangent spaces $\Omega^1(M,\mathfrak{g})$.

Let $P\times_G^{\pi}E$ be an associated vector bundle. Let $m\in M$ and $u\in T_mM$. Choose $p\in P$ with $m=[p]=\pi(p)$. For any connexion $A\in\Gamma^1(P)$ the horizontal subspace T_p^AP yields an isomorphism

$$T_p\pi : T_p^AP\to T_mM.$$

Hence there exists a unique horizontal tangent vector $u^A\in T_p^AP$ such that $T_p(\pi)u^A = u$. Given a section Φ, apply u^A to the smooth function $\tilde{\Phi}:P\to E$ we obtain $u^A\cdot d_p\tilde{\Phi}\in E$. Then

$$u\cdot d_m^A\Phi = [p,u^A\cdot d_p\tilde{\Phi}]$$

is independent of the choice of p [**?**, Section III.1, Lemma on p. 115], and we obtain the **covariant differential** as a 1-form $d^A\Phi$. The map

$$d^A:\Omega^0(P\overset{\pi}{\underset{G}{\times}}E)\to\Omega^1(P\overset{\pi}{\underset{G}{\times}}E),\quad \Phi\mapsto d^A\Phi$$

satisfies the Leibniz rule

$$d^A(f\Phi) = df \wedge \Phi + f \cdot d^A\Phi$$

for all sections $\Phi \in \Omega^0(P \times_G^\pi E)$ and functions $f \in \Omega^0(M, \mathbf{K})$. On the other hand, given a vector field $X \in \Gamma(TM)$ we define the **covariant derivative** d_X^A acting on sections. The two notions are related by

$$X \cdot d^A\Phi = d_X^A \cdot \Phi.$$

The value at a given point $m \in M$ is denoted by

$$(X \cdot d^A\Phi)_m = (d_X^A \cdot \Phi)_m = X_m \cdot (d_m^A\Phi)$$

Thus there is a canonical mapping

$$\Omega^1(P) \times \Omega^0(P \overset{\pi}{\underset{G}{\times}} E) \to \Omega^1(P \overset{\pi}{\underset{G}{\times}} E), \quad (A, \Phi) \mapsto d_A\Phi.$$

If $P \times_H E \to M$ is a holomorphic vector bundle one can also consider the anti-linear part

$$\overline{\partial}^A : \Omega^0(P \underset{H}{\times} E) \to \Omega^{0,1}(P \underset{H}{\times} E).$$

Proposition 1.4.2. *For any tangent metric* $\boldsymbol{g}$ *there is a unique* **Levi-Civita connexion** $\eth\boldsymbol{g}$ *which satisfies*

$$d_X\boldsymbol{g}(Y,Z) = \boldsymbol{g}(d_X^{\eth\boldsymbol{g}}Y, Z) + \boldsymbol{g}(Y, d_X^{\eth\boldsymbol{g}}Z)$$

and is torsion-free, i.e.,

$$d_X^{\eth\boldsymbol{g}}Y - d_Y^{\eth\boldsymbol{g}}X = [X,Y].$$

It is given by

$$2\boldsymbol{g}(d_X^{\eth\boldsymbol{g}}Y, Z) = d_X\boldsymbol{g}(Y,Z) + d_Y\boldsymbol{g}(Z,X) - d_Z\boldsymbol{g}(Y,X) + \boldsymbol{g}(Z,[X,Y]) - \boldsymbol{g}(Y,[X,Z]) - \boldsymbol{g}([Y,Z],X)$$

Proof. Combining the two properties yields

$$d_X\boldsymbol{g}(Y,Z)+d_Y\boldsymbol{g}(Z,X)-d_Z\boldsymbol{g}(Y,X) = \boldsymbol{g}(d_X^{\eth\boldsymbol{g}}Y,Z)+\boldsymbol{g}(Y,d_X^{\eth\boldsymbol{g}}Z)+\boldsymbol{g}(d_Y^{\eth\boldsymbol{g}}Z,X)+\boldsymbol{g}(Z,d_Y^{\eth\boldsymbol{g}}X)-\boldsymbol{g}(d_Z^{\eth\boldsymbol{g}}Y,X)-\boldsymbol{g}(Y,d_Z^{\eth\boldsymbol{g}}X)$$

$$= \boldsymbol{g}(d_X^{\eth\boldsymbol{g}}Y+d_Y^{\eth\boldsymbol{g}}X,Z)+\boldsymbol{g}(Y,d_X^{\eth\boldsymbol{g}}Z-d_Z^{\eth\boldsymbol{g}}X)+\boldsymbol{g}(d_Y^{\eth\boldsymbol{g}}Z-d_Z^{\eth\boldsymbol{g}}Y,X) = \boldsymbol{g}(2d_X^{\eth\boldsymbol{g}}Y+[Y,X],Z)+\boldsymbol{g}(Y,[X,Z])+\boldsymbol{g}([Y,Z],X)$$

□

The definition of the Levi-Civita connexion $\eth\boldsymbol{g}$ is analogous to the exterior derivative

$$d\omega(X,Y,Z) = d_X\omega(Y,Z) - d_Y\omega(X,Z) + d_Z\omega(X,Y) - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X)$$

$$= d_X\omega(Y,Z) + d_Y\omega(Z,X) - d_Z\omega(Y,X) + \omega(Z,[X,Y]) - \omega(Y,[X,Z]) - \omega([Y,Z],X)$$

of a 2-form ω.

Now let $P = N \times_\sim^\beta H$ be a cocycle H-bundle on $M = N/R$. Then the adjoint bundle is

$$(N \overset{\beta}{\underset{\sim}{\times}} H) \overset{\mathrm{ad}}{\underset{G}{\times}} \mathfrak{h} = N \overset{\beta,\mathrm{ad}}{\underset{R}{\times}} \mathfrak{h}.$$

Hence the affine space $\Omega^1(N \times_\sim^\beta H)$ of all H-connexions on $N \times_\sim^\beta H$ has the tangent space

$$T_A(\Omega^1(N \overset{\beta}{\underset{\sim}{\times}} H)) = \Omega^1(N \overset{\beta,\mathrm{ad}}{\underset{R}{\times}} \mathfrak{h}$$

at any $A \in \Omega^1(N \times_\sim^\beta H)$.

1.4.1 Covered manifolds

For covered manifolds, connexions are constructed as follows. A connexion A on $\mathcal{V} \times^{\beta}_{\sim} E$ is given by the covariant differential

$$d_A : \Omega^0(\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} E) \to \Omega^1(\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} E).$$

Given $v \in T_m M$ and a section $\Phi \in \Omega^0(\mathcal{V} \times^{\beta}_{\sim} E)$ we have local representatives $(v \cdot d^A_m \Phi)^a \in E$ for $m \in V_a$.

Proposition 1.4.3. *A family $m \mapsto A^a_m$ of $\mathfrak{gl}(E)$-valued 1-forms on V_a such that*

$$A^a = \beta^a_b\Big(d\beta^b_a + A^b\ \beta^b_a\Big), \quad A^a_m = \beta^a_b(m)\Big(d_m\beta^b_a + A^b_m\ \beta^b_a(m)\Big)$$

for $m \in V_a \cap V_b$, as an identity of linear functionals $T_m M \to \mathfrak{gl}(E)$, defines a (global) connexion A on $\mathcal{V} \times^{\beta}_{\sim} E$ with covariant derivative

$$(v \cdot d^A \Phi)^a_m = v \cdot d_m \Phi^a + (v \cdot A^a_m)\Phi^a_m.$$

Here $v \cdot d_m \Phi^a \in E$ and $v \cdot A^a_m \in \mathcal{L}(E)$. For $E = \mathbf{C}$, a family of 1-forms A^a_m on V_a, satisfying

$$A^a_m - A^b_m = \frac{d_m \beta^b_a}{\beta^b_a(m)} = d_m \log \beta^b_a$$

for all $m \in V_a \cap V_b$, yields a global connexion A on $\mathcal{V} \times^{\beta}_{\sim} \mathbf{C}$ with covariant derivative (**??**).

Proof. In order to define a global connexion, we need to check the compatibility relation

$$(v \cdot d^A_m \Phi)^a = \beta^a_b(m)(v \cdot d^A_m \Phi)^b$$

for $m \in V_a \cap V_b$ and $v \in T_m M$. The condition (**??**) becomes

$$v \cdot A^a_m = \beta^b_a(m)\Big(v \cdot d_m \beta^a_b + (v \cdot A^b_m)\ \beta^a_b(m)\Big)$$

with $v \cdot d_m \beta^a_b \in \mathcal{L}(E)$. Since $v \cdot d_m \Phi^b = v \cdot d_m(\beta^b_a \Phi^a) = (v \cdot d_m \beta^b_a)\Phi^a(m) + \beta^b_a(m)(v \cdot d_m \Phi^a)$ by the product rule, we have

$$\beta^a_b(m)(v \cdot d_m \Phi^b) = \beta^a_b(m)(v \cdot d_m \beta^b_a)\ \Phi^a(m) + v \cdot d_m \Phi^a.$$

Hence (**??**) implies

$$(v \cdot d^A \Phi)^a(m) = v \cdot d_m \Phi^a + (v \cdot A^a_m)\Phi^a(m) = v \cdot d_m \Phi^a + \beta^a_b(m)\Big(v \cdot d_m \beta^b_a + (v \cdot A^b_m)\ \beta^b_a(m)\Big)\Phi^a(m)$$

$$= v \cdot d_m \Phi^a + \beta^a_b(m)(v \cdot d_m \beta^b_a)\Phi^a(m) + \beta^a_b(m)(v \cdot A^b_m)\ \Phi^b(m) = \beta^a_b(m)\Big(v \cdot d_m \Phi^b + (v \cdot A^b_m)\Phi^b(m)\Big) = \beta^a_b(m)(v \cdot d^A \Phi)^b(m).$$

For $E = \mathbf{C}$, we have

$$\beta^a_b(m)\ A^b_m\ \beta^b_a(m) = A^b_m\ \beta^a_b(m)\ \beta^b_a(m) = A^b_m.$$

Thus (**??**) simplifies to (**??**). □

The space $\Omega^1(\mathcal{V} \times^{\beta}_{\sim} GL(E))$ of all connexions on $\mathcal{V} \times^{\beta}_{\sim} E$ is an affine space, with tangent space

$$T_A(\Omega^1(\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} GL(E)) = \Omega^1(\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} \mathfrak{gl}(E)).$$

In fact, let (A^a_1) and (A^a_2) be two connexions on $\mathcal{V} \times^{\beta}_{\sim} E$. Then

$$A^a := A^a_1 - A^a_2$$

is a smooth mapping $V_a \cap V_b \to \mathfrak{gl}(E)$ such that

$$A^a = \beta^a_b\ \Lambda^b\ \beta^b_a$$

on $V_a \cap V_b$. Thus (A^a) defines a global $\mathfrak{gl}(E)$-valued 1-form on M.

Proposition 1.4.4. *Let (χ_a) be a partition of unity subordinate to (V_a). Then the family*

$$A^a = \sum_c \chi_c \; \beta_a^c (d\beta_c^a)$$

defines a global connexion on $\mathcal{V} \times_{\sim}^{\beta} E$.

Proof. On $V_a \cap V_b \cap V_c$ we have

$$\beta_a^c(d\beta_c^a) = -(d\beta_a^c)\beta_c^a = -(d(\beta_a^b\beta_b^c))\beta_c^a = -(d\beta_a^b)\beta_b^c\beta_c^a - \beta_a^b(d\beta_b^c)\beta_c^a$$

$$= -(d\beta_a^b)\beta_b^a - \beta_a^b(d\beta_b^c)\beta_c^b\beta_b^a = \beta_a^b(d\beta_b^a) + \beta_a^b\beta_b^c(d\beta_c^b)\beta_b^a = \beta_a^b\Big(d\beta_b^a + \beta_b^c(d\beta_c^b)\beta_b^a\Big).$$

it follows that

$$A^a = \sum_c \chi_c\beta_a^c(d\beta_c^a) = \sum_c \chi_c\beta_a^b\Big(d\beta_b^a + \beta_b^c(d\beta_c^b)\beta_b^a\Big) = \beta_a^b\Big(d\beta_b^a + \sum_c \chi_c\beta_b^c(d\beta_c^b)\beta_b^a\Big) = \beta_a^b(d\beta_b^a + A^b\beta_b^a)$$

□

For the tangent bundle, a family $m \mapsto \mathbf{A}_m^a$ of $\mathfrak{gl}(L)$-valued 1-forms on V_a such that

$$\mathbf{A}^a = \frac{\partial\sigma_b}{\partial\sigma_a}\Big(d\frac{\partial\sigma_a}{\partial\sigma_b} + \mathbf{A}^b \; \frac{\partial\sigma_a}{\partial\sigma_b}\Big)$$

on $V_a \cap V_b$, defines a global tangent connexion $\mathbf{A}$ on $M = \mathcal{V}/R$, with covariant derivative

$$(v \cdot d^{\mathbf{A}} X)_m^a = v \cdot d_m X^a + (v \cdot \mathbf{A}_m^a) X_m^a.$$

Here $v \cdot d_m X^a \in L$ and $v \cdot \mathbf{A}_m^a \in \mathcal{L}(L)$.

If M is a complex manifold, we consider holomorphic vector bundles over M defined by holomorphic cocycles β_b^a.

Theorem 1.4.5. *Let M be a complex manifold, with a metric on (h^a) on $\mathcal{V} \times_{\sim}^{\beta} E$. Then the family*

$$(\eth h)_m^a := (h_m^a)^{-1} \; \partial_m h^a$$

of $(1,0)$-forms induces a (unique) connexion $\eth h$ on $\mathcal{V} \times_{\sim}^{\beta} E$ which satisfies

$$d_X(\xi|\eta) = (d_X^{\eth h}\xi|\eta) + (\xi|d_X^{\eth h}\eta)$$

for all real vector fields $X \in \Gamma_1(M_{\mathbf{R}})$, and the ('torsion-free') condition

$$\overline{\partial}^{\eth h}\Phi = 0$$

for all holomorphic sections $\Phi \in \Gamma(\mathcal{V} \times_{\sim}^{\beta} E)$. For $E = \mathbf{C}$, given a ^{0}metric (h^a) on the line bundle $\mathcal{V} \times_{\sim}^{\beta} \mathbf{C}$, the family

$$(\eth h)_m^a := \frac{\partial_m h^a}{h_m^a} = \partial_m \log h^a$$

of $(1,0)$-forms induces the Chern connexion $\eth h$ on $\mathcal{V} \times_{\sim}^{\beta} \mathbf{C}$.

Proof. Since we take the $\mathbf{C}$-linear Wirtinger derivative $\partial_m h^a$, it follows that $(\eth h)^a$ is a $(1,0)$-form with values in $\mathfrak{gl}(E)$. Since β_a^b is holomorphic in m we have $\partial_m \beta_a^{b*} = 0$ and $\partial_m \beta_a^b = d_m \beta_a^b$. Applying the product rule to $h_m^a = \beta_a^b(m)^* \; h_m^b \; \beta_a^b(m)$ we obtain

$$\partial_m h^a = \beta_a^b(m)^*\Big((\partial_m h^b)\beta_a^b(m) + h_m^b \; (\partial_m \beta_a^b)\Big) = \beta_a^b(m)^*\Big((\partial_m h^b)\beta_a^b(m) + h_m^b \; (d_m \beta_a^b)\Big).$$

It follows that

$$(\eth h)^a_m = (h^a_m)^{-1}\ \partial_m h^a = \Big(\beta^a_b(m)(h^b_m)^{-1}\beta^a_b(m)^*\Big)\ \beta^b_a(m)^*\ \Big((\partial_m h^b)\beta^b_a(m) + h^b_m\ (d_m\beta^b_a)\Big)$$

$$= \beta^a_b(m)\Big((h^b_m)^{-1}(\partial_m h^b)\beta^b_a(m) + d_m\beta^b_a\Big) = \beta^a_b(m)\Big((\eth h)^b_m\ \beta^b_a(m) + d_m\beta^b_a\Big).$$

Thus (**??**) is satisfied. For the second assertion, let $\Phi = (\Phi^a)$ be a holomorphic section. Then the Φ^a are holomorphic and hence $\overline{\partial}_m\Phi^a = 0$. It follows that

$$(v\cdot d^{\eth h}_m\Phi)^a = v\cdot d_m\Phi^a + (h^a_m)^{-1}(v\cdot\partial_m h^a)\ \Phi^a(m) = v\cdot\partial_m\Phi^a + (h^a_m)^{-1}(v\cdot\partial_m h^a)\ \Phi^a(m)$$

is $\mathbf{C}$-linear in v. Therefore the anti-linear part $(\overline{\partial}^{\eth h}_m\Phi)^a$ vanishes for all a and hence $\overline{\partial}^{\eth h}\Phi = 0$. □

For a tangent metric $(\boldsymbol{h}^a)$ the family

$$(\eth\boldsymbol{h})^a_m := \frac{\partial_m\boldsymbol{h}^a}{\boldsymbol{h}^a_m}$$

of $(1,0)$-forms induces the Chern connexion $\eth\boldsymbol{h}$ on the tangent bundle $\mathcal{V}\times^{\beta}_{\sim} L \equiv TM$. A complex manifold M endowed with a tangent ^{0}metric is called a **hermitian manifold**. For a cocycle description, endow L with an inner product $(\xi|\eta)$.

• **Jordan manifolds**

Example 1.4.6. For the holomorphic tangent bundle of $\mathbf{P}^1$, with metric (**??**), the general formula (**??**) yields the connexion 1-form

$$(\eth\boldsymbol{h})^0_z = \frac{\partial\boldsymbol{h}^0_z}{\boldsymbol{h}^0_z} = (1+z\overline{z})^2\ \frac{\partial}{\partial z}(1+z\overline{z})^{-2}dz = -2(1+z\overline{z})^2\ (1+z\overline{z})^{-3}\ \overline{z}\ dz = \frac{-2\overline{z}}{1+z\overline{z}}\ dz$$

of type $(1,0)$.

1.4.2 Homogeneous manifolds

We first construct some vector fields on $M = G/H$. Consider the left translation action

$$g^L g' := gg'$$

of G on itself. For $\gamma\in\mathfrak{g}$ define a vector field $\gamma^L\in\Gamma(TG)$ by

$$\gamma^L_g := (T_e g^L)\gamma = \gamma\cdot(T_e g^L) \in\ T_g G$$

Lemma 1.4.7. *The vector field γ^L on G is left-invariant, i.e. for each $g\in G$ the left translation g^L on G satisfies*

$$g^L_*\gamma^L = \gamma^L.$$

Proof. This follows from

$$(g^L_*\gamma^L)_{gg'} = (T_{g'}g^L)\gamma^L_{g'} = (T_{g'}g^L)(T_e g'^L)\gamma = T_e(g^L\circ g'^L)\gamma = T_e((gg')^L)\gamma = \gamma^L_{gg'}.$$

□

Consider the left translation action $g \mapsto g^\lambda$ of G on G/H given by

$$g^\lambda(g'H) := gg'H.$$

Then the canonical projection $\pi : G \to G/H$ satisfies

$$\pi \circ g^L = g^\lambda \circ \pi$$

for all $g \in G$. For $\gamma \in \mathfrak{g}$ define a vector field $\gamma^\lambda \in \Gamma(G/H)$ by

$$\gamma^\lambda_{gH} := (T_e g^\lambda)(T_e \pi)\gamma$$

Lemma 1.4.8. *The vector field γ^λ on G/H is left-invariant, i.e. for each $g \in G$ the left translation g^λ on G/H satisfies*

$$g^\lambda_* \gamma^\lambda = \gamma^\lambda.$$

Proof. This follows from

$$(g^\lambda_* \gamma^\lambda)_{gg'H} = (T_{g'H} g^\lambda)\gamma^\lambda_{g'H} = (T_{g'H} g^\lambda)(T_H {g'}^\lambda)(T_e\pi)\gamma = T_H(g^\lambda \circ {g'}^\lambda)(T_e\pi)\gamma = T_H((gg')^\lambda)(T_e\pi)\gamma = \gamma^\lambda_{gg'}.$$

□

Lemma 1.4.9. *For all $\gamma \in \mathfrak{g}$ we have*

$$\pi_* \gamma^L = \gamma^\lambda,$$

i.e.,

$$\gamma^\lambda_{gH} = (T_g\pi)\gamma^L_g.$$

Proof.

$$\gamma^\lambda_{gH} := (T_e g^\lambda)(T_e \pi\gamma) = (T_g\pi)(T_e g^L \gamma)$$

□

Lemma 1.4.10. *The left-invariant vector field $\tilde{\gamma}$ satisfies*

$$(d_{\gamma^L}\ f)_g = \partial^0_t\ f_{g\exp(t\gamma)}$$

Proof.

$$\partial^0_t\ f_{g\exp(t\gamma)} = \partial^0_t\ (f \circ g^L)(\exp(t\gamma)) = d_e(f \circ g^L)\gamma = (d_g f)(T_e g^L)\gamma = \gamma^L_g \cdot d_g f = (d_{\gamma^L}\ f)_g.$$

□

Let $M = G/H$. Then we have a commuting diagram

$$\begin{array}{ccc}
T_{gH}M & \xleftarrow{T_H \underset{\sim}{g}} & T_H M \\
\uparrow{\scriptstyle T_g\pi} & & \uparrow{\scriptstyle T_e\pi} \\
T_g G & \xleftarrow{T_e L_g} & T_e G \\
\uparrow & & \uparrow \\
T^\Theta_g G & \xleftarrow[T_e L_g]{} & \mathfrak{m}
\end{array}$$

Lemma 1.4.11. *For $\eta \in \mathfrak{h}$ we have*

$$d_{\eta^L}\ \tilde{\Phi} = -\eta^\pi\ \tilde{\Phi}$$

Proof. It follows from (**??**) and Lemma (**??**) that

$$(d_{\eta^L}\ \tilde{\Phi})_g = \partial_t^0\ \tilde{\Phi}_{g\exp(t\eta)} = \partial_t^0\ \exp(t\eta)^{-\pi}\ \tilde{\Phi}_g = -\eta^\pi\ \tilde{\Phi}_g.$$

□

Consider a vector space splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

which is Ad_H-invariant. Thus $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$, but not necessarily $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. For $\gamma \in \mathfrak{g}$ we write $\gamma_\mathfrak{h}$ and $\gamma_\mathfrak{m}$ for the projections. Given an H-module (E,π) we consider the corresponding infinitesimal action

$$\eta^\pi := \partial_t^0\ \exp(t\eta)^\pi$$

for all $\eta \in \mathfrak{h}$.

Proposition 1.4.12. *The left-invariant connexion A associated with a splitting* (**??**) *has the covariant derivative*

$$(d^A_{\gamma^\lambda}\ \Phi)^\sim = d_{\gamma^L}\tilde{\Phi} + \gamma^\pi_\mathfrak{h}\ \tilde{\Phi}, \quad (\gamma^\lambda \cdot d^A\Phi)^\sim_g = \gamma^L_g \cdot d_g\tilde{\Phi} + \gamma^\pi_\mathfrak{h}\ \tilde{\Phi}_g$$

for all $\gamma \in \mathfrak{g}$.

Proof. For $M = G/H$ every tangent vector in $T_{gH}M$ can be written as

$$\gamma^\lambda_{gH} = (\gamma \cdot T_e\pi) \cdot (T_H g^\lambda)$$

for a uniquely determined $\gamma \in \mathfrak{m}$. Then $\gamma^L_g = \gamma \cdot (T_e g^L)$ belongs to $(T_e g^L)\mathfrak{m} = T^A_g G$, since the connexion is left-invariant, and the projection is

$$(T_g\pi)\gamma^L_g = (T_g\pi)(T_e g^L)\gamma = (T_H g^\lambda)(T_e\pi)\gamma = \gamma^\lambda_{gH}.$$

Therefore $\gamma^L_g = (\gamma^\lambda_{gH})^A$ is the horizontal lift of γ^λ. Now (**??**) implies

$$(d^A_{\gamma^\lambda}\ \Phi)_{gH} = \gamma^\lambda_{gH} \cdot d^A_{gH}\Phi = [g, (\gamma^\lambda_{gH})^A \cdot d_g\tilde{\Phi}] = [g, \gamma^L_g \cdot d_g\tilde{\Phi}] = [g, (d_{\gamma^L}\tilde{\Phi})_g].$$

Equivalently, we have $(d^A_{\gamma^\lambda}\ \Phi)^\sim = d_{\gamma^L}\tilde{\Phi}$ for all $\gamma \in \mathfrak{m}$. This implies

$$(d^A_{\gamma^\lambda}\ \Phi)^\sim = d_{\gamma^L_\mathfrak{m}}\ \tilde{\Phi} = d_{\gamma^L}\ \tilde{\Phi} - d_{\gamma^L_\mathfrak{h}}\ \tilde{\Phi}$$

for all $\gamma \in \mathfrak{g}$, since, by (**??**), we have $\gamma^\lambda = 0$ on M for all $\gamma \in \mathfrak{h} = \text{Ker}\ T_e\pi$ and hence both sides of (**??**) vanish. Applying (**??**) to $\eta := \gamma_\mathfrak{h}$, the assertion follows. □

1.5 2-Geometry: Curvature

For every G-connexion $A \in \Omega^1(P)$ the covariant derivative (**??**) has a canonical extension

$$d^A : \Omega^j(P \overset{\pi}{\underset{G}{\times}} E) \to \Omega^{j+1}(P \overset{\pi}{\underset{G}{\times}} E)$$

for $j \geq 0$, satisfying a graded Leibniz rule

$$d^A(\vartheta \wedge \Phi) = d\vartheta \wedge \Phi + (-1)^i \vartheta \wedge d^A\Phi$$

for all $\Phi \in \Omega^j(P \times^\pi_G E)$ and $\vartheta \in \Omega^i(M, \mathbf{K})$. Thus there is a canonical mapping

$$\Omega^1(P) \times \Omega^j(P \overset{\pi}{\underset{G}{\times}} E) \to \Omega^{j+1}(P \overset{\pi}{\underset{G}{\times}} E), \quad (A, \Phi) \mapsto d_A\Phi.$$

If $P \times_H E \to M$ is a holomorphic vector bundle one can also consider the anti-linear part

$$\overline{\partial}^A : \Omega^{p,q}(P \underset{H}{\times} E) \to \Omega^{p,q+1}(P \underset{H}{\times} E).$$

Proposition 1.5.1. *The square*

$$d^A d^A : \Omega^0(P \underset{H}{\times} E) \to \Omega^2(P \underset{H}{\times} E)$$

can be written as

$$d^A(d^A\Phi) = (d^A A) \wedge \Phi.$$

for the **curvature 2-form** $d^A A \in \Omega^2(P \times_H^{\mathrm{ad}} \boldsymbol{h})$ *More generally, the square*

$$d^A d^A : \Omega^p(P \underset{H}{\times} E) \to \Omega^{p+2}(P \underset{H}{\times} E)$$

is given by

$$d^A d^A(\vartheta \otimes \Phi) = (d^A A) \wedge (\vartheta \otimes \Phi)$$

Proof. Using $\otimes$ also for multiplication by functions, we have

$$d^A(d^A(f \otimes \Phi)) = d^A(df \otimes \Phi + f \otimes d^A\Phi) = d(df) \otimes \Phi - df \wedge d^A\Phi + df \wedge d^A\Phi + f \otimes (d^A d^A \Phi) = f \otimes (d^A d^A)\Phi$$

since $ddf = 0$. Thus $d^A d^A$ commutes with multiplication by functions f and is therefore a multiplication by a 2-form with values in the bundle $P \times_H^{\mathrm{ad}} \mathfrak{h}$. □

For a matrix group the curvature is given by

$$d^A A = dA + [A \wedge A].$$

Thus the curvature depends in a non-linear, quadratic manner on A. For abelian groups. we have $d^A A = dA$.

For a holomorphic vector bundle with metric h we have the Chern connexion $\eth h$, with covariant derivative $d^{\eth h}$ and curvature $d^{\eth h}\eth h$.

1.5.1 Covered manifolds

For a covered manifold M this looks as follows. For any vector field $X \in \Gamma(TM)$ we put

$$(X \cdot d^A\Phi)^a_m := (X_m \cdot d^A\Phi)^a.$$

Then the family $(X \cdot d^A\Phi)^a$ of smooth maps $V_a \to E$ is a localized section. The curvature of A is defined by

$$d^A_X(d^A_Y\Phi) - d^A_Y(d^A_X\Phi) - d^A_{[X,Y]}\Phi = d^A A(X,Y) \cdot \Phi$$

Proposition 1.5.2. *The curvature of* (A^a) *is given by the family*

$$(u,v) \cdot (d^A A)^a_m := v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot d_m A^a) + [u \cdot A^a_m, v \cdot A^a_m].$$

of $\mathfrak{gl}(E)$*-valued 2-forms. Here* $[S,T] = ST - TS$ *is the commutator in* $\mathfrak{gl}(E)$*. For* $E = \mathbf{C}$ *the curvature of* (A^a) *simplifies to*

$$(u,v) \cdot (d^A A)^a_m := v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot d_m A^a) = (dA^a)_m(u,v).$$

Proof. Let X, Y be smooth vector fields on M. Then

$$(Y_m \cdot d^A_m\Phi)^a = Y_m \cdot d_m\Phi^a + (Y_m \cdot A^a_m)\Phi^a_m$$

and hence

$$(X_m \cdot d^A_m(Y \cdot d^A\Phi))^a = X_m \cdot d_m(Y_m \cdot d^A_m\Phi)^a + (X_m \cdot A^a_m)(Y_m \cdot d^A_m\Phi)^a$$

$$= X_m \cdot d_m\Big(Y_m \cdot d_m\Phi^a + (Y_m \cdot A^a_m)\Phi^a_m\Big) + (X_m \cdot A^a_m)\Big(Y_m \cdot d_m\Phi^a + (Y_m \cdot A^a_m)\Phi^a_m\Big)$$
$$= (X_m \cdot d_m Y)\cdot d_m\Phi^a + (X_m, Y_m)d^2_m\Phi^a + ((X_m \cdot d_m Y)\cdot A^a_m)\Phi^a_m + (Y_m \cdot (X_m \cdot d_m A^a))\Phi^a_m$$
$$+(X_m \cdot A^a_m)(Y_m \cdot A^a_m)\Phi^a_m + (Y_m \cdot A^a_m)(X_m \cdot d_m\Phi^a) + (X_m \cdot A^a_m)(Y_m \cdot d_m\Phi^a).$$

For the commutator we obtain, using symmetry of the second derivative $d^2_m\Phi^a$ and the symmetry of the last two summands,

$$(X_m \cdot d^A_m(Y \cdot d^A\Phi))^a - (Y_m \cdot d^A_m(X \cdot d^A\Phi))^a$$
$$= (X_m \cdot d_m Y - Y_m \cdot d_m X)\cdot d_m\Phi^a + ((X_m \cdot d_m Y - Y_m \cdot d_m X)\cdot A^a_m)\Phi^a_m$$
$$+(Y_m \cdot (X_m \cdot d_m A^a) - X_m \cdot (Y_m \cdot d_m A^a))\Phi^a_m + \Big((X_m \cdot A^a_m)(Y_m \cdot A^a_m) - (Y_m \cdot A^a_m)(X_m \cdot A^a_m)\Big)\Phi^a_m$$
$$= [X,Y]_m \cdot d_m\Phi^a + ([X,Y]_m \cdot A^a_m)\Phi^a_m$$
$$+(Y_m \cdot (X_m \cdot d_m A^a) - X_m \cdot (Y_m \cdot d_m A^a))\Phi^a_m + [X_m \cdot A^a_m, Y_m \cdot A^a_m]\Phi^a_m.$$

For a line bundle, the commutator part $[u \cdot A^a_m, v \cdot A^a_m]$ vanishes. □

Proposition 1.5.3. *For a holomorphic metric vector bundle* $\mathcal{V} \times^{\beta}_{\sim} E$ *the Chern connexion* $((\eth h)^a)$ *satisfies*

$$\partial(\eth h)^a = (\eth h)^a \wedge (\eth h)^a$$

and

$$\overline{\partial}(\eth h)^a = (d^{\eth h}\eth h)^a.$$

In other words, the exterior differential $d^A = \partial^A + \overline{\partial}^A$ *has the* $(2,0)$*-part* $(\eth h)^a \wedge (\eth h)^a$ *and the* $(1,1)$*-part is given by the curvature* $(d^{\eth h}\eth h)^a$.

Proof. Consider first the wedge product. Since

$$(Y \cdot (\eth h)^a)_m = (h^a_m)^{-1}(Y_m\ \partial_m h^a)$$

we have

$$(\eth h)^a_m \wedge (\eth h)^a_m = [X_m \cdot (\eth h)^a_m, Y_m \cdot (\eth h)^a_m] = [(h^a_m)^{-1}(X_m\ \partial_m h^a), (h^a_m)^{-1}(Y_m\ \partial_m h^a)]$$
$$= (h^a_m)^{-1}(X_m\ \partial_m h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) - (h^a_m)^{-1}(Y_m\ \partial_m h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a).$$

and hence

$$h^a_m((\eth h)^a_m \wedge (\eth h)^a_m) = (X_m\ \partial_m h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) - (Y_m\ \partial_m h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a).$$

Therefore $(\eth h)^a \wedge (\eth h)^a$ is a differential form of type $(2,0)$ since both X and Y involve holomorphic Wirtinger derivatives. Consider now the exterior differential

$$(X,Y)d\Theta = d_X(Y \cdot (\eth \boldsymbol{h})) - d_Y(X \cdot (\eth \boldsymbol{h})) - [X,Y]\cdot(\eth \boldsymbol{h})$$

for vector fields X, Y. The product and quotient rules imply

$$d_X(Y \cdot (\eth h)^a)_m = (h^a_m)^{-1}(d_X \cdot (Y_m\ \partial_m h^a)) - (h^a_m)^{-1}(d_X h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a).$$

Therefore

$$h^a_m\, d_X(Y\cdot(\eth h)^a)_m = d_X\cdot(Y_m\,\partial_m h^a) - (d_X h^a)(h^a_m)^{-1}(Y_m\,\partial_m h^a) = (X_m d_m Y)(\partial_m h^a) + Y_m(X_m d_m \partial_m h^a) - (d_X h^a)(h^a_m)^{-1}(Y_$$

It follows that

$$h^a_m\ ((X,Y)d\Theta) = (X_m d_m Y - Y_m d_m X)\ \partial_m h^a + Y_m(X_m d_m\partial_m h^a) - X_m(Y_m d_m\partial_m h^a)$$

$$-(d_X h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) + (d_Y h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a) - [X,Y]_m\ \partial_m h^a$$

$$= Y_m(X_m d_m \partial_m h^a) - X_m(Y_m d_m \partial_m h^a) - (X_m d_m h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) + (Y_m d_m h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a),$$

since the first and last terms cancel. Subtracting (**??**) we obtain the curvature

$$h^a_m\ ((X,Y)\Omega) = h^a_m\ ((X,Y)d\Theta - (\eth h)^a_m \wedge (\eth h)^a_m)$$

$$= Y_m(X_m d_m \partial_m h^a) - X_m(Y_m d_m \partial_m h^a) - (X_m \overline{\partial}_m h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) + (Y_m \overline{\partial}_m h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a).$$

Finally, the second holomorphic derivatives $Y_m(X_m\ \partial_m\partial_m h^a) = X_m(Y_m\ \partial_m\partial_m h^a)$ vanish by Schwarz' theorem. Therefore

$$h^a_m\ ((X,Y)\Omega^a)$$

$$= Y_m(X_m \overline{\partial}_m \partial_m h^a) - X_m(Y_m \overline{\partial}_m \partial_m h^a) - (X_m \overline{\partial}_m h^a)(h^a_m)^{-1}(Y_m\ \partial_m h^a) + (Y_m \overline{\partial}_m h^a)(h^a_m)^{-1}(X_m\ \partial_m h^a).$$

It follows that Ω is a differential form of type $(1,1)$, involving only mixed derivatives. In summary, $d(\eth h)$ has the $(1,1)$-part $d^{\eth h}\eth h$ and the $(2,0)$-part $(\eth h) \wedge (\eth h)$. Since $(\eth h)$ is of type $(1,0)$, $d(\eth h)^a$ has no $(0,2)$-part, and the assertion follows. □

Proposition 1.5.4. *For hermitian holomorphic line bundles the curvature* $(1,1,)$*-form* $(d^{\eth h}\eth h)^a$ *is closed.*

Proof. The curvature is given by

$$(d^{\eth h}\eth h)^a = \overline{\partial}(\eth h)^a = \overline{\partial}\partial \log h^a.$$

Since $\overline{\partial}^2 = \partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$, it follows that

$$d(\overline{\partial}\partial \log h^a) = (\overline{\partial} + \partial)(\overline{\partial}\partial \log h^a) = \overline{\partial}\overline{\partial}(\partial \log h^a) + \partial\overline{\partial}(\partial \log h^a) = -\overline{\partial}\partial\partial(\partial \log h^a) = 0$$

□

1.5.2 Homogeneous manifolds

Consider the invariant connexions on homogeneous vector bundles over G/H given by a splitting (**??**) of the Lie algebra $\mathfrak{g}$.

Proposition 1.5.5. *For* $\gamma, \delta \in \mathfrak{m}$ *the curvature is given by the 'multiplication operator'*

$$(d^A A(\underset{\sim}{\gamma}, \underset{\sim}{\delta})\ \Phi)^\sim = -[\gamma,\delta]^{\pi}_{\mathfrak{h}}\ \tilde{\Phi}.$$

Proof. For $\gamma, \delta \in \mathfrak{m}$ we have vanishing $\mathfrak{h}$-projection. Hence (**??**) implies

$$(d^A_{\underset{\sim}{\gamma}}(d^A_{\underset{\sim}{\delta}}\ \Phi) - d^A_{\underset{\sim}{\delta}}(d^A_{\underset{\sim}{\gamma}}\ \Phi) - d^A_{[\underset{\sim}{\gamma},\underset{\sim}{\delta}]}\ \Phi)^\sim$$

$$= d_{\tilde{\gamma}}\ \widetilde{d^A_{\underset{\sim}{\delta}}\ \Phi} - d_{\tilde{\gamma}}\ \widetilde{d^A_{\underset{\sim}{\gamma}}\ \Phi} - d_{[\tilde{\gamma},\tilde{\delta}]}\tilde{\Phi} - [\gamma,\eta]^{\pi}_{\mathfrak{h}}\tilde{\Phi}$$

$$= d_{\tilde{\gamma}}\ d_{\tilde{\delta}}\ \tilde{\Phi} - d_{\tilde{\gamma}}\ d_{\tilde{\gamma}}\ \tilde{\Phi} - d_{[\tilde{\gamma},\tilde{\delta}]}\tilde{\Phi} - [\gamma,\eta]^{\pi}_{\mathfrak{h}}\tilde{\Phi} = -[\gamma,\eta]^{\pi}_{\mathfrak{h}}\tilde{\Phi}.$$

Here we used that $\pi_*\tilde{\gamma} = \underset{\sim}{\gamma}$ and $\pi_*\tilde{\delta} = \underset{\sim}{\delta}$ implies $\pi_*[\tilde{\gamma},\tilde{\delta}] = [\underset{\sim}{\gamma},\underset{\sim}{\delta}]$, so that $[\tilde{\gamma},\tilde{\delta}]$ is a horizontal lift of $[\underset{\sim}{\gamma},\underset{\sim}{\delta}]$. □

Chapter 2

Classical Phase Spaces

2.1 Symplectic Manifolds and Kähler Manifolds

A 2-form $\omega \in \Omega^2(M)$ on a smooth manifold M is called **symplectic** if $d\omega = 0$ and ω is non-degenerate, i.e. for each $m \in M$ the canonical map

$$\omega_m : T_m M \to T_m^* M,$$

arising as a special case of (**??**), is a linear isomorphism. Alternatively (in the finite-dimensional case), $\omega_m(u, v) = 0$ for all $v \in T_m M$ implies $u = 0$. A **symplectic manifold** is a manifold M endowed with a symplectic 2-form ω. Then $\dim M = 2n$ is even. The **Liouville measure** is defined by the $2n$-form

$$\frac{1}{n!}\omega^n.$$

Proposition 2.1.1. *Let Q be a real manifold (configuration space). Then the cotangent bundle*

$$M = T^*Q$$

is a symplectic manifold (phase space), with symplectic form

$$\omega_{x,\xi}(\dot{x}, \dot{\xi}, \dot{y}, \dot{\eta}) = \dot{x}\dot{\eta} - \dot{y}\dot{\xi}$$

for all $\dot{x}, \dot{y} \in T_x Q$ and $\dot{\xi}, \dot{\eta} \in T_x^* Q$

Proof. Let $\pi : T^*Q \to Q$ denote the canonical projection. Then $T_{x,\xi}\pi : T_{x,\xi}(T^*Q) \to T_x Q$. Define a global 1-form $\vartheta \in \Omega^1(T^*Q)$ by

$$\vartheta_{x,\xi} v := \xi((T_{x,\xi}\pi)v).$$

for all $v \in T_{x,\xi}(T^*Q)$. Thus we apply $\xi \in T_x^* Q$ to $(T_{x,\xi}\pi)v \in T_x Q$. Then

$$\omega := d\vartheta$$

is closed, since $d^2 = 0$, and non-degenerate. □

The symplectic manifold T^*Q is given in its **real polarization**. We will work instead with **complex polarizations**. This is crucial for harmonic analysis but also quantum field theory.

Lemma 2.1.2. *Let $(M, J, \boldsymbol{h})$ be a hermitian complex manifold. Then*

$$\omega_m(u + \overline{u}, v + \overline{v}) := \frac{\boldsymbol{h}_m(u, v) - \boldsymbol{h}_m(v, u)}{2i}$$

is a (not necessarily closed) non-degenerate 2-form $\omega \in \Omega^2(M, \mathbf{R})$*, satisfying*

$$\omega_m(J_m X, J_m Y) = \omega_m(X, Y)$$

for all $X, Y \in T_m^{\mathbf{R}} M$*.*

Proof. Since $J(u + \overline{u}) = iu + \overline{iu}$ we have

$$\omega_m(J_m(u + \overline{u}), J_m(v + \overline{v})) = \omega_m(iu + \overline{iu}, iv + \overline{iv}) = \frac{\boldsymbol{h}_m(iu, iv) - \boldsymbol{h}_m(iv, iu)}{2i}$$

$$= \frac{\boldsymbol{h}_m(u, v) - \boldsymbol{h}_m(v, u)}{2} = \omega_m(u + \overline{u}, v + \overline{v})$$

□

Define a Riemannian metric $\boldsymbol{g}$ on M by

$$\boldsymbol{g}_m(X, Y) := \omega_m(J_m X, Y).$$

Then

$$\boldsymbol{g}_m(u+\overline{u}, v+\overline{v}) = \omega_m(J_m(u+\overline{u}), v+\overline{v}) = \omega_m(iu+\overline{iu}), v+\overline{v}) = \frac{\boldsymbol{h}_m(iu, v) + \boldsymbol{h}_m(v, iu)}{2i} = \frac{\boldsymbol{h}_m(u, v) + \boldsymbol{h}_m(v, u)}{2}.$$

Then

$$\boldsymbol{g}_m(u + \overline{u}, u + \overline{u}) = \boldsymbol{h}_m(u, u).$$

If $\boldsymbol{h}$ is a ^{0}metric (positive definite), it follows that $\boldsymbol{g}_m(X, X) > 0$ for all $0 \neq X \in T_m^{\mathbf{R}} M$. The hermitian metric $\boldsymbol{h}$ can be recovered from ω and $\boldsymbol{g}$ via

$$\boldsymbol{h}_m(u, v) = \boldsymbol{g}_m(u + \overline{u}, v + \overline{v}) + i\omega_m(u + \overline{u}, v + \overline{v}).$$

Thus on a symplectic manifold (M, ω) the formula (**??**) yields a 1-1 correspondence between almost complex structures J and Riemannian (pseudo)-metrics $\boldsymbol{g}$. An almost complex structure J on (M, ω) is called **compatible** if (**??**) is a positive-definite Riemannian metric. By Proposition **??** every symplectic manifold has a compatible almost complex structure, which however may not be integrable. This leads to the important

Definition 2.1.3. *The following equivalent definitions define a* ^{0}**Kähler manifold***:*

- *A symplectic manifold* (M, ω) *with a compatible almost complex structure* J *which is* **integrable** *(vanishing Nijenhuis tensor) and hence, by the Newlander-Nirenberg theorem, is a complex structure.*
- *A* ^{0}*hermitian manifold* M *such that the resulting 2-form* ω *is* **closed**
- *A* ^{0}*-hermitian manifold such that the tangent Chern connexion* $\eth\boldsymbol{h}$ *and the Levi-Civita connexion* $\eth\boldsymbol{g}$ *coincide (after proper identification)*

Example 2.1.4. For $Q = \mathbf{R}^n$ we take $M = \mathbf{C}^n$ with it standard complex structure J. The hermitian metric

$$\boldsymbol{h}_z(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 1$$

introduced in (**??**) leads to

$$\omega_z(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = \omega_z(i\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big), \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}) = \frac{1}{2i}(\boldsymbol{h}_z\Big(i\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) - \boldsymbol{h}_z\Big(\frac{\partial}{\partial z}, i\frac{\partial}{\partial z}\Big)) = \frac{1}{2i}(2i) = 1.$$

It follows that $\boldsymbol{h}$ induces the symplectic form

$$\omega = dp_j \wedge dq^j,$$

when we identify $q = x$ and $p = y$. In differential form language we have

$$\omega = dy \wedge dx = \frac{dz - d\overline{z}}{2i} \wedge \frac{dz \wedge d\overline{z}}{2} = \frac{1}{4i}(dz \wedge d\overline{z} - d\overline{z} \wedge dz) = \frac{1}{2i} dz \wedge d\overline{z}.$$

Thus $\boldsymbol{h}_z$ corresponds to the $(1,1)$-form $dz \wedge d\overline{z}$.

• Jordan manifolds

The Bergman metric $\boldsymbol{h}_m(u,v) = \text{tr } D(B_{z,z}^{-1}u, v)$ is positive definite and we obtain a symplectic form

$$\omega_m(u + \overline{u}, v + \overline{v}) := \frac{\boldsymbol{h}_m(u,v) - \boldsymbol{h}_m(v,u)}{2i}$$

which is closed, as will be shown later. Hence the Jordan manifolds

$$\check{Z} \subset Z \subset \hat{Z}$$

are Kähler manifolds.

• Restricted Grassmannian

Proposition 2.1.5. *On the space $\mathcal{S}$ of symmetries the imaginary symplectic form*

$$\omega = \text{tr} s \ ds \ ds = s_i^j \ ds_j^k \wedge ds_k i$$

is closed.

• Loop groups

For the (parallelizable) loop space $\Gamma(\mathbf{S}, G)$ the tangent space $\Gamma(\mathbf{S}, \mathfrak{g})$ has a class of hermitian Sobolev type metrics

$$(u|v)^k = \int_{\mathbf{S}} ds \ ((\Delta^k u)_s | v_s).$$

For $k = 0$ this gives the basic L^2-metric

$$(u|v)^0 = \int_{\mathbf{S}} ds \ (u_s | v_s).$$

For $k = 1/2$ one obtains a Kähler metric

$$(u|v)^{1/2} = \int_{\mathbf{S}} ds \ ((|D|u)_s | v_s)$$

with Kähler form

$$\omega_e(u,v) = \frac{1}{2\pi} \int_{\mathbf{S}} ds \ (u_s' | v_s).$$

There is also a 1-metric

$$(u|v)^1 = \int_{\mathbf{S}} ds \ ((\Delta u)_s | v_s) = \int_{\mathbf{S}} ds \ (u_s' | v_s')$$

• **Conformal blocks**

Proposition 2.1.6. *Let S be a compact oriented surface. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then the affine space $\Omega^1(S,G)$ of all connexions A on the trivial G-bundle $S\times G$ carries the symplectic form*

$$\omega_A(\dot A_1,\dot A_2)=\int\limits_S \mathrm{tr}[\dot A_1\wedge\dot A_2]$$

where $\dot A\in T_A(\Omega^1(S,G))=\Omega^1(S,\mathfrak{g})$.

Proof. We write

$$\Lambda=\lambda_i\otimes\gamma^i$$

for scalar 1-forms $\lambda_i\in\Omega^1(S)$ and a basis $\gamma^i\in\mathfrak{g}$. Choose a U-invariant inner product $\mathrm{tr}[\gamma,\gamma']$ on $\mathfrak{g}$, for example the negative Killing form in the semi-simple case. Then the scalar 2-form

$$\mathrm{tr}[\Lambda\wedge\Lambda']:=\lambda_i\wedge\lambda'_j\ \mathrm{tr}[\gamma^i,\gamma^j]\in\Omega^2(S)$$

is independent of the choice of basis γ^i and can be integrated over S. for $U=SU_n(\mathbf{C})$ we use $-\mathrm{tr}\gamma\gamma'$ □

Is this of complex type?

2.1.1 Homogeneous manifolds

Let G be a Lie group with a (right) action

$$M\times G\to M,\quad (m,g)\mapsto m\underset{\sim}{g}$$

on a manifold M. The corresponding infinitesimal action

$$M\times\mathfrak{g}\to TM,\quad (m,u)\mapsto \underset{\sim}{u}_m$$

of the Lie algebra $\mathfrak{g}$ is defined by

$$\underset{\sim}{u}_m=\partial_t^0(m\cdot\exp(tu)).$$

Here $\exp:\mathfrak{g}\to G$ is the exponential map. For any $m\in M$ the stabilizer subgroup

$$G_m:=\{g\in G:\ m\cdot g=m\}$$

has the Lie algebra

$$\mathfrak{g}_m:=\{u\in\mathfrak{g}:\ \underset{\sim}{u}_m=0\}.$$

The quotient manifold

$$G^m:=G_m\backslash G$$

has the tangent space

$$T_{\boldsymbol{m}}(G^m)=\{\underset{\sim}{u}_m:\ u\in\mathfrak{g}\}=\mathfrak{g}_m\backslash\mathfrak{g}$$

Lemma 2.1.7.

$$\underset{\sim}{u}_m(T_m\underset{\sim}{g})=\underset{\sim}{Ad_g^{-1}u}_{\,m\cdot g}$$

Proof.

$$\underset{\sim}{u}_m(T_m\underset{\sim}{g})=\partial_t^0(m\cdot\exp(tu))(T_m\underset{\sim}{g})=\partial_t^0\Big((m\cdot\exp(tu))\cdot g\Big)=\partial_t^0\Big(m\cdot(\exp(tu)\ g)\Big)$$

$$=\partial_t^0\Big(m\cdot g\cdot(g^{-1}\exp(tu)\ g)\Big)=\partial_t^0\Big(m\cdot g\cdot\exp(t\ Ad_g^{-1}u)\Big)=\underset{\sim}{Ad_g^{-1}u}_{\,m\cdot g}$$

□

Put $Int_{\boldsymbol{g}}(g') := gg'g^{-1}$. Then

$$Ad_g := T_e(Int_g)$$

defines an action of G on the Lie algebra $\mathfrak{g} = T_eG$. The **co-adjoint action** $\mathfrak{g}^* \times G \to \mathfrak{g}^*$ on the linear dual space $M := \mathfrak{g}^*$ is defined by

$$(m \circ Ad_g)u := m(Ad_g u).$$

for all $m \in \mathfrak{g}^*$ and $u \in \mathfrak{g}$. Put

$$ad_u v = [u, v].$$

Lemma 2.1.8. *For any $m \in \mathfrak{g}^*$ the stabilizer subgroup*

$$G_m := \{g \in G : \ m \circ Ad_g = m\}$$

has the Lie algebra

$$\mathfrak{g}_m := \{u \in \mathfrak{g} : \ m \circ ad_u = 0\}.$$

Proof. Let $u \in \mathfrak{g}$ and let $g_t \in G$ be a smooth curve with $g_0 = e$ and $\partial_t^0 g_t = u$. Then

$$\partial_t^0 (m \circ (Adg_t)|v) = \partial_t^0 m((Adg_t)v)) = m(\partial_t^0 (Adg_t)v) = m[u, v] = (m \circ \ ad_u)v$$

Since $v \in \mathfrak{g}$ is arbitrary, it follows that

$$\partial_t^0 m \circ \ Adg_t = m \circ \ ad_u.$$

Regarding $m \circ \ Ad_{g_t}$ as a curve in the orbit $G^m = G_m \backslash G$ it follows that

$$T_m(G^m) = \{m \circ \ ad_u : \ u \in \mathfrak{g}\}.$$

□

For $u \in \mathfrak{g}$ let

$$\underset{\sim}{u}_m := u + \mathfrak{g}_m \in T_m G^m$$

denote the equivalence class. For each $\xi \in \mathfrak{g}^*$ we have the action

$$(\xi \circ (Adg)|v) := (\xi|(Adg)v)$$

for all $v \in \mathfrak{g}$. Now we define

$$\omega_\xi(\xi \circ (adu)|\xi \circ (adv)) := \xi[u, v]$$

Theorem 2.1.9. *For $m \in \mathfrak{g}^*$ define a bilinear form ω_m on $T_m G^m$ by*

$$\omega_m(\underset{\sim}{u}_m, \underset{\sim}{v}_m) := m[u, v]$$

for all $u, v \in \mathfrak{g}$. This is well-defined and yields a G-invariant symplectic form on the coadjoint orbit G^m.

Proof. Suppose $u, u' \in \mathfrak{g}$ and $v, v' \in \mathfrak{g}$ satisfy $u - u' \in \mathfrak{g}^m$ and $v - v' \in \mathfrak{g}^m$. Then $m \circ ad(u - u') = 0 = m \circ ad(v - v')$ and

$$m[u, v] - m[u', v'] = m[u - u', v] + m[u', v - v'] = (m \circ \ ad_{u-u'})v - (m \circ \ ad_{v-v'})u' = 0.$$

This shows that $m[u, v]$ depends only on the equivalence class $\underset{\sim}{u}_m, \underset{\sim}{v}_m$. Hence (**??**) is well-defined.

The tangent space $T_m G_m$ consists of all linear functionals $\underset{\sim}{u}_m = m \circ \ ad_u$, for $u \in \mathfrak{g}$. Suppose that $\underset{\sim}{u}_m \in T_m G_m$ belongs to the radical of ω_m. Then

$$(m \circ \ ad)u(v) = m(ad_u v) = m[u, v] = \omega_m(\underset{\sim}{u}_m, \underset{\sim}{v}_m) = 0$$

for all $v \in \mathfrak{g}$. Thus $m \circ ad_u = 0$ as a tangent vector to G_m. Therefore ω is non-degenerate.

To show that ω is G-invariant, we apply (**??**) and obtain

$$(\underset{\sim}{u}_m, \underset{\sim}{v}_m)(\underset{\sim}{g}^*\omega)_m = (\underset{\sim}{u}_m(T_m \underset{\sim}{g}), \underset{\sim}{v}_m(T_m \underset{\sim}{g}))\omega_{m\cdot g} = (\underbrace{Ad_g^{-1}u}_{m\cdot g}, \underbrace{Ad_g^{-1}v}_{m\cdot g})\omega_{m\cdot g}$$

$$= (m\cdot g)[Ad_g^{-1}u, Ad_g^{-1}v] = m(Ad_g[Ad_g^{-1}u, Ad_g^{-1}v]) = m[u,v] = (\underset{\sim}{u}_m, \underset{\sim}{v}_m)(\underset{\sim}{g}^*\omega)_m.$$

Thus we have

$$\underset{\sim}{g}^*\omega = \omega$$

for all $g \in G$.

Every $u \in \mathfrak{g}$ induces a vector field $\underset{\sim}{u}$ on G_m by

$$m \cdot \exp(t\underset{\sim}{u}) = m \circ Ad_{\exp(tu)}.$$

For fixed $v \in \mathfrak{g}$ consider the smooth function

$$f_m^v := m|v.$$

Then

$$(\underset{\sim}{u}_\partial f^v)_m = \partial_t^0 f^v_{m\cdot\exp(t\underset{\sim}{u})} = \partial_t^0 f^v_{m\circ\ Ad_{\exp(tu)}}$$

$$= (\partial_t^0 f_{m\circ\ Ad_{\exp(tu)}})v = m(\partial_t^0 Ad_{\exp(tu)}v) = m[u,v].$$

Since

$$\omega_m(\underset{\sim}{v}_m, \underset{\sim}{w}_m) = f_m^{[v,w]}$$

it follows that $(\underset{\sim}{u}_\partial \omega(\underset{\sim}{v}, \underset{\sim}{w}))_m = m|[u[v,w]]$ and the Jacobi identity implies

$$\underset{\sim}{u}_\partial \omega(\underset{\sim}{v}, \underset{\sim}{w}) + \underset{\sim}{v}_\partial \omega(\underset{\sim}{w}, \underset{\sim}{u}) + \underset{\sim}{w}_\partial \omega(\underset{\sim}{u}, \underset{\sim}{v}) = 0.$$

On the other hand, we have

$$[\underset{\sim}{u}, \underset{\sim}{v}] = \underbrace{[u,v]}$$

and hence

$$\omega_m([\underset{\sim}{u}, \underset{\sim}{v}]_m, \underset{\sim}{w}_m) = \omega_m(\underbrace{[u,v]}_m, \underset{\sim}{w}_m) = m[[u,v]w].$$

Using the Jacobi identity again, we obtain

$$\omega_m([\underset{\sim}{u}, \underset{\sim}{v}]_m, \underset{\sim}{w}_m) + \omega_m([\underset{\sim}{v}, \underset{\sim}{w}]_m, \underset{\sim}{u}_m) + \omega_m([\underset{\sim}{w}, \underset{\sim}{u}]_m, \underset{\sim}{v}_m) = m([[u,v]w] + [[v,w]u] + [[w,u]v]) = 0.$$

In summary, $d\omega(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}) = 0$. Thus $d\omega = 0$. $\square$

For $u_0, u_1, u_2 \in \mathfrak{g}$ we consider the right invariant vector fields

$$\underset{\sim}{u}_g := u \cdot T_e(R_g)$$

acting on G and also on $G^\xi = G_\xi \backslash G$. Consider the function

$$f(m) := m|[u^1, u^2]$$

on the orbit. Then

$$f(o \cdot g_t^0) = (o \cdot g_t^0)|[u^1, u^2] = o|g_t^0 \cdot [u^1, u^2]$$

and therefore

$$(\underset{\sim}{u}^0 \cdot f)(o) = \partial_t^0 f(o \cdot g_t^0) = o|\partial_t^0 g_t^0 \cdot [u^1, u^2] = o|[u^0[u^1, u^2]].$$

Thus the first three terms sum up to zero by the Jacobi identity. For the second type we have

$$\omega_m([\underset{\sim}{u}, \underset{\sim}{v}]_m, \underset{\sim}{w}_m) = \omega_m(\underbrace{[u,v]}_m, \underset{\sim}{w}_m) := m|[[u,v],w].$$

Thus the last three terms sum up to zero by the Jacobi identity.

• **Jordan manifolds**

*projective space
*Grassmannian

Proposition 2.1.10. *Let G be a real semi-simple Lie group of hermitian type, with maximal compact subgroup K. Then the 'symmetric domain' $\check{Z} = G/K$ is a coadjoint orbit, whose (Kostant-Kirillov)-symplectic structure agrees with(**??**). Moreover, the compact dual space (conformal hull)*

$$\hat{Z} = G^{\mathbf{C}}/K^{\mathbf{C}} \cdot \overline{Z}$$

is a compact Kähler manifold, and ?? we have

$$(\check{Z}, \omega) = (\hat{G}^{\mathbf{C}}/G_{+}^{\mathbf{C}}, \mathrm{Im} h)$$

Proof. Define $m : \dot{\mathfrak{g}} \to i\mathbf{R}$ by

$$m(\gamma) = (iz\frac{\partial}{\partial z}|\gamma_0')$$

□

Proposition 2.1.11. *Let $\hat{G}$ be a simply-connected compact Lie group, with maximal torus $\hat{T}$. Then the full flag manifold*

$$\hat{G}/\hat{T} = \hat{T} = \hat{G}_m$$

is a coadjoint orbit for the linear functional $m : \hat{\mathfrak{g}} \to i\mathbf{R}$ given by

$$mY := \rho Y_0.$$

Here $Y \mapsto Y_0$ is the projection onto $\hat{\mathfrak{t}}$.

Proof. With respect to the root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\beta \in \Delta} \mathfrak{g}_\beta$$

we write elements in $\mathfrak{g}$ as

$$Y = Y_0 + \sum_{\beta \in \Lambda} Y_\beta.$$

Define a linear form $m : \mathfrak{g} \to \mathbf{C}$ by

$$mY := \rho Y_0.$$

Let $X = X_0 + \sum_{\alpha > 0} (X_\alpha - X_\alpha^*) \in \hat{\mathfrak{g}}$ satisfy $m \circ ad_X = 0$. Let $\beta > 0$ and $Y \in \mathfrak{g}_{-\beta}$ be arbitrary. Then

$$[X, Y] = [X_0, Y] + \sum_{\alpha > 0} [X_\alpha - X_\alpha^*, Y] = -(\beta X_0)Y + \sum_{\alpha > 0} ([X_\alpha, Y] - [X_\alpha^*, Y])$$

has the $\mathfrak{t}$-projection

$$[X, Y]_0 = [X_\beta, Y] = c \cdot H_\beta.$$

Since $\beta > 0$ we have $\rho H_\beta > 0$. Therefore $0 = (m \circ ad_X)Y = \rho[X, Y]_0 = c \cdot \rho H_\beta$ implies $c = 0$. Hence $[X_\beta, Y] = 0$ for all $Y \in \mathfrak{g}_{-\beta}$, showing that $X_\beta = 0$ for $\beta > 0$. Thus $X = X_0 \in \mathfrak{t}$. □

complex structure on coadjoint orbits

Lemma 2.1.12. *For each $w \in W$*

$$G_{>}^{w} \cdot T^{\mathbf{C}} \cdot G_{<}^{w} \subset G^{\mathbf{C}}$$

is open.

Moreover

$$G/T = G^{\mathbf{C}}/G^{\mathbf{C}}_{+}$$

is a compact Kähler manifold, and we have

$$(G/T, \omega) = (G^{\mathbf{C}}/G^{\mathbf{C}}_{+}, \mathrm{Im}h)$$

*Peirce manifolds as coadjoint orbits

• **Restricted Grassmannian**

• **Loop groups**

Let G be a simply-connected and simply laced (ADE) Lie group. Put $\mathbf{S} := \mathbf{S}^1$ and let

$$L = \mathcal{C}^\infty(\mathbf{S}, G)$$

with Lie algebra

$$\Lambda := \mathbf{C}^\infty(\mathbf{S}, \mathfrak{g}).$$

Then the (smooth) dual is

$$\Lambda^+ := \mathbf{C}^\infty(\mathbf{S}, \mathfrak{g}^*)$$

under the pairing

$$(m|\gamma) := \int\limits_{\mathbf{S}} ds\ m_s \gamma_s$$

for all $m \in \Lambda, m \in \mathfrak{g}^*$. The coadjoint action is

Its orbit of 0 is the loop space

$$\Omega(\mathbf{S}) = \{m \in \mathbf{C}^\infty(\mathbf{S}, \mathfrak{g}^*) :\ m_0 = m_{2\pi}\}$$

It carries the symplectic form

$$\omega(\xi, \eta) := \frac{1}{2\pi} \int\limits_0^{2\pi} ds(\xi'(s), \eta(s))$$

• **Conformal blocks**

Proposition 2.1.13. *For the affine symplectic space $\Omega^1(S, G)$ of G-connexions on a compact oriented surface S we consider the group*

$$\Omega^0(S, G)$$

acting by **gauge transformations**

$$g \cdot A := gAg^{-1} + g^{-1}dg.$$

This action preserves the symplectic structure (**??**).

Thus $\Omega^1(S, G)$ becomes a $\Omega^0(S, G)$-equivariant symplectic manifold. The Lie algebra of $\Omega^0(S, G)$ is identified with $\Omega^0(S, \mathfrak{g})$ under the pointwise Lie bracket. Define a pairing $\Omega^0(S, \mathfrak{g}) \otimes \Omega^2(S, \mathfrak{g}) \to \mathbf{R}$ by

$$(\gamma, \Theta) \mapsto \int\limits_{\Sigma} \mathrm{tr}[\gamma \cdot \Theta].$$

Here we write

$$\Theta = \vartheta \otimes \eta$$

for some 2-form $\vartheta \in \Omega^2(S,\mathbf{R})$ and $\eta \in \mathfrak{g}$. Then the $\mathfrak{g}$-valued 2-form

$$[\Theta \cdot \gamma] = \vartheta\ [\eta,\gamma]$$

gives rise to a scalar 2-form

$$\mathrm{tr}[\Theta \cdot \gamma] = \vartheta\ \mathrm{tr}[\eta,\gamma]$$

which can be integrated over S. Via this pairing we identify $\Omega^2(S,\mathfrak{g})$ with a subspace of the dual space $\Omega^0(S,\mathfrak{g})^*$. The full continuous dual should be of distribution type.

2.2 Hamiltonian vector fields, Poisson bracket

$$\mathcal{C}^\infty(M,\mathbf{R})$$

Hamiltonian vector fields: For any function $f \in \mathcal{C}^\infty(M,\mathbf{R})$ define a vector field $\tilde{f}$ on M by

$$\omega_m(\tilde{f}_m, Y_m) := d_m(f)Y_m$$

for all $Y \in \Gamma(M, T^*M)$. Then the Poisson bracket is defined by

$$\widetilde{\{f_1,f_2\}} = [\tilde{f}_1, \tilde{f}_2]$$

We say that f_1, f_2 are in involution if $\widetilde{\{f_1,f_2\}} = 0$. Completely integrable classical systems $f_1,\ldots,f_n$ in pairwise involution. classical dynamics: Geodesic flow on T^*Q multi-flow: A-action, $G = KAN$ Iwasawa decomposition
Prequantization $f \mapsto f + i\nabla_f$ quantum dynamics: $e^{i\Delta}$ on $L^2(Q)$, quantization of geodesic flow multi-dynamics: Berezin transform for real Jordan manifolds

2.3 Moment Map and Classical Reduction

Coadjoint orbits, moment map and symplectic quotient

$$T^*Q \times \mathrm{Diff}(X) \to T^*Q, \quad \sigma \mapsto T^*\sigma$$

is a symplectic action with moment map

$$\mu : T^*X \to \Gamma_1(X)^+$$

$$\mu_{x,\xi} v = \xi v_x$$

for all $v \in \Gamma_1(X)$

A symplectic manifold (M,ω) endowed with a smooth (right) action $M \times G \to M, g \mapsto R_g$ preserving ω

$$R_g^*\omega = \omega$$

is called a **G-equivariant** symplectic manifold. Let M be a symplectic manifold endowed with a symplectic G-action. The associated infinitesimal action of the Lie algebra $\mathfrak{g}$ defines a tangent vector

$$\gamma_m := \partial_t^0(g_t \cdot m) \in T_m M$$

for every $m \in M$. Here $g_t \in G$ is a smooth curve with $g_0 = I$ and $\partial_t^0 g_t = \gamma$.

Definition 2.3.1. *A smooth map*

$$\mu : M \to \mathfrak{g}^*$$

is called a **moment map** *if for each $v \in \mathfrak{g}$ the smooth function $\mu^v : M \to \mathbf{R}$, defined by*

$$\mu^v(m) := \mu(m)v$$

for the standard pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbf{R}$, has the differential

$$d_m(\mu^v) = \omega_m \underset{\sim}{v}_m$$

for all $m \in M$. Here $\omega_m \underset{\sim}{v}_m \in T^1_m M$ since $\underset{\sim}{v}_m \in T_m M$ and $\omega_m \in T^2_m M$. Thus

$$t \cdot d_m \mu^v = \omega_m(t, \underset{\sim}{v}_m)$$

for all $u \in T_m M$.

Formally, we have $d\mu = \omega$. This 'explains' that a moment map is unique up to a constant if M is connected

Theorem 2.3.2. *Let $\mu : M \to \mathfrak{g}^*$ be a Hamiltonian G-action. Suppose that G acts freely and properly on $\mu^{-1}(0)$. Then*

$$M//G := \mu^{-1}(0)/G$$

is a smooth manifold of dimension $\dim M - 2\dim G$, *which carries a unique symplectic form $\omega_0 \in \Omega^2(M//G)$ satisfying*

$$p^*\omega_= \iota^*\omega,$$

where $p : \mu^{-1}(0) \to M$ is the inclusion map and $p : \mu^{-1}(0) \to M//G$ is the canonical projection.

Proposition 2.3.3. *Suppose that μ is a moment map for the symplectic G-action on M. Then, for each $\gamma \in \mathfrak{g}$ the smooth function $\mu\gamma$ has the Hamiltonian vector field γ_M acting on M. (One says that the G-action is hamiltonian)*

***Symplectic quotient

• Jordan manifolds

Example 2.3.4. The torus action $\mathbf{C}^n \times \mathbf{T}^n \to \mathbf{C}^n$ is a hamiltonian action. We have the Lie algebra $\mathfrak{t} = i\mathbf{R}^n$ has the dual space

$$\mathfrak{t}^+ = {}_n\mathbf{R}$$

and the moment map $\mu : \mathbf{C}^n \to {}_n\mathbf{R}$ has the form

$$\mu(m) = (|z^1|^2, \ldots, |z^n|^2)$$

• Restricted Grassmannian

Example 2.3.5.

$$Gr_{res}(H) = GL_{res}(H)/B_{res}$$

has the symplectic structure

$$\omega = \frac{i}{4}\mathrm{tr}\Phi\ d\Phi\ d\Phi$$

where $\Phi^2 = 1$. Hence

$$\Phi\ d\Phi = -d\Phi\ F$$

and therefore

$$d\omega = \mathrm{d}\Phi^3 = \mathrm{tr}\Phi^2(d\Phi^3) = -\mathrm{tr}\Gamma(d\Phi)^3\Phi = -\mathrm{tr}(d\Phi)^2\Phi^2 = -d\omega$$

so ω is closed.

The moment map is

$$\mu(\Phi)u = -\mathrm{tr}(\Phi u)$$

where $u \in \mathfrak{g}(H)$ satisfies $u^* = \epsilon u \epsilon$. This follows from the computation

$$2\mathrm{tr}\Phi[u, \Phi]d\Phi = -d\mathrm{tr}u\Phi.$$

Example 2.3.6. Let Q be a Riemannian manifold. then T^*Q is a symplectic quotient. In particular, for $Q = \mathbf{R}^n$, it follows that $T^*\mathbf{R}^n$ is a symplectic quotient.

Remark 2.3.7. *Every classical physical system is a symplectic quotient.*

- **Conformal blocks**

Theorem 2.3.8. *The action of $\Gamma^0(S, G)$ on $\Gamma^1(S, G)$ by gauge transformations has a moment map*

$$\Gamma^1(S, G) \to \Gamma^2(S, \mathfrak{g}) \subset \Gamma^0(S, \mathfrak{g})^*$$

given by the curvature

$$\mu_A = \eth\Theta = d\Theta + [\Theta \wedge \Theta] \in \Gamma^2(S, \mathfrak{g})$$

Proof. We have to show that for each $\gamma \in \Gamma^0(S, \mathfrak{g})$ the smooth function

$$(\mu\gamma)(A) := (\mu(A)|\gamma) = (\eth\Theta|\gamma) = \int_\Sigma \mathrm{tr}[\gamma \cdot \eth\Theta]$$

of the argument $A \in \Gamma^1(S, G)$ has the differential

$$d_A(\mu\gamma)\dot{A} = \omega_A(\gamma_A, \dot{A}) = \int_S \mathrm{tr}[\gamma_A \wedge \dot{A}]$$

where $\dot{A}$ and the value γ_A of the vector field at A belong to $T_A(\Gamma^1(S, G)) = \Gamma^1(S, \mathfrak{g})$. For the left hand side consider a curve A_t with $A_0 = A$ and $\partial_t^0 A_t = \dot{A}$. Since

$$\partial_t^0(d^{A_t}A_t) = \partial_t^0(dA_t + [A_t \wedge A_t]) = d\dot{A} + [\dot{A} \wedge A] = \eth\dot{A}$$

it follows that

$$d_A(\mu\gamma)\dot{A} = \partial_t^0 \int_S \mathrm{tr}[\gamma \cdot \eth\Theta] = \int_S \mathrm{tr}[\gamma \cdot \partial_t^0\eth\Theta] = \int_S \mathrm{tr}[\gamma \cdot \eth\dot{A}].$$

For the right hand side, consider a curve $g_t \in \Gamma^0(S, G)$ with $g_0 = I$ and $\partial_t^0 g_t = \gamma$. Differentiating

$$g_t \cdot A = g_t A g_t^{-1} - g_t^{-1} dg_t$$

at $t = 0$ we obtain, using Schwarz rule to exchange the diffentiation in t and on S,

$$\gamma_A = \partial_t^0(g_t \cdot A) = \gamma A - A\gamma - d\gamma = -\eth\gamma \in \Omega^1(S, \mathfrak{g}) \equiv T_A(\Omega^1(S, G)).$$

Since $\eth$ is a graded derivation after applying the trace, it follows that

$$-\int_S \mathrm{tr}[\gamma_A \cdot \dot{A}] = -\int_S \mathrm{tr}[\eth\gamma \wedge \dot{A}] = \int_S \mathrm{tr}[\gamma \wedge \eth\dot{A}].$$

□

Corollary 2.3.9. *$\mu^{-1}(0) = \{A : d^A A = 0\}$ consists of all* **flat** *C-connexions on S, and the symplectic quotient $\Omega^1(S,G)//\Omega^0(S,G)$ agrees with the non-abelian 1-cohomology*

$$H^1_c(S,G) = \mu^{-1}(0)/\Omega^0(S,G)$$

which can be identified with $\mathrm{Hom}(\pi_1(S),C)/C$. This is a **compact** *symplectic orbifold.*

Note that in the non-abelian case higher order cohomology cannot be defined directly (higher categories).

Theorem 2.3.10. *(Narasimhan-Seshadri) The symplectic quotient*

$$H^1(S,C) := \Omega^1(S,C)//\Gamma^0(S,C) = \Omega^1_{flat}(S,C)/C = \mathrm{Hom}(\pi_1(S),C)/C$$

is the space of all flat C-connexions on S, modulo conjugation by C. It is an orbifold with smooth part consisting of all irreducible connexions.

Theorem 2.3.11. *Fix a complex structure τ on S. Then the complex-analytic quotient $H^1(S_\tau, C^{\mathbf{C}})$ consists of all semi-stable holomorphic vector bundles over S_τ. It is an complex-analytic space, with regular part consisting of all stable vector bundles.*

2.3.1 Homogeneous Manifolds

For a coadjoint orbit G^m we have

Proposition 2.3.12. *For any $m \in \mathfrak{g}^*$, the inclusion $\iota : G^m \to \mathfrak{g}^*$ is a moment map for the co-adjoint action.*

Proof. We have to show that for each $v \in \mathfrak{g}$ the mapping $\iota^v_m := \iota_m v = mv$ has the differential

$$\underset{\sim}{u}_m(d_m\iota^v) = \omega_m(\underset{\sim}{u}_m, \underset{\sim}{v}_m).$$

This follows from

$$\underset{\sim}{u}_m(d_m\iota^v) = (\partial^0_t\ m \cdot \exp(tu))(d_m\iota^v) = \partial^0_t \iota^v_{m\cdot\exp(tu)} = \partial^0_t(m \cdot \exp(tu))v$$

$$= \partial^0_t m(Ad_{\exp(tu)}v) = \partial^0_t m((\exp(t\ ad_u)v)) = m(ad_u v) = m[u,v] = \omega_m(\underset{\sim}{u}_m, \underset{\sim}{v}_m)$$

□

2.4 Quantum line bundles

We call a symplectic form ω **integral**, if

$$\frac{1}{2\pi i}\int_S \omega \in \mathbf{Z}$$

for all 2-cycles $S \subset M$. This means that $\frac{\omega}{2\pi i} \in H^2(M,\mathbf{Z})$.

Theorem 2.4.1. *Let $\frac{\omega}{2\pi i} \in H^2(M,\mathbf{Z})$ be an integral symplectic form. Then there exists a complex* **prequantum line bundle** *endowed with a hermitian metric $\boldsymbol{h}$ and a metric connection A with curvature $d^A A = \omega$ (called the first Chern class). Conversely, the integrality condition is also necessary for the existence of a prequantum line bundle.*

Proof. Choose a Leray open cover V_a of M, meaning that all finite intersections are contractible. Since $d\omega = 0$, the Poincaré Lemma implies that for each a there exists a potential $A^a \in \Omega^1(V_a, i\mathbf{R})$ such that

$$dA^a = \omega|_{V_a}.$$

Then $d(A^a - A^b)|_{V_a\cap V_b} = \omega - \omega = 0$. Applying the Poincaré Lemma again there exist functions $\ell_b^a \in \Omega^0(V_a \cap V_b, i\mathbf{R})$ such that

$$(A^a - A^b)|_{V_a\cap V_b} = d\ell_b^a.$$

Put

$$\kappa_b^a := \exp(\ell_b^a) \in \mathbf{T}$$

Since $\omega \in H^1(M, 2\pi i\mathbf{Z})$ is integral, it follows that

$$(\ell_b^a + \ell_c^b + \ell_a^c)|_{V_a\cap V_b\cap V_c} \in 2\pi i\mathbf{Z}.$$

Therefore the cocycle property

$$(\kappa_b^a\ \kappa_c^b\ \kappa_a^c)|_{V_a\cap V_b\cap V_c} = \exp(\ell_b^a + \ell_c^b + \ell_a^c) = 1$$

holds. Hence we obtain a $\mathbf{T}$-bundle $\mathcal{V} \times_{\sim}^{\kappa} \mathbf{T}$ and the associated line bundle

$$\mathcal{V} \underset{\sim}{\overset{\kappa}{\times}} \mathbf{C} = (\mathcal{V} \underset{\sim}{\overset{\kappa}{\times}} \mathbf{T}) \underset{\mathbf{T}}{\times} \mathbf{C} = \{\langle m, \phi\rangle_a = \langle m, \beta_b^a(m)\phi\rangle_b :\ m \in V_a \cap V_b,\ \phi \in \mathbf{C}\}$$

for the standard $\mathbf{T}$-representation $\mathbf{C}$. By corollary 3.2.11 it carries the hermitian metric

$$(\langle m, \phi\rangle_a | \langle m, \psi\rangle_a) = \overline{\phi}\psi.$$

Since

$$\kappa_b^a(d\kappa_a^b) = d\ell^a - d\ell^b = A^a - A^b,$$

the family A^a defines a connexion A. By (**??**) the curvature of (A^a) is given by

$$(d^A A)_m^a(u, v) = v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot A^b) = (dA^a)_m(u, v) = \omega_m(u, v).$$

Hence $d^A A = \omega$. □

For a hermitian holomorphic line bundle, the curvature

$$\omega^a = \overline{\partial}(\boldsymbol{h}^a)^{-1}\partial\boldsymbol{h}^a$$

defines an integer cohomology class

$$c_1(\mathcal{L}) := \frac{1}{2\pi i}\omega \in H^2(M, \mathbf{Z})$$

called the (first) Chern class. This is a conformal invariant: If the hermitian metric $(\boldsymbol{h}^a)$ is changed by a conformal factor $\acute{\boldsymbol{h}}^a := e^f\ \boldsymbol{h}^a$, where $f \in \Gamma(M, \mathbf{R})$, then $\acute{A}^a := (\acute{\boldsymbol{h}}^a)^{-1}\partial\acute{\boldsymbol{h}}^a = A^a + \partial f$ and therefore

$$\acute{\omega}^a = \overline{\partial}\acute{A}^a = \omega^a + \overline{\partial}\partial f.$$

Since $\overline{\partial}\partial f = d\frac{\partial - \overline{\partial}}{2}\ f$, it follows that $\frac{1}{2\pi i}\acute{\omega} = \frac{1}{2\pi i}\omega$ in $H^2(M, \mathbf{Z})$.

On a Kähler manifold M a **quantum line bundle** is a holomorpic hermitian line bundle whose Chern connexion has curvature ω. We may also consider the scale of all k-th powers, with the inverse $\frac{1}{k}$ being interpreted as Planck's constant.

Lemma 2.4.2. *If a hermitian holomorphic line bundle (LL, h) on a Kähler manifold satisfies*

$$\boldsymbol{h}_m(u,v) = \overline{\partial}_v \partial_u \log h_m$$

then ω_m is the curvature of the Chern connexion $\eth h$. Thus $(\mathcal{L}, h, \eth h)$ becomes a (pre)-quantum line bundle.

On a complex manifold M a smooth function $\ell : M \to \mathbf{R}$ is called **plurisubharmonic** (in short, plush) if the Levi form

$$(\partial_i \overline{\partial}_j \ell(m)) = (\frac{\partial^2}{\partial z^i \ell \overline{\partial} z^j}(m))$$

is positive (semi-definite). If (**??**) is strictly positive, the ℓ is called strictly plurisubharmonic. In this case the $(1,1)$-form

$$\partial\overline{\partial}\ell = \sum_{i,j} \partial_i \overline{\partial}_j \ell$$

on M is a strictly positive (imaginary) symplectic form on M. Consider the hermitian metric $h_m := \exp \ell(m)$ on the holomorphic line bundle.

• Jordan manifolds

Example 2.4.3. For the holomorphic tangent bundle on $\mathbf{P}^1$ Proposition **??** and (**??**) yield the curvature $(1,1)$-form

$$\overline{\partial}\mathbf{A}^0 = -2\frac{\partial}{\partial \overline{z}} \frac{\overline{z}}{(1+z\overline{z})^2}\ d\overline{z} \wedge dz = -2\frac{1\cdot(1+z\overline{z}) - \overline{z}z}{(1+z\overline{z})^2}\ d\overline{z}\wedge dz = -2\frac{d\overline{z}\wedge dz}{(1+z\overline{z})^2}.$$

Lemma 2.4.4.

$$\int_{\mathbf{S}^2} \frac{d\overline{z}\wedge dz}{(1+z\overline{z})^2} = 2\pi i.$$

Proof. We have

$$d\overline{z}\wedge dz = (dx - i\ dy)\wedge(dx + i\ dy) = 2i dx\wedge dy.$$

Using polar coordinates $z = r\ e^{is}$ and putting $u := r^2$ we obtain

$$\frac{1}{2\pi i}\int_{\mathbf{S}^2} \frac{d\overline{z}\wedge dz}{(1+z\overline{z})^2} = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{dxdy}{(1+x^2+y^2)^2}$$

$$= \frac{1}{\pi}\int_0^{2\pi} ds \int_0^\infty \frac{r\ dr}{(1+r^2)^2} = 2\int_0^\infty \frac{r\ dr}{(1+r^2)^2} = \int_0^\infty \frac{du}{1+u^2} = \frac{-1}{1+u}|_0^\infty = 1.$$

□

As a consequence we have

$$\int_{\mathbf{S}^2} \omega = -4\pi i.$$

Therefore the symplectic form ω of the holomorphic tangent bundle is integral, but this bundle is not the 'minimal' line bundle associated with an integral symplectic form.

• **Loop groups**

The corresponding prequantum line bundle is the central extension viewed as a circle bundle over $\Omega(\mathbf{S})$.

Passing to Jordan manifolds, the **quasi-determinant** $\Delta_{z,w}$ of an irreducible metric Jordan triple Z satisfies

$$\det B_{z,w} = \Delta^p_{z,w},$$

where p is a numerical invariant called the genus. For matrices $Z = \mathbf{K}^{r\times s}$ we have

$$\Delta_{z,w} = \det(I_r - zw^*) = \det(I_s - w^* z).$$

We also have the addition formula

$$\Delta_{z,u}\,\Delta_{z^u,v} = \Delta_{z,u+v},$$

which is not trivial since a p-th root is involved. It follows that the map $\delta : R \to \mathbf{C}^*$ defined by

$$\delta^{w,b}_{z,a} := \Delta_{z,a-b}$$

is a cocycle with values in $\mathbf{C}^*$. This cocycle and its integer power δ^n induces a line bundle

$$Z^2 \underset{\sim}{\overset{\delta,n}{\times}} \mathbf{C} := (Z^2 \underset{\sim}{\overset{\delta}{\times}} \mathbf{C}^*) \underset{\mathbf{C}^*}{\overset{n}{\times}} \mathbf{C} := \{[m,\phi]_a = [m^{a-b}, \Delta^{-n}_{z,a-b}\phi]_b :\ \phi\in\mathbf{C},\ \Delta_{z,a-b}\neq 0\}$$

over $\hat{Z}$. A holomorphic section $\Phi \in \mathcal{O}(Z^2 \times^{\delta,n}_{\sim} \mathbf{C})$ has the local trivializations

$$\Phi_{[m,a]} = [m, a, \Phi^a(m)]$$

for $a \in Z$, where $\Phi^a : Z \to \mathbf{C}$ are holomorphic functions satisfying the compatibility condition

$$\Phi^b(z^{a-b}) = \Delta^{-n}_{z,a-b}\,\Phi^a(m)$$

whenever $a, b \in Z$ satisfy $\Delta_{z,a-b} \neq 0$. Since $Z \subset \hat{Z}$ is a dense open subset via the embedding $z \mapsto z^0 = [z,0]$, a section $\Phi \in \mathcal{O}(Z^2 \times^{\delta,n}_{\sim} \mathbf{C})$ is uniquely determined by its trivialization $\underline{\Phi} := \Phi^0$. Thus via the mapping $\Phi \mapsto \underline{\Phi}$ we may identify $\mathcal{O}(Z^2 \times^{\alpha^n}_{\sim} \mathbf{C})$ with a vector space of entire functions on Z. Later, this will be determined explicitly.

Proposition 2.4.5. *For an irreducible ^{0}metric Jordan triple Z, the $\check{G}$-invariant Bergman metric on $\check{Z} \subset Z$ is given by*

$$(u|v)_z = \mathrm{tr}\ D(B^{-1}_{z,z}u, v) = \partial_u\overline{\partial}_v \log \Delta^{-p}_{z,z}.$$

It follows that $\check{Z} \times^{\delta,-p} \mathbf{C}$, endowed with the ^{0}metric

$$([z,u]|[z,v]) := \Delta^{-p}_{z,z}\ (u|v)$$

is the (pre)-quantum bundle for the ^{0}Kähler manifold $\check{Z}$. Similarly, the $\hat{G}$-invariant Bergman metric on $\hat{Z} \supset Z$ is given by

$$(u|v)_z = \mathrm{tr}\ D(B^{-1}_{z,-z}u, v) = \partial_u\overline{\partial}_v \log \Delta^{p}_{z,-z}.$$

It follows that $Z^2 \times^{\delta,p} \mathbf{C}$, endowed with the ^{0}metric

$$([z,u]|[z,v]) := \Delta^{p}_{z,-z}\ (u|v)$$

is the (pre)-quantum bundle for the ^{0}Kähler manifold $\hat{Z}$.

Proof. We carry out the proof for the matrix case $Z = \mathbf{C}^{r\times s}$, where $\Delta_{z,w} = \det(I_r - zw^*)$ and $p = r+s$. We have

$$\det a \det z = \det(az) = (\det \circ L_a)(z)$$

and hence

$$\det a \det_e' u = (\det \circ L_a)_e' u = \det_a'(au).$$

It follows that

$$\det_a' v = \det a \ \det_e'(a^{-1}v) = \det a \ \mathrm{tr}(a^{-1}v).$$

Therefore

$$\partial_v \log \det(a) = \mathrm{tr}(a^{-1}v).$$

In the non-compact setting we obtain

$$\overline{\partial}_v \log \det(1 - zz^*) = \mathrm{tr}(1 - zz^*)^{-1}\overline{\partial}_v(1 - zz^*) = -\mathrm{tr}(1 - zz^*)^{-1}zv^*.$$

Therefore

$$-\partial_u\overline{\partial}_v \log \det(1-zz^*) = \mathrm{tr}\ \partial_u((1-zz^*)^{-1}zv^*) = \mathrm{tr}\ \Big(-(1-zz^*)^{-1}\partial_u(1-zz^*)(1-zz^*)^{-1}zv^*+(1-zz^*)^{-1}uv^*\Big)$$

$$= \mathrm{tr}\ \Big((1 - zz^*)^{-1}uz^*(1 - zz^*)^{-1}zv^* + (1 - zz^*)^{-1}uv^*\Big) = \mathrm{tr}\ (1 - zz^*)^{-1}\Big(uz^*(1 - zz^*)^{-1}zv^* + uv^*\Big)$$

$$= \mathrm{tr}\ (1 - zz^*)^{-1}\Big(uz^*z(1 - z^*z)^{-1}v^* + u(1 - z^*z)(1 - z^*z)^{-1}v^*\Big)$$

$$= \mathrm{tr}\ (1 - zz^*)^{-1}u(1 - z^*z)^{-1}v^* = \mathrm{tr}\ (B_{z,z}^{-1}u)v^* = \frac{1}{p}\mathrm{tr}\ D(B_{z,z}^{-1}u, v)$$

It follows that

$$\mathrm{tr}\ D(B_{z,z}^{-1}u, v) = -p\partial_u\overline{\partial}_v \log \det(1 - zz^*) = dl_u\overline{\partial}_v \log \det(1 - zz^*)^{-p}.$$

In the compact setting we have $\Delta_{z,-w} = \det(I_r + zw^*)$ and obtain

$$\overline{\partial}_v \log \det(1 + zz^*) = \mathrm{tr}((1 + zz^*)^{-1}(\overline{\partial}_v(1 + zz^*))) = \mathrm{tr}((1 + zz^*)^{-1}zv^*).$$

Therefore

$$\partial_u\overline{\partial}_v \log \det(1+zz^*) = \mathrm{tr}\ \partial_u((1+zz^*)^{-1}zv^*) = \mathrm{tr}\ \Big(-(1+zz^*)^{-1}\partial_u(1+zz^*)(1+zz^*)^{-1}zv^*+(1+zz^*)^{-1}uv^*\Big)$$

$$= \mathrm{tr}\ \Big((1 + zz^*)^{-1}uz^*(1 + zz^*)^{-1}zv^* + (1 + zz^*)^{-1}uv^*\Big) = \mathrm{tr}\ (1 + zz^*)^{-1}\Big(- uz^*(1 + zz^*)^{-1}zv^* + uv^*\Big)$$

$$= \mathrm{tr}\ (1 + zz^*)^{-1}\Big(- uz^*z(1 + z^*z)^{-1}v^* + u(1 + z^*z)(1 + z^*z)^{-1}v^*\Big)$$

$$= \mathrm{tr}\ (1 + zz^*)^{-1}uz^*z(1 + z^*z)^{-1}v^* = \mathrm{tr}\ (B_{z,-z}^{-1}u)v^* = \frac{1}{p}\mathrm{tr}\ D(B_{z,-z}^{-1}u, v)$$

It follows that

$$\mathrm{tr}\ D(B_{z,-z}^{-1}u, v) = p\partial_u\overline{\partial}_v \log \det(1 + zz^*) = \partial_u\overline{\partial}_v \log \det(1 + zz^*)^p.$$

□

For $Z = \mathbf{C}$ the calculation simplifies to

$$\partial_u\overline{\partial}_v \log(1 - z\overline{z}) = \partial_u \frac{-z\overline{v}}{1 - z\overline{z}} = \frac{-u\overline{v}(1 - z\overline{v}) + z\overline{v}(-u\overline{z})}{(1 - z\overline{z})^2} = -\frac{u\overline{v}}{(1 - z\overline{z})^2} = -\boldsymbol{h}_z(u|v)$$

• **Conformal blocks**

Determinant line bundle Fix a complex structure S_τ. Then $\Omega^1(S,G)$ acquires a complex structure J and the covariant derivative d^A of A has a $(0,1)$-part $\overline{\partial}^A$.

Lemma 2.4.6. $(\Omega^1(S,G),J)$ *can be identified with the space*

$$H^1(S_\tau, G^{\mathbf{C}})$$

of all holomorphic $G^{\mathbf{C}}$*-bundles over* S_τ.

Holomorphic Quillen determinant line bundle over $H^1(S_\tau, G^{\mathbf{C}})$ with connexion whose curvature is the Kähler form.

$$\mathcal{L}_A = \det H^1(S_\tau, E_A) \otimes \overline{\det H^0(S_\tau, E_A)}$$

metric defined by regularized determinants of Laplacians.

Chapter 3

Quantum State Spaces

3.1 Reproducing kernels

On the other hand, we obtain an anti-holomorphic map

$$\mathcal{K} : M \to \mathbf{P}(\mathcal{O}(M \times^{\beta} \mathbf{C}))$$

by

$$w \mapsto \mathcal{K}_w \in \mathcal{O}(M \times^{\beta} \mathbf{C})$$

• **Jordan manifolds**

Example 3.1.1. Consider the projective space $\mathbf{P}^d$. For $0 \le i \le d$ put

$$V_i := \{[\zeta] \in \mathbf{P}^d : \ \zeta^i \neq 0\}$$

where $[\zeta] := \mathbf{C}\zeta$ for $0 \neq \mu \in \mathbf{C}^{d+1}$. Define $\beta^i_j : V_i \cap \mathbf{P}^d_j \to \mathring{G}$ by

$$\beta^i_j[\zeta] := \frac{\zeta^i}{\zeta^j}.$$

Note that the fraction depends only on $[\zeta]$. Since the cocycle identity is satisfied, we obtain a $\mathbf{C}^{\times}$-bundle

$$\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} \mathbf{C}^{\times} = \{[[\zeta], h]_i = [[\zeta], h\beta^j_i[\zeta]]_j\} = \{[[\zeta], h]_i = [[\zeta], \frac{\zeta^i}{\zeta^j} h]_j\}$$

over $\mathbf{P}^d = \mathcal{V}/R$. For each $m \in \mathbf{N}$, let $\mathbf{C}_m[\zeta]$ be the space of all m-homogeneous polynomials $\psi(\zeta)$ in $\zeta = (\zeta^0, \ldots, \zeta^d) \in \mathbf{C}^{d+1}$. For $\psi \in \mathbf{C}_m[\zeta]$, define a holomorphic function $\psi^i : V_i \to \mathbf{C}$ by

$$\psi^i([\zeta]) := \frac{1}{(\zeta^i)^m} \ \psi(\zeta).$$

This depends only on $[\zeta]$ since ψ is m-homogeneous. For $[\zeta] \in V_i \cap V_j$ we have

$$\psi^i([\zeta]) := \frac{(\zeta^j)^m}{(\zeta^i)^m} \ \psi^j(\zeta)$$

by definition. Hence the finite family (ψ^i) defines a holomorphic section of

$$\mathcal{V} \underset{\sim}{\overset{\beta,}{\times}} m\mathbf{C} = (\mathcal{V} \underset{\sim}{\overset{\beta}{\times}} \mathbf{C}^{\times}) \underset{\mathbf{C}^{\times}}{\overset{m}{\times}} \mathbf{C}.$$

Thus we obtain a linear map

$$\mathbf{C}_m[\zeta] \to \mathcal{O}(\mathbf{P}^d \times^\beta \mathbf{C}_m), \quad \psi \mapsto (\alpha^i)$$

which is a $GL_{d+1}(\mathbf{C})$-equivariant isomorphism. After 'symmetry breaking,' $\mathbf{C}^{d+1}$, is isomorphic to the space of all polynomials of degree $\leq d$ on $\mathbf{C}^d$. For $0 \leq a \leq d$ we define a polynomial ψ^a in d variables by

$$\psi^a(z^0, \ldots, \hat{z}^a, \ldots, z^d) := \psi(z^0, \ldots, 1^a, \ldots, z^d)$$

If $\zeta^a \neq 0$ then

$$\psi(\zeta) = \frac{1}{(\zeta^a)^m} \psi^a(\frac{\zeta^0}{\zeta^a}, \ldots, \frac{\hat{\zeta^a}}{\zeta^a}, \frac{\zeta^d}{\zeta^a})$$

It follows that

$$\psi^a = \psi^b \circ \sigma^a_b.$$

Conversely, let $\psi^a,\ 0 \leq a \leq d$ be polynomials in d variables of degree $\leq m$ such that (**??**) holds. Then there is a unique section $\psi \in \mathcal{O}(\mathcal{U} \times_{\sim}^{\sigma,m} \mathbf{C})$ satisfying (**??**). It follows that $\mathcal{O}(\mathcal{U} \times_{\sim}^{\sigma,m} \mathbf{C})$ can be identified with the space of all m-homogeneous polynomials in $\zeta = (\zeta^0, \ldots, \zeta^d)$. This space is irreducible under the natural action of $SL_{d+1}(\mathbf{C})$. For $d = 2$ we obtain the tangent bundle and

$$\mathcal{O}(\mathcal{U} \underset{\sim}{\overset{\sigma,2}{\times}} \mathbf{C}) = \mathcal{O}_1(\mathbf{P}^d).$$

Example 3.1.2. The **tautological bundle** $\mathcal{T}$ over the Grassmannian $M = \mathbf{G}_r(\mathbf{K}^{r+s})$ has the fibre U over $U \in M$. Consider the dual bundle $\mathcal{T}^*$ and the line bundle $\wedge^r \mathcal{T}^*$, whose fibre over U consists of all alternating r-multilinear maps from U to $\mathbf{K}$. For any index chain $1 \leq i_1 < i_2 < \ldots < i_r \leq r+s$ of length r there is a section $\sigma^{i_1,\ldots,i_r}$ of $\wedge^r \mathcal{T}^*$, defined by

$$U \mapsto \sigma_U^{i_1,\ldots,i_r}(v_1 \wedge \ldots \wedge v_r) := \det(v_j|\beta_{i_k})_{j,k=1}^r$$

for all $v_1, \ldots, v_r \in U$. For another $w \in \mathbf{C}^{r \times s}$ we put $v_j := (\beta_j, \beta_j z) \in U$ and $\beta'_k := \beta_k$ for $1 \leq k \leq r$, and $\beta''_k := \beta_k w$. Then

$$(v_j|(\beta'_k, \beta''_k) = (\beta_j|\beta_k) + (\beta_j z|\beta_k w) = (\beta_j(1 + zw^*)|\beta_k)$$

showing that the trivialization $\underline{\sigma}_{z,w} := \sigma^w_{\mathcal{G}(m)} = \det(1 + zw^*)$. Comparing with (**??**) we see that

$$\mathcal{Z} \underset{R}{\times} \mathbf{C} \equiv \wedge^r \mathcal{T}^*$$

and hence $\mathcal{Z} \times_R^n \mathbf{C}$ is the n-th power of $\wedge^r \mathcal{T}^*$. The action of $\hat{G}$ on $H^2_\pi(Z, E)$ is given by

$$(U_g^{-1}\Phi)(\zeta) := (\partial_\zeta g)^{-\pi}\ \Phi(g\zeta)$$

for all $\Phi \in H^2_\pi(\hat{Z}, E)$. Here we use the fact that $\partial_\zeta g \in \mathring{K}$.

Example 3.1.3. In the rank 1 case $Z = \mathbf{C}^{1 \times d}$ the homogeneous line bundle $Z^2 \times_\sigma^{\alpha^n} \mathbf{C}$ over $\hat{Z} = \mathbf{P}^d$ has holomorphic sections $\mathcal{K}_w$ with affine trivialization

$$\underline{\mathcal{K}}_w(m) = (1 + (z|w))^n,$$

where $w \in \mathbf{C}$ is arbitrary and $(z|w)$ denotes the inner product. Thus $\mathcal{O}(Z^2 \times_\sigma^{\alpha^n} \mathbf{C}) \equiv H_n(Z, \mathbf{C})$ consists of all polynomials in z of degree $\leq n$, or equivalently, of all n-homogeneous polynomials in $d+1$ variables, under the natural action of $\hat{G} = SU(d+1)$. For $d = 1$ this space is also described by entire functions f_0, f_∞ on $\mathbf{C}$ satisfying the compatibility condition

$$f_\infty(-\frac{1}{z}) = z^{-n}\ f_0(m)$$

for all $m \in \mathbf{C}^*$.

More explicitly, for any $w \in Z$ there exists a global holomorphic section $\mathcal{K}_w\xi \in \mathcal{O}(Z^2 \times_R^{\delta^n} \mathbf{C})$ with local trivializations

$$m \mapsto \mathcal{K}^a_{z,w} = D^n_{z,a-w},$$

since the relation (**??**) implies

$$D^n_{z,a-b}\, \mathcal{K}^b_{z^{a-b},w} = D^n_{z,a-b} D^n_{z^{a-b},b-w} = D^n_{z,a-w} = \mathcal{K}^a_{z,w}.$$

Proposition 3.1.4. *There is a natural $\hat{G}$-equivariant isomorphism*

$$\mathcal{O}(Z^2 \overset{\delta,n}{\underset{\sim}{\times}} \mathbf{C}) \equiv \mathcal{P}^n(Z) := \sum_{\boldsymbol{m} \leq n} \mathcal{P}_{\boldsymbol{m}}(Z).$$

Proof. Using the Faraut-Korányi formula we obtain

$$\underline{\mathcal{K}}_{z,w} = \mathcal{K}^0_{z,w} = D^n_{z,-w} = \sum_{\boldsymbol{m} \leq n} (-n)_{\boldsymbol{m}} \mathcal{K}^{\boldsymbol{m}}(z,-w) = \sum_{\boldsymbol{m} \leq n} (-1)^{|\boldsymbol{m}|} (-n)_{\boldsymbol{m}} \mathcal{K}^{\boldsymbol{m}}(z,w).$$

□

As a special case of (**??**) the action of $\hat{G}$ on $H^2_n(Z, \mathbf{C})$ is given by

$$(U_g^{-1}\Phi)(\zeta) := \det(\partial_\zeta g)^{-n/p}\, \Phi(g\zeta)$$

for all $\Phi \in H^2_n(\hat{Z}, \mathbf{C})$. Since

$$\det B_{z,w} = D^p_{z,w},$$

where p is the genus of Z, the cocycles (**??**) and (**??**) are related by

$$\det \beta^{w,b}_{z,a} = (\delta^{w,b}_{z,a})^p.$$

On the level of principal bundles this implies

$$Z^2 \overset{\delta^p}{\underset{\sigma}{\times}} \mathbf{C}^* = Z^2 \overset{\det\circ\beta}{\underset{\sigma}{\times}} \mathbf{C}^* = \mathring{G} \overset{\det\circ\partial_0}{\underset{\mathring{G}_0}{\times}} \mathbf{C}^*$$

for the p-th power cocycle δ^p. As a special case consider the determinant character $\delta_k := \det_Z k$ of K. Then (**??**) implies

$$\hat{G} \overset{\delta}{\underset{K}{\times}} \mathbf{C} = Z^2 \overset{\alpha^p}{\underset{\sim}{\times}} \mathbf{C}.$$

In this sense, the line bundle $Z^2 \times^\alpha_\sim \mathbf{C}$ is more fundamental.

Proposition 3.1.5. *Let (E, π) be a holomorphic representation of $\mathring{K}$. Then, for any $w \in Z$ there exists a global holomorphic section $\mathcal{K}_w\xi \in \mathcal{O}(Z^2 \times^\pi_\sim E)$ with local trivializations*

$$m \mapsto \mathcal{K}^a_{z,w}\xi = B^\pi_{z,a-w}\xi.$$

In particular, we have

$$\underline{\mathcal{K}}_{z,w}\xi = \mathcal{K}^0_{z,w}\xi = B^\pi_{z,-w}\xi.$$

Proof. This follows from (**??**) which implies

$$B^\pi_{z,a-b}\, \mathcal{K}_{z^{a-b},w}\xi = B^\pi_{z,a-b} B^\pi_{z^{a-b},b-w}\xi = B^\pi_{z,a-w}\xi = \mathcal{K}^a_{z,w}\xi.$$

□

3.2 Compact Lie Groups and Borel-Weil-Bott Theorem

Theorem 3.2.1. *For a metric Jordan triple Z, the associated Jordan manifolds $\check{Z} \subset Z \subset \hat{Z}$*

$$\dot{Z} = \dot{G}/K = \dot{G}/\dot{G}_m,$$

are coadjoint orbits, for the linear functional $m : \dot{\mathfrak{g}} \to i\mathbf{R}$ defined by

$$mX = \mathrm{tr}\partial_0 X$$

where $\partial_0 X = X'(0) \in \mathring{\mathfrak{k}} \subset \mathfrak{gl}(Z)$.

Proof. The complexified Lie algebra $\mathring{\mathfrak{g}} = \dot{\mathfrak{g}} \otimes \mathbf{C}$ consists of all vector fields

$$X = a + \ell + b^*$$

where $a \in Z$ is the constant vector field,

$$b^*(z) = \frac{1}{2}\{z; b; z\}$$

is a quadratic vector field given by the Jordan triple product, and $\ell = \ell(z) \in \mathring{\mathfrak{k}}$ is a linear vector field. Define an invariant inner product on $\mathring{\mathfrak{g}}$ by

$$\langle a + \ell + b^* | \alpha + \lambda + \beta^* \rangle := \mathrm{tr}(D(a, \beta) + \ell\lambda + D(\alpha, b))$$

for all $a, b, \alpha, \beta \in Z$ and $\ell, \lambda \in \mathring{\mathfrak{k}}$. Via the inner product we may identify $\mathring{\mathfrak{g}}$ and its dual $\mathring{\mathfrak{g}}^*$. We have to find an element $m \in \mathfrak{g}^* \approx \mathfrak{g}$ such that

$$\mathfrak{k} = \dot{\mathfrak{g}}_m = \{X = a + \ell + b^* : \ m \circ ad_X = 0\}.$$

Since $\dot{Z}$ is circular, we have

$$I := z\frac{\partial}{\partial z} \in \mathring{\mathfrak{k}}.$$

This element generates the center of $\mathring{\mathfrak{k}}$ and corresponds to the identity on Z. Now define

$$mX = m(a + \ell + b^*) = \langle I|X\rangle = \mathrm{tr}\ell.$$

The commutator

$$[X, Y] := d_X Y - d_Y X$$

yields $[a, \alpha] = 0$ and $[b^*, \beta^*] = 0$. Moreover $[\ell, \alpha] = \ell\alpha$ as a constant vector field. Moreover,

$$[\ell, \beta^*] = \frac{1}{2}[\ell z, \{z; \beta; z\}] = \{\ell z; \beta, z\} - \frac{1}{2}\ell\{z; \beta; z\} = -\frac{1}{2}\{z; \ell^*\beta; z\} = -(\ell^*\beta)^*$$

as a quadratic vector field. Finally,

$$[a, b^*] = [a, \frac{1}{2}\{z; b; z\}] = \{a; b; z\} = D(a, b)z = D(a, b)$$

viewed as a linear vector field. Therefore

$$ad_X(\alpha + \lambda + \beta^*) = [a + \ell + b^*, \alpha + \lambda + \beta^*] = ([\ell, \alpha] - [\lambda, a]) + ([a, \beta^*] + [\ell, \lambda] - [\alpha, b^*]) + ([\ell, \beta^*] - [\lambda, b^*])$$
$$= (\ell\alpha - \lambda a) + (D(a, \beta) + [\ell, \lambda] - D(\alpha, b^*)) + ((\lambda^* b)^* - (\ell^*\beta)^*).$$

Therefore

$$(m \circ ad_X)(\alpha + \lambda + \beta^*) = \mathrm{tr}(D(a, \beta) + [\ell, \lambda] - D(\alpha, b)) = \mathrm{tr}(D(a, \beta) - D(\alpha, b))$$

since $[\ell, \lambda]$ is a commutator in $\mathring{\mathfrak{k}}$. For $X \in \dot{\mathfrak{g}}$ we need $b = \epsilon a, \beta = \epsilon\alpha$ where $\boldsymbol{e} = \pm$. Thus

$$(m \circ ad_X)(\alpha + \lambda + \epsilon\beta^*) = \epsilon\mathrm{tr}(D(a, \alpha) - D(\alpha, a)).$$

By polarization, it follows that $m \circ ad_X = 0$ if and only if $xtrD(a, \alpha) = 0$ for all $\alpha \in Z$. Since Z is non-degenerate, this means $a = 0$ and therefore $X \in \mathfrak{k}$. □

Consider a compact complex projective manifold M, with structure sheaf $\mathcal{O}$. Let $\mathcal{O}^q$ denote the sheaf of germs of holomorphic sections of the q-th exterior power T^qM. Then $\mathcal{O}^n$ belongs to the canonical bundle of n-forms.

For any holomorphic vector bundle V over M, let $\mathcal{O} \otimes V$ denote the sheaf of germs of holomorphic sections of V, and let $\mathcal{O}^p \otimes V$ denote the sheaf of germs of holomorphic p-form sections of V. Since M is compact, the sheaf cohomology groups $H^q(M, \mathcal{O}^p \otimes V)$ are finite-dimensional complex vector spaces. The **Serre duality theorem** states that $H^q(M, \mathcal{O} \otimes V)$ is in duality with $H^{n-q}(M, \mathcal{O}^n \otimes V^*)$, where V^* is the dual vector bundle of V. Thus

$$H^q(M, \mathcal{O} \otimes V)^* = H^{n-q}(M, \mathcal{O}^n \otimes V^*).$$

• **Jordan manifolds**

Example 3.2.2. The group $\mathring{G} = SL_{1+n}(\mathbf{C})$ contains the parabolic subgroup

$$\mathring{G}_- = \{p = \begin{pmatrix} p_0^0 & b \\ 0 & d \end{pmatrix}\}$$

fixing the line $\mathbf{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\mathring{G}/\mathring{G}_- = \mathbf{P}^n$. For integers m, consider the homogeneous line bundle

$$\mathcal{O}(m) = \mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C} = \{[g, \phi] = [gp, (p_0^0)^m \phi]\}$$

associated with the character $p \mapsto (p_0^0)^m$ of $\mathring{G}_-$. Its holomorphic sections

$$H^0(\mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = \begin{cases} \mathbf{C}_m[\zeta^0, \ldots, \zeta^n] & m \geq 0 \\ 0 & m < 0 \end{cases}.$$

Then $\mathring{G} \times_{\mathring{G}_-}^{n+1} \mathbf{C} = \wedge^n(TM)$ and

$$\mathring{G} \overset{-n-1}{\underset{\mathring{G}_-}{\times}} \mathbf{C} = \wedge^n(T^*M)$$

is the canonical bundle. By a theorem of Serre, we have

$$H^q(\mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = 0$$

for $0 < q < n$. For the n-th cohomology, we apply Serre duality:

$$H^n(\mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C})^* = H^0((\mathring{G} \overset{-n-1}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) \otimes (\mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C})^*) = H^0((\mathring{G} \overset{-n-1}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) \otimes (\mathring{G} \overset{-m}{\underset{\mathring{G}_-}{\times}} \mathbf{C})) = H^0(\mathring{G} \overset{-n-m-1}{\underset{\mathring{G}_-}{\times}} \mathbf{C}).$$

Thus

$$H^n(\mathring{G} \overset{m}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = \begin{cases} 0 & n + m \geq 0 \\ \mathbf{C}_{-n-m-1}[\zeta]^* & n + m < 0 \end{cases},$$

where $\mathbf{C}_k[\zeta]^*$ carries the contragredient representation.

3.2.1 Borel Subgroups and full Flag Manifolds

A semisimple complex Lie algebra $\mathring{\mathfrak{g}}$ with maximal torus $\mathring{\mathfrak{t}} \subset \mathring{\mathfrak{g}}$ has a **root decomposition**

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{t}} \oplus \sum_{\alpha \in \Delta} \mathring{\mathfrak{g}}_\alpha.$$

For every root $\alpha \in \Delta$ there exists a unique 'coroot' $H_\alpha \in [\mathring{\mathfrak{g}}_\alpha, \mathring{\mathfrak{g}}_{-\alpha}]$ satisfying $\alpha H_\alpha = 2$. The **weight lattice**

$$\mathring{T}^* := \{\lambda \in \mathring{\mathfrak{t}}^* : \lambda H_\alpha \in \mathbf{Z} \; \forall \; \alpha \in \Delta\}$$

is a free abelian group of rank $\dim \mathring{\mathfrak{t}}$, containing the roots. The elements of $\mathring{T}^*$ correspond to characters of the group $\mathring{T}$ under taking 'logarithms,' whence the notation. The **Weyl group** $\mathring{W}$ acts on Δ and on $\mathring{T}^*$.

Fix a subset $\Delta_+ \subset \Delta$ of positive roots. There exists a unique element $w_0 \in W$ satisfying $w_0 \Delta_+ = -\Delta_+$. Define

$$\mathring{T}^*_+ := \{\lambda \in \mathring{T}^* : \lambda H_\alpha \geq 0 \; \forall \; \alpha \in \Delta_+\}.$$

The half-sum ρ of positive roots belongs to $\mathring{T}^*_+$, since $\rho H_\sigma = 1$ for each simple (positive) root σ. Define

$$\mathfrak{g}_> := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_< := \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

Then we have the 'Gauss decomposition'

$$\mathring{\mathfrak{g}} = \mathfrak{g}_< \oplus \mathring{\mathfrak{t}} \oplus \mathfrak{g}_>.$$

On the Lie group level this implies that

$$G_< \cdot \mathring{T} \cdot G_> \subset \mathring{G}$$

is a dense open subset. Here $\mathring{G}$ is assumed to be simply-connected, containing a maximal complex torus $\tilde{T}$. Let $g \mapsto g^*$ be an involution such that

$$\hat{\mathfrak{g}} = \{\gamma \in \mathring{\mathfrak{g}} : \gamma^* = -\gamma\}$$

is the Lie algebra of a compact form $\hat{G}$ of $\mathring{G}$.

The **Theorem of the highest weight** is the following:

Theorem 3.2.3. *For every $\lambda \in \mathring{T}^*_+$ there is a finite-dimensional irreducible $\mathring{\mathfrak{g}}$-module, denoted by $\underline{\mathring{G}}_\lambda$, with highest weight λ, and every finite-dimensional irreducible $\mathring{\mathfrak{g}}$-module is isomorphic to $\underline{\mathring{G}}_\lambda$ for a unique $\lambda \in \mathring{T}^*_+$.*

Thus any choice of positive roots yields a bijection

$$\mathring{T}^*_+ \to \mathring{G}^*,$$

where the right-hand side denotes the (discrete) set of all finite-dimensional irreducible $\mathring{\mathfrak{g}}$-modules.

3.2.2 0-Cohomology: Borel-Weil theorem

Lemma 3.2.4. *For every $G^{\mathbf{C}}$-module E there is a G-equivariant mapping*

$$E \to \mathcal{O}(G^{\mathbf{C}}, \mathbf{C}), \quad (\xi, \eta) \mapsto \xi^* \eta$$

defined by

$$(\xi^* \eta)_g := (\xi | g^\pi \eta)$$

for all $\xi, \eta \in E$. We have

$$\rho_{\boldsymbol{g}}(\xi^* \eta) = \xi^*(g^\pi \eta)$$

Proof.

$$(\rho_{\boldsymbol{g}}(\xi^* \eta))_{g_1} = (\xi^* \eta))_{g_1 g} = (\xi | (g_1 g)^\pi \eta) = (\xi | g_1^\pi g^\pi \eta) = (\xi^*(g^\pi \eta))_{g_1}$$

□

Assume that

$$b_+^\pi \xi = \chi(b_+)\xi$$

is a highest weight vector. Then $\overline{b}_-^\pi \xi$

$$(\xi^*\eta)_{b_- g} = (\xi|(b_- g)^\pi \eta) = (\xi|b_-^\pi\ g^\pi \eta) = (b_-^{\pi *}\xi|g^\pi \eta)$$

$$= (\overline{b}_-^\pi \xi|g^\pi \eta) = \chi(\overline{b}_-)(\xi|g^\pi \eta) = \overline{\chi(b_-)}(\xi|g^\pi \eta) = \overline{\chi(b_-)}(\xi^*\eta)_g.$$

In particular,

$$(\xi^*\eta)_{tg} = \overline{\chi(t)}(\xi^*\eta)_g = t^{-\chi}(\xi^*\eta)_g.$$

This shows that

$$\xi^*\eta \in \mathcal{O}(G^{\mathbf{C}} \underset{B_-}{\overset{\chi}{\times}} \mathbf{C}).$$

Consider the simply-connected complex Lie group $\mathring{G}$ with Lie algebra $\mathring{\mathfrak{g}}$.

Theorem 3.2.5. Borel-Weil Theorem: *Let $\lambda \in \mathring{T}^*$ and consider the induced line bundle $\mathring{G} \times_{\mathring{G}_-}^{\lambda} \mathbf{C}$, with trivial action of $\mathring{G}_<$. Then*

$$H^0(\mathring{G} \underset{\mathring{G}_-}{\overset{\lambda}{\times}} \mathbf{C}) = \begin{cases} \underline{\mathring{G}}_\lambda & \lambda \in \mathring{T}_+^* \\ 0 & \lambda \notin \mathring{T}_+^* \end{cases}$$

Here $\underline{\mathring{G}}_\lambda$ is 'the' irreducible $\mathring{G}$-module with highest weight λ.

Proof. The holomorphic sections $H^0(\mathring{G} \times_{\mathring{G}_-}^{\lambda} \mathbf{C})$ are identified with the subspace

$$\{f \in \mathcal{O}(\mathring{G}, \mathbf{C}) :\ f_{gb} = b^{-\lambda}\ f_g \quad \forall\ b \in \mathring{G}_-\}.$$

Assume first that $H^0(\mathring{G} \times_{\mathring{G}_-}^{\lambda} \mathbf{C}) \neq 0$. Then there exists a (non-zero) highest weight vector $f^0 \in H^0(\mathring{G} \times_{\mathring{G}_-}^{\lambda} \mathbf{C})$ satisfying

$$a \ltimes f^0 = a^\chi f^0,\ f_{ag}^0 = a^{-\chi}\ f_g^0$$

for all $a \in \mathring{G}^+$, where χ is a character of $\mathring{G}^+$. For $c \in \mathring{G}^>$, $t \in \mathring{T}$, $d \in \mathring{G}_<$ we have $(ct)^\chi = t^\chi$ and $(td)^\lambda = t^\lambda$. Hence (**??**) and (**??**) imply

$$f_{ctd}^0 = (ct)^{-\chi}\ f_d^0 = t^{-\chi}\ f_e^0 = (td)^{-\lambda} f_c^0 = t^{-\lambda} f_e^0.$$

If $f_e^0 = 0$ then f^0 vanishes on the dense open subset $\mathring{G}^>\ \mathring{T}\mathring{G}_< \subset \mathring{G}$. Hence $f^0 = 0$ by continuity, a contradiction. Thus $f_e^0 \neq 0$ and (**??**) shows $\lambda = \chi$. Since χ is dominant, $\lambda \in \mathring{T}_+^*$ and the second assertion follows.

Conversely let $\lambda \in \mathring{T}_+^*$ be a dominant weight such that (**??**) holds. Let $\underline{\mathring{G}}_\lambda$ be an irreducible $\mathring{G}$-module with highest weight λ. Consider the involution $g \mapsto g^*$ on $\mathring{G}$ such that the compact real form $\hat{G}$ acts unitarily. Let $v^0 \in \underline{G}^\lambda$ be a non-zero highest weight vector (unique up to a scalar multiple). For any $v \in \underline{\mathring{G}}_\lambda$ define a holomorphic function $\tilde{v}$ on $\mathring{G}$ by

$$\tilde{v}_g := (v^0|g^{-\lambda}v).$$

For all $b = ct \in \mathring{G}_-$ we have $b^* \in \mathring{G}^+$ and hence

$$b^{-\lambda *}v^0 = b^{*-\lambda}v^0 = t^{-\lambda}v^0.$$

It follows that

$$\tilde{v}_{gb} = (v^0|(gb)^{-\lambda}v) = (v^0|b^{-\lambda}(g^{-\lambda}v)) = (b^{*-\lambda}v^0|g^{-\lambda}v) = (b^{*-\lambda}v^0|g^{-\lambda}v) = t^{-\lambda}(v^0|g^{-\lambda}v) = t^{-\lambda}\tilde{v}_g.$$

Therefore, via the identification (**??**), we have $\tilde{v} \in H^0(\mathring{G} \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$. The computation

$$(g \cdot \tilde{v})_{g'} = \tilde{v}_{g^{-1}g'} = (v^0|(g^{-1}g')^{-\lambda}v) = (v^0|(g')^{-\lambda}(g^\lambda v)) = \widetilde{gv}_{g'}$$

for $g, g' \in G$ shows

$$g \cdot \tilde{v} = \widetilde{gv}.$$

It follows that the $\mathbf{C}$-linear mapping

$$\underline{\mathring{G}}_\lambda \to H^0(\mathring{G} \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}), \quad v \mapsto \tilde{v}$$

is $\mathring{G}$-equivariant. We have to show that it is an isomorphism. Suppose that $\tilde{v} = 0$ for some $v \in \underline{\mathring{G}}_\lambda$. Then

$$(g^* v^0|v) = (v^0|g^\lambda v) = \tilde{v}_{g^{-1}} = 0$$

for all $g \in G$. By irreducibility, the orbit $\mathring{G}^\lambda v^0$ is total in $\underline{\mathring{G}}_\lambda$. It follows that $v = 0$. Thus (**??**) is also injective, and the range of (**??**) is a $\mathring{G}$-submodule of $H^0(\mathring{G} \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$. For $c \in \mathring{G}^>$, $t \in \mathring{T}$, $d \in \mathring{G}_<$ we have $c^{-\lambda}v^0 = v^0 = d^{-\lambda *}v^0$ and $t^{-\lambda}v^0 = t^{-\lambda}v^0$. It follows that

$$\tilde{v}^0_{ctd} = (v^0|(ctd)^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}c^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}v^0)$$

$$= t^{-\lambda}(v^0|d^{-\lambda}v^0) = t^{-\lambda}(d^{-\lambda *}v^0|v^0) = t^{-\lambda}(v^0|v^0).$$

On the other hand, let $f^0 \in H^0(\mathring{G} \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$ be any highest weight vector. Then $c^{-\lambda}f^0 = f^0$ and hence

$$f^0_{ctd} = (c^{-\lambda}f^0)_{td} = f^0_{td} = t^{-\lambda}f^0_e.$$

Since $\mathring{G}^> \mathring{T}\mathring{G}_<$ is dense in $\mathring{G}$, a continuity argument shows

$$f^0 = \frac{f^0_e}{(v^0|v^0)}\tilde{v}^0.$$

Thus all highest weight vectors in $H^0(\mathring{G} \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$ are proportional. Since distinct irreducible summands would contain distinct highest weight vectors, $H^0(\mathring{G} \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$ is irreducible. Therefore (**??**) defines a $\mathring{G}$-equivariant isomorphism. □

3.2.3 Parabolic subgroups and flag manifolds

We now pass from a maximal torus to an arbitrary torus. Let $\Pi \subset \Delta$ be a set of simple (positive) roots. Let $\Phi \subset \Pi$ be any subset, including the empty set $\Phi = \emptyset$. Then

$${}^\Phi\mathring{\mathfrak{t}} := \{H \in \mathring{\mathfrak{t}} : \ \alpha H = 0 \ \forall \ \alpha \in \Phi\}$$

is a subtorus whose centralizer

$$\mathfrak{c} := \{X \in \mathring{\mathfrak{g}} : \ [X, {}^\Phi\mathring{\mathfrak{t}}] = 0\}$$

is a reductive Lie algebra, with Levi decomposition

$$\mathfrak{c} = {}^\Phi\mathring{\mathfrak{t}} \oplus \mathring{\mathfrak{g}}^\Phi$$

Its semi-simple commutator ideal $\mathring{\mathfrak{g}}^\Phi$ has itself a Gauss decomposition

$$\mathring{\mathfrak{g}}^\Phi = \mathring{\mathfrak{g}}^\Phi_< \oplus \mathring{\mathfrak{t}}^\Phi \oplus \mathring{\mathfrak{g}}^\Phi_>$$

where

$$\overset{\circ}{\mathfrak{t}}^{\Phi} :=< H_\alpha : \ \alpha \in \Delta \cap \mathbf{Z} \cdot \Phi >\subset \overset{\circ}{\mathfrak{t}}$$

and

$$\overset{\circ}{\mathfrak{g}}^{\Phi}_{>} = \sum_{\alpha\in\Delta_+\cap\mathbf{Z}\cdot\Phi} \overset{\circ}{\mathfrak{g}}_\alpha, \quad \overset{\circ}{\mathfrak{g}}^{\Phi}_{<} = \sum_{\alpha\in\Delta_+\cap\mathbf{Z}\cdot\Phi} \overset{\circ}{\mathfrak{g}}_{-\alpha}.$$

On the other hand, define

$$^{\Phi}\overset{\circ}{\mathfrak{g}}_{>} := \sum_{\alpha\in\Delta_+\backslash\mathbf{Z}\cdot\Phi} \mathfrak{g}_\alpha, \quad ^{\Phi}\overset{\circ}{\mathfrak{g}}_{<} := \sum_{\alpha\in\Delta_+\backslash\mathbf{Z}\cdot\Phi} \mathfrak{g}_{-\alpha}.$$

Then the parabolic subalgebra is

$$^{\Phi}\overset{\circ}{\mathfrak{g}}_{-} = {}^{\Phi}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi} \oplus {}^{\Phi}\overset{\circ}{\mathfrak{g}}_{<} = {}^{\Phi}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi}_{<} \oplus \overset{\circ}{\mathfrak{t}}^{\Phi} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi}_{>} \oplus {}^{\Phi}\overset{\circ}{\mathfrak{g}}_{<} = \overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}_{<} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi}_{>} = \mathfrak{g}_{-} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi}_{>}$$

since $\overset{\circ}{\mathfrak{t}} = \overset{\circ}{\mathfrak{t}}^{\Phi} \oplus {}^{\Phi}\overset{\circ}{\mathfrak{t}}$ and $\overset{\circ}{\mathfrak{g}}^{\Phi}_{<} \oplus {}^{\Phi}\overset{\circ}{\mathfrak{g}}_{<} = \overset{\circ}{\mathfrak{g}}_{<}$.

Thus in the non-empty case $\Phi \neq \emptyset$ the reductive torus centralizer ${}^{\Phi}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\Phi}$ plays the role of the torus $\overset{\circ}{\mathfrak{t}}$ and ${}^{\Phi}\overset{\circ}{\mathfrak{g}}_{<}$ is the unipotent radical. Compared to the line bundles in the case $\Phi = \emptyset$, we now have vector bundles since the semi-simple part $\overset{\circ}{\mathfrak{g}}^{\Phi}$ has higher dimensional irreducible highest weight representations.

In the special case $\Phi = \emptyset$ we have ${}_{\emptyset}\overset{\circ}{\mathfrak{t}} = 0$, ${}^{\emptyset}\overset{\circ}{\mathfrak{t}} = \overset{\circ}{\mathfrak{t}}$, ${}_{\emptyset}\overset{\circ}{\mathfrak{g}} = 0$, since $\mathfrak{t}$ is maximal. Therefore

$$^{\emptyset}\overset{\circ}{\mathfrak{g}}_{-} = \overset{\circ}{\mathfrak{t}} \oplus \sum_{\alpha\in\Delta_+} \overset{\circ}{\mathfrak{g}}_{-\alpha} = \overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}_{<} = \overset{\circ}{\mathfrak{g}}_{-}$$

is a Borel subalgebra. In the opposite case $\Phi = \Pi$ we have ${}_{\Pi}\overset{\circ}{\mathfrak{t}} = \overset{\circ}{\mathfrak{t}}, {}^{\Pi}\overset{\circ}{\mathfrak{t}} = 0$ and hence ${}_{\Pi}\overset{\circ}{\mathfrak{g}} = {}^{\Pi}\overset{\circ}{\mathfrak{g}}_{-} = \mathfrak{g}$ is the full Lie algebra.

3.2.4 q-Cohomology: Bott's Theorem

For passing from 0-cohomology to q-cohomology, in case λ is not dominant, we use reflections by simple roots. Let $\Phi = \{\sigma\}$, where $\sigma \in \Pi$ is a simple root. Then there is a splitting

$$\overset{\circ}{\mathfrak{t}} = {}^{\sigma}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{t}}^{\sigma}$$

where $\overset{\circ}{\mathfrak{t}}^{\sigma} := \mathbf{C} \cdot H_\sigma$ and ${}^{\sigma}\overset{\circ}{\mathfrak{t}} := \{H \in \overset{\circ}{\mathfrak{t}} : \ \sigma H = 0\}$. The torus centralizer

$$\mathfrak{c} := \{X \in \overset{\circ}{\mathfrak{g}} : \ [X, {}^{\sigma}\overset{\circ}{\mathfrak{t}}] = 0\} = {}^{\sigma}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\sigma}$$

is a reductive Lie algebra, and the Gauss decomposition of its semi-simple commutator ideal $\overset{\circ}{\mathfrak{g}}^{\sigma}$ simplifies to

$$\overset{\circ}{\mathfrak{g}}^{\sigma} = \mathfrak{g}_{-\sigma} \oplus \mathbf{C} \cdot H_\sigma \oplus \mathfrak{g}_\sigma \equiv \mathfrak{sl}_2(\mathbf{C}),$$

since $\overset{\circ}{\mathfrak{g}}^{\sigma}_{<} = \mathfrak{g}_{-\sigma}$ and $\overset{\circ}{\mathfrak{g}}^{\sigma}_{>} = \mathfrak{g}_\sigma$. On the other hand, define

$$^{\sigma}\mathfrak{g}_{>} := \sum_{\alpha\in\Delta_+\backslash\sigma} \mathfrak{g}_\alpha, \quad ^{\sigma}\mathfrak{g}_{<} := \sum_{\alpha\in\Delta_+\backslash\sigma} \mathfrak{g}_{-\alpha}.$$

The parabolic subalgebra is

$$^{\sigma}\overset{\circ}{\mathfrak{g}}_{-} = {}^{\sigma}\overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\sigma} \oplus {}^{\sigma}\mathfrak{g}_{<} = \overset{\circ}{\mathfrak{t}} \oplus \overset{\circ}{\mathfrak{g}}^{\sigma} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_{-\alpha} = \overset{\circ}{\mathfrak{g}}^{\sigma} \oplus \mathfrak{g}_{-}.$$

Thus we have added one positive root space $\mathfrak{g}_\sigma$ to the Borel subalgebra $\mathfrak{g}_-$. Since $\dim \mathfrak{g}_\alpha = 1$ we have

$$^{\sigma}\overset{\circ}{G}_{-}/\overset{\circ}{G}_{-} = \mathbf{P}^1.$$

Take $E_\sigma \in \mathfrak{g}_\sigma,\ F_\sigma \mathfrak{g}_{-\sigma}$ with $[E_\sigma, F_\sigma] = H_\sigma$. Since $\mathring{G}$ is supposed to be simply-connected, ${}^\sigma\mathring{G}_-$ has a Levi decomposition

$$^\sigma\mathring{G}_- = {}^\sigma\mathring{T}\ \mathring{G}^\sigma\ {}^\sigma\mathring{G}_<$$

where the semi-simple part $\mathring{G}^\sigma \equiv \mathrm{SL}_2(\mathbf{C})$ has the Lie algebra

$$\mathring{\mathfrak{g}}^\sigma =< E_\sigma, F_\sigma, H_\sigma >$$

and the complex torus ${}^\sigma\mathring{T} \subset \mathring{T}$ has the Lie algebra

$$^\sigma\mathring{\mathfrak{t}} := \{H \in \mathring{\mathfrak{t}} :\ \sigma H = 0\}.$$

Let $\lambda \in T^*$ satisfy $m := \lambda H_\sigma \geq 0$. Let

$$_\sigma\underline{\mathring{\mathfrak{g}}}_m :=< v_m, v_{m-2}, \ldots, v_{2-m}, v_{-m} >$$

be the $m+1$-dimensional 'spin' representation of $_\sigma\mathring{\mathfrak{g}} \equiv \mathfrak{sl}_2(\mathbf{C})$. Then

$$H_\sigma v_k = k\ v_k,\ \ E_\sigma v_k \in \mathbf{C}v_{k+2},\ \ F_\sigma v_k \in \mathbf{C}v_{k-2}$$

for all k, putting $v_k = 0$ if $|k| > m$. Since $\mathring{G}$ is assumed to be simply-connected, one can show that $_\sigma\mathring{G} \equiv \mathrm{SL}_2(\mathbf{C})$ and hence the infinitesimal action on $_\sigma\underline{\mathring{\mathfrak{g}}}_m$ can be integrated to an action π of $_\sigma\mathring{G}$ denoted by $_\sigma\underline{\mathring{G}}_m$. The highest weight vector v_m satisfies $p(H_\sigma)v_m = p(m)v_m$ for all polynomials p and hence

$$\exp(zH_\sigma)^\pi v_m = e^{zm} v_m$$

for all $z \in \mathbf{C}$. Now suppose that $t \in T^\sigma \cap {}_\sigma\mathring{G}$. Then $t = \exp(zH_\sigma)$ for some $z \in \mathbf{C}$. This implies

$$t^\lambda \cdot v_m = \exp(zH_\sigma)^\lambda \cdot v_m = e^{z\lambda H_\sigma} \cdot v_m = e^{zm} v_m = \exp(zH_\sigma)^\pi v_m = t^\pi v_m.$$

Since $_\sigma\mathring{G}$ centralizes ${}^\sigma\mathring{T}$, it follows that

$$t^\pi(s^\pi v_m) = (ts)^\pi v_m = (st)^\pi v_m = s^\pi t^\pi v_m = s^\pi(t^\lambda v_m) = t^\lambda \cdot (s^\pi v_m)$$

for all $s \in {}_\sigma\mathring{G}$. Since the set $_\sigma\mathring{G}^\pi v_m$ is total in $_\sigma\underline{\mathring{G}}_m$, it follows that

$$t^\pi v = t^\lambda v$$

for all $v \in {}_\sigma\underline{\mathring{G}}_m$. Thus the two representations agree on $_\sigma\mathring{G} \cap {}^\sigma\mathring{T}$ and therefore induce an irreducible representation of $_\sigma\mathring{G}\ {}^\sigma\mathring{T}$ which extends trivially to a representation of ${}^\sigma\mathring{G}_-$. We denote this module by ${}^\sigma\underline{\mathring{G}}_m^-$.

Lemma 3.2.6. *Let $\lambda \in T^*$ satisfy $m := \lambda H_\sigma \geq 0$. Then there is an exact sequence of $\mathring{G}_-$-modules*

$$0 \to M \to {}^\sigma\underline{\mathring{G}}_\lambda^- \to \underline{\mathring{G}}_\lambda^- \to 0$$

such that

$$\begin{cases} M = 0 & m = 0 \\ M = \underline{\mathring{G}}_{s_\sigma\lambda}^- & m = 1 \\ 0 \to \underline{\mathring{G}}_{s_\sigma\lambda}^- \to M \to {}^\sigma\underline{\mathring{G}}_{\lambda-\sigma}^- \to 0 & m \geq 2 \end{cases}$$

Proof. Define a $\mathring{G}_-$-submodule

$$M :=< v_{m-2}, \ldots, v_{2-m}, v_{-m} >$$

Since

$$\underline{\mathring{G}}_\lambda^\sigma / M =< v_m >$$

with $tv_m = \Phi(t)v_m$ and $\vartheta \cdot v_m = \dot{\Phi}v_m = \lambda v_m$, we obtain an exact sequence

$$0 \to M \to \underline{\mathring{G}}_\lambda^\sigma \to \mathbf{C}_\lambda \to 0.$$

If $m = \lambda H_\sigma = 0$ then $M = 0$ by definition. If $m = \lambda H_\sigma = 1$ then

$$M = < v_{-1} > = \mathbf{C}_{\lambda - \sigma} = \mathbf{C}_{s_\sigma \lambda}$$

since in general v_k has weight $\lambda - \frac{m-k}{2}\sigma$. If $m = \lambda H_\sigma \geq 2$ then $< v_{-m} >$ is a $\mathring{G}_-$-submodule of M isomorphic to $\mathbf{C}_{\lambda - m\sigma} = \mathbf{C}_{s_\sigma \lambda}$. The quotient module is

$$M / < v_{-m} > = \underline{\mathring{G}}_{\lambda-\sigma}^\sigma.$$

This yields the exact sequence

$$0 \to \mathbf{C}_{s_\sigma \lambda} \to M \to \underline{\mathring{G}}_{\lambda-\sigma}^\sigma.$$

□

Lemma 3.2.7. *Let $\pi : G \to GL(E)$ be a holomorphic representation and consider the restricted representation $\pi : H \to GL(E)$. Then the map*

$$G \underset{H}{\times} E \to G/H \times E, \ [g, v] \mapsto (gH, g^\pi v)$$

is an isomorphism.

Proof. The calculation

$$[g, v] = [gh, h^{-\pi} v] \mapsto (ghH, (gh)^\pi h^{-\pi} v) = (gH, g^\pi v)$$

shows that the map (**??**) is well-defined. It is clearly surjective. To show injectivity, let $(gH, g^\pi v) = (g_1 H, g_1^\pi v_1)$. Then $h := g^{-1} g_1 \in H$ and $h^{-\pi} v = g_1^{-\pi} g^\pi v = v_1$. Thus $[g, v] = [gh, h^{-\pi} v] = [g_1, v_1]$. □

Proposition 3.2.8. *Let V be a (holomorphic) ${}^\sigma\mathring{G}_-$-module and let $\lambda \in T^*$ satisfy $\lambda H_\sigma = -1$. Then*

$$H^k(G \underset{\mathring{G}_-}{\times} (\underline{\mathring{G}}_\lambda^- \otimes V)) = 0 \quad \forall \ k \geq 0.$$

Proof. We have

$${}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} (\underline{\mathring{G}}_\lambda^- \otimes V)) = ({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}_\lambda^-) \otimes ({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} V)$$

and the condition $\lambda H_\sigma = -1$ implies that

$${}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}_\lambda^- = \mathcal{O}(-1)$$

as a line bundle over ${}^\sigma\mathring{G}_- / \mathring{G}_- \equiv \mathbf{P}^1$. By Proposition **??** it follows that

$$H^q({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}_\lambda^-) = H^q(\mathbf{P}^1, \mathcal{O}(-1)) = 0$$

for $q = 0, 1$. On the other hand,

$${}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} V = \mathbf{P}^1 \times V$$

is a trivial vector bundle by Lemma 3.2.7, since V carries a representation of ${}^\sigma\mathring{G}_- \supset \mathring{G}_-$. Therefore

$$H^q({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} (\underline{\mathring{G}}_\lambda^- \otimes V)) = H^q(({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}_\lambda^-) \otimes ({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} V)) = 0$$

for all q. The fibration

$$ {}^\sigma\mathring{G}_-/\mathring{G}_- \to G/\mathring{G}_- \to G/{}^\sigma\mathring{G}_- $$

induces the **Leray spectral sequence**

$$ H^{p+q}(G \underset{\mathring{G}_-}{\times} (\underline{\mathring{G}}^-_\lambda \otimes V)) = H^p(G \underset{{}^\sigma\mathring{G}_-}{\times} H^q({}^\sigma\mathring{G}_- \underset{\mathring{G}_-}{\times} (\underline{\mathring{G}}^-_\lambda \otimes V)). $$

The assertion follows. □

For each simple root σ, the reflection $s_\sigma \in W$ acts on $\mathbf{t}^*$ by

$$ s_\sigma(\lambda) := \lambda - (\lambda H_\sigma)\sigma = \lambda - (\lambda|\sigma)\sigma. $$

These reflections generate the Weyl group $W_T(G)$. Define an affine action of W on $\mathbf{t}^*$ by

$$ w \cdot \lambda := w(\lambda + \rho) - \rho. $$

Lemma 3.2.9. *Let $\sigma \in \Pi$ be a simple root and $\lambda \in T^*$ such that $(\lambda + \rho)H_\sigma \geq 0$. Then, as G-modules,*

$$ H^k(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) \equiv H^{k+1}(G \overset{s_\sigma\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) \quad \forall\, k \in \mathbf{Z}. $$

Proof. Assume $m = (\lambda + \rho)H_\alpha \geq 2$. Then Lemma 3.2.6 yields exact $\mathring{G}_-$-module sequences

$$ 0 \to M \to {}^\sigma\underline{\mathring{G}}^-_{\lambda+\rho} \to \underline{\mathring{G}}^-_{\lambda+\rho} \to 0, $$

$$ 0 \to \underline{\mathring{G}}^-_{s_\sigma(\lambda+\rho)} \to M \to {}^\sigma\underline{\mathring{G}}^-_{\lambda+\rho-\sigma} \to 0 $$

Tensoring with $\underline{\mathring{G}}^-_{-\rho}$ yields exact $\mathring{G}_-$-module sequences

$$ 0 \to M \otimes \underline{\mathring{G}}^-_{-\rho} \to {}^\sigma\underline{\mathring{G}}^-_{\lambda+\rho} \otimes \underline{\mathring{G}}^-_{-\rho} \to \underline{\mathring{G}}^-_{\lambda} \to 0, $$

$$ 0 \to \underline{\mathring{G}}^-_{s_\sigma\cdot\lambda} \to M \otimes \underline{\mathring{G}}^-_{-\rho} \to {}^\sigma\underline{\mathring{G}}^-_{\lambda+\rho-\sigma} \otimes \underline{\mathring{G}}^-_{-\rho} \to 0. $$

The corresponding sequences of holomorphic $\mathring{G}$-module sheaves are also exact. Since $\rho H_\sigma = 1$, Proposition 3.2.8 yields

$$ H^k(G \underset{\mathring{G}_-}{\times} ({}^\sigma\underline{\mathring{G}}^-_\mu \otimes \underline{\mathring{G}}^-_{-\rho})) = 0 $$

for $\mu = \lambda + \rho$ and $\mu = \lambda + \rho - \sigma$. Therefore the corresponding exact cohomology sequence implies

$$ H^k(G \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_\lambda) \equiv H^{k+1}(G \underset{\mathring{G}_-}{\times} (M \otimes \underline{\mathring{G}}^-_{-\rho})) \equiv H^{k+1}(G \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{s_\sigma\cdot\lambda}) $$

for all $k \in \mathbf{Z}$. □

Lemma 3.2.10. *Let $\lambda \in \mathring{T}^*$ with $\lambda + \rho \in \mathring{T}^*_+$. Then, as $\mathring{G}$-modules, for all $w \in W$*

$$ H^k(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+|w|}(G \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) \quad \forall\, k \in \mathbf{Z} $$

Proof. The proof uses induction over $\ell \geq 1$. For $\ell = 1$, we have $w = s_\sigma$ for some simple root σ and Lemma 3.2.9 applies. Now let $w = s_0 \cdots s_\ell$ be a product of minimal length $\ell + 1$, with $s_k = s_{\alpha_k}$ for simple roots α_k. Suppose we have

$$ s_{k-1} \cdots s_1 \alpha_0 = \alpha_k $$

for some $1 \le k \le \ell$. Then $(s_{k-1} \cdots s_1)s_0(s_1 \cdots s_k) = s_k$ and hence

$$w = s_0 \cdots s_{k-1}s_{k+1} \cdots s_\ell$$

has length $\le \ell$, a contradiction. Thus (**??**) cannot happen for any k. Since

$$s_\sigma(\Delta^+ - \sigma) = \Delta^+ - \sigma$$

for any simple root σ, it follows that $w' := s_1 \cdots s_\ell$ satisfies

$$w'^{-1}\alpha_0 = s_\ell \cdot s_1\alpha_0 \in \Delta^+.$$

Putting $\sigma = \alpha_0$ we have

$$(w' \cdot \lambda + \rho)H_\sigma = w'(\lambda + \rho)H_\sigma = (\lambda + \rho)H_{w'^{-1}\sigma} \ge 0$$

since $\lambda + \rho \in \mathring{\mathbf{T}}^*_+$. Applying the induction hypothesis to w', of length $\le \ell$, and Lemma 3.2.9 to $w' \cdot \lambda$ we obtain $\mathring{G}$-module isomorphisms

$$H^k(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell}(G \overset{w'\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell+1}(G \overset{s_\sigma\cdot w'\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell+1}(G \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C})$$

□

Corollary 3.2.11. *Let $\lambda + \rho \in \mathring{T}^*_+$. Then*

$$H^k(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = 0 \quad \forall\ k > 0.$$

Proof. An element $w \in W$ of maximal length satisfies $w(\Delta^+) = \Delta^-$. This implies that $\ell = \ell(w) = \dim_{\mathbf{C}} \mathring{G}/\mathring{G}_-$. Applying Lemma 3.2.10 we obtain

$$H^k(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell}(G \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = 0$$

for $k > 0$, since $k + \ell > \dim_{\mathbf{C}} \mathring{G}/\mathring{G}_-$. □

A linear form $\mu \in \mathring{T}^*$ is called **regular**, if $\mu H_\alpha \neq 0$ for all $\alpha \in \Delta$. Then there exists a unique $w = w_\mu \in \mathring{W}$ such that $w(\mu) \in \mathring{T}^*_+$.

Theorem 3.2.12. *(Bott) Let ${}^\Phi\underline{\mathring{G}}^-_\lambda$ be irreducible with highest weight λ. If $\lambda + \rho$ is singular, then*

$$H^k(\mathring{G} \underset{{}^\Phi\mathring{G}_-}{\times} {}^\Phi\underline{\mathring{G}}^-_\lambda) = 0$$

for all $k \ge 0$. If $\lambda + \rho$ is regular, let $w \in W$ be the unique element such that $w(\lambda + \rho) \in \mathring{T}^*_+$. Then

$$H^k(\mathring{G} \underset{{}^\Phi\mathring{G}_-}{\times} {}^\Phi\underline{\mathring{G}}^-_\lambda) = \begin{cases} \underline{\mathring{G}}_{w\cdot\lambda} & k = |w| \\ 0 & k \neq |w| \end{cases}.$$

Here $\underline{\mathring{G}}_{w\cdot\lambda}$ is 'the' irreducible $\mathring{G}$-module of highest weight $w \cdot \lambda$.

Proof. We first consider line bundles over $\mathring{G}_-$. ($\Phi = \emptyset$.). Choose $w \in W$ with $w \cdot \lambda + \rho = w(\lambda + \rho) \in \mathring{T}^*_+$. Assume first that $\lambda + \rho$ is singular. Then $w \cdot \lambda + \rho = w(\lambda + \rho)$ is also singular. Thus there exists a simple root σ with $(w\cdot\lambda+\rho)H_\sigma = 0$. Hence $(w\cdot\lambda)H_\sigma = -\rho H_\sigma = -1$. Applying Lemma 3.2.10 and Proposition 3.2.8 (for $V = \mathbf{C}$) we obtain

$$H^k(\mathring{G} \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell}(\mathring{G} \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = 0.$$

Now let $\lambda+\rho$ be regular. Then w is unique. Since $w\cdot\lambda+\rho \in \mathring{T}^*_+$, Lemma 3.2.10 implies

$$H^{k+\ell}(\mathring{G} \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^{k+\ell}(\mathring{G} \overset{w^{-1}\cdot w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^k(\mathring{G} \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C})$$

for all $k \in \mathbf{Z}$. For $k < 0$ this vanishes trivially. For $k > 0$ this vanishes by Corollary 3.2.11. For $k = 0$ we obtain

$$H^\ell(\mathring{G} \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^0(\mathring{G} \overset{w\cdot\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = \underline{G}_{w\cdot\lambda}$$

by the Borel-Weil Theorem 3.2.5. Here we use that $w\cdot\lambda \in \mathring{T}^*_+$ since $w\cdot\lambda+\rho$ is regular, so that $(w\cdot\lambda+\rho)H_\sigma \geq 1$ for all simple roots σ, and therefore $(w\cdot\lambda)H_\sigma = (w\cdot\lambda+\rho)H_\sigma - \rho H_\sigma = (w\cdot\lambda+\rho)H_\sigma - 1 \geq 0$. □

The final step in the proof is achieved by

Proposition 3.2.13. *Let ${}^\Phi\underline{\mathring{G}}^-_\lambda$ be an irreducible holomorphic representation of ${}^\Phi\mathring{G}_-$ with highest weight λ. Then, as $\mathring{G}$-modules,*

$$H^k(\mathring{G} \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_\lambda) = H^k(\mathring{G} \underset{{}^\Phi\mathring{G}_-}{\times} {}^\Phi\underline{\mathring{G}}^-_\lambda) \quad \forall\ k \geq 0.$$

Proof. We first show that

$$H^0({}^\Phi\mathring{G}_- \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = {}^\Phi\underline{\mathring{G}}^-_\lambda$$

is an irreducible ${}^\Phi\mathring{G}$-module of highest weight λ. The parabolic subgroup ${}^\Phi\mathring{G}_-$ has a Levi decomposition

$${}^\Phi\mathring{G}_- = {}^\Phi\mathring{T}\ {}^\Phi\mathring{G}\ {}^\Phi\mathring{G}_<$$

where ${}^\Phi\mathring{G}$ is semi-simple and connected, ${}^\Phi\mathring{T} \subset \mathring{T}$ is a complex torus and ${}^\Phi\mathring{G}_<$ is the unipotent radical of ${}^\Phi\mathring{G}_-$. We have

$${}^\Phi\mathring{G}_- = {}^\Phi\mathring{T}\ {}^\Phi\mathring{G}\ {}^\Phi\mathring{G}_< = {}^\Phi\mathring{T}\ {}^\Phi\mathring{G}_<\ {}^\Phi\mathring{T}\ {}^\Phi\mathring{G}^>\ {}^\Phi\mathring{G}_< = \mathring{T}\ {}^\Phi\mathring{G}^>\ \mathring{G}_< = {}^\Phi\mathring{G}^>\ \mathring{G}_-.$$

Since the unipotent radical always acts trivially we have to check the actions of ${}^\Phi\mathring{G}$ and ${}^\Phi\mathring{T}$ on $H^0({}^\Phi\mathring{G}_- \times^\lambda_{\mathring{G}_-} \mathbf{C})$. The semi-simple Lie group ${}^\Phi\mathring{G}$ has the Borel subgroup

$${}^\Phi\mathring{G}_- = {}^\Phi\mathring{T}\ {}^\Phi\mathring{G}_<.$$

It follows that

$${}^\Phi\mathring{G}_-/\mathring{G}_- = {}^\Phi\mathring{G}^> = {}^\Phi\mathring{G}/{}^\Phi\mathring{G}_-.$$

Hence the inclusion map $\iota : {}^\Phi\mathring{G} \to {}^\Phi\mathring{G}_-$ induces a biholomorphic map $\iota : {}^\Phi\mathring{G}/{}^\Phi\mathring{G}_- \to {}^\Phi\mathring{G}_-/\mathring{G}_-$ satisfying

$$\iota^*({}^\Phi\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_\lambda) = {}^\Phi\mathring{G} \overset{\lambda'}{\underset{{}^\Phi\mathring{G}_-}{\times}} \mathbf{C},$$

where $\lambda' := \lambda|_{\mathring{\mathfrak{t}}^\Phi}$. This implies

$$H^0({}^\Phi\mathring{G}_- \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = H^0({}^\Phi\mathring{G} \overset{\lambda'}{\underset{{}^\Phi\mathring{G}_-}{\times}} \mathbf{C})$$

as ${}^{\Phi}\mathring{G}$-modules. Applying the first part of the proof (Bott's theorem for line bundles) to ${}^{\Phi}\mathring{G}/{}^{\Phi}\mathring{G}_-$ it follows that

$$H^0({}^{\Phi}\mathring{G} \overset{\lambda'}{\underset{{}^{\Phi}\mathring{G}_-}{\times}} \mathbf{C}) = {}^{\Phi}\underline{\mathring{G}}_{\lambda'}$$

is an irreducible ${}^{\Phi}\mathring{G}$-module of highest weight λ'. Let $f^0 \in {}^{\Phi}\underline{\mathring{G}}_{\lambda'}$ be a highest weight vector. Since $H^0({}^{\Phi}\mathring{G}_- \times^{\lambda}_{\mathring{G}_-} \mathbf{C})$ is irreducible under ${}^{\Phi}\mathring{G}$, it is 'a fortiori' irreducible under ${}^{\Phi}\mathring{G}_-$. In order to find its highest weight, recall that

$$H^0({}^{\Phi}\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda}) = \{f \in \mathcal{O}({}^{\Phi}\mathring{G}_-) : \; f(pb) = b^{-\Phi} \; f(b) \; \forall \; p \in {}^{\Phi}\mathring{G}_-, \; b \in \mathring{G}_-\}.$$

For $t \in {}^{\Phi}\mathring{T}$ and $p = scu$, with $s \in {}^{\Phi}\mathring{G}, \; c \in {}^{\Phi}\mathring{T}, \; u \in {}^{\Phi}\mathring{G}_<$ we have $t^{-1}sc = sct^{-1}$ since ${}^{\Phi}\mathring{T}$ ${}^{\Phi}\mathring{G}$ is the centralizer of ${}^{\Phi}\mathring{T}$ in $\mathring{G}$. Let $\lambda'' := \lambda_{{}^{\Phi}\mathring{\mathfrak{t}}}$. Then

$$(t \cdot f^0)_p = f^0(t^{-1}p) = f^0(t^{-1}scu) = f^0(t^{-1}sc) = f^0(sct^{-1}) = t^{\lambda''} \; f^0(sc) = t^{\lambda''} \; f^0(scu) = t^{\lambda''} \; f^0(p).$$

Hence f^0 is a highest weight vector for the weight $\lambda = (\lambda', \lambda'')$ under the action of $\mathring{T} = {}^{\Phi\circ}T \; {}^{\Phi\circ}T$. Therefore (**??**) holds. Under the inclusion map $\iota : {}^{\Phi}\mathring{G}_-/\mathring{G}_- \to G/\mathring{G}_-$ the pull-back is the homogeneous line bundle

$$\iota^*(G \overset{\lambda}{\underset{\mathring{G}_-}{\times}} \mathbf{C}) = {}^{\Phi}\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda}.$$

Applying the Leray spectral sequence

$$H^{p+q}(\mathring{G} \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda}) = H^p(\mathring{G} \underset{{}^{\Phi}\mathring{G}_-}{\times} H^q({}^{\Phi}\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda}))$$

to the special case $q = 0$ and using (**??**) yields the assertion

$$H^k(\mathring{G} \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda}) = H^k(\mathring{G} \underset{{}^{\Phi}\mathring{G}_-}{\times} H^0({}^{\Phi}\mathring{G}_- \underset{\mathring{G}_-}{\times} \underline{\mathring{G}}^-_{\lambda})) = H^k(\mathring{G} \underset{{}^{\Phi}\mathring{G}_-}{\times} {}^{\Phi}\underline{\mathring{G}}^-_{\lambda}).$$

□

Let G/T be a compact flag manifold, where T is the centralizer of a torus. Consider the complexified Lie algebra

$$\mathfrak{g}^{\mathbf{C}},$$

with Cartan subalgebra $\mathfrak{t}^{\mathbf{C}}$ and Weyl group $W := N(T)/T$. For every $w \in W$ we obtain a Borel subalgebra $\mathfrak{g}^w \subset \mathfrak{g}^{\mathbf{C}}$ such that

$$\mathfrak{g}^w_+ \cap \mathfrak{g}^w_- = \mathfrak{t}^{\mathbf{C}}.$$

$$T \subset G$$

torus, centralizer $C_G(T)$ $G/C_G(T)$ flag domain

3.3 Compact Kähler Manifolds and Kodaira Embedding Theorem

3.3.1 Chern Classes, Divisors and Positivity

Recall that for a positive definite hermitian metric

$$\sum_{i,j} h_{ij} dz^i d\overline{z}^j$$

with $(h_{ij}) > 0$ positive definite, the associated $(1,1)$-form

$$-i\omega := \sum_{i,j} h_{ij} dz^i \wedge d\overline{z}^j$$

is called a **Kähler form** if $d\omega = 0$.

Lemma 3.3.1. *On a ball $U \subset \mathbf{C}^n$ a $(1,1)$-form ω is closed if and only if $-i\omega = \partial\overline{\partial}K$ for some smooth real function K.*

Proof. □

Corollary 3.3.2. *The $(1,1)$-form ω associated with a ^{0}metric h is a Kähler form if and only if for a covering (V_a) there exist smooth functions $K_a : V_a \to \mathbf{R}$ such that*

$$-i\omega|_{V_a} = \partial\overline{\partial}K_a$$

for all a.

Proof. $d\omega = 0$ if and only if $\omega|_{V_a}$ is closed for all a. □

A smooth function $K : M \to \mathbf{R}$ is called ^{0}**plurisubharmonic** if the hermitian **Levi form**

$$\sum_{i,j} \frac{\partial^2 K}{\partial z^i \partial \overline{z}^j}\ dz^i d\overline{z}^j$$

is ^{0}positive. In this case

$$-i\omega = \partial\overline{\partial}K = \sum_{i,j} \frac{\partial^2 K}{\partial z^i \partial \overline{z}^j}\ dz^i \wedge d\overline{z}^j$$

is a Kähler form.

Proposition 3.3.3. *$\mathbf{P}^n$ is a Kähler manifold.*

Proof. For $0 \leq a \leq n$ let $z' = (z^0, \ldots, \hat{z}^a, \ldots, z^n)$ with $z^j = \frac{\zeta^j}{\zeta^a}$. Define $K_a : V_a \to \mathbf{R}$ by

$$K_a[\zeta] = \log(1 + (z'|z')) = \log(1 + \sum_{j \neq a} |z^j|^2) = \log(1 + \sum_{j \neq a} |\frac{\zeta^j}{\zeta^a}|^2) = \log|\zeta|^2 - \log|\zeta^a|^2.$$

Then on $V_a \cap V_b$ we have

$$K_a[\zeta] - K_b[\zeta] = \log|\frac{\zeta^b}{\zeta^a}|^2 = \log|\sigma_a^b(z)|^2 = \log\sigma_a^b + \log\overline{\sigma}_a^b$$

evaluated on $U_a \cap U_b$. Hence $\partial\overline{\partial}K_a = \partial\overline{\partial}K_b$ on $U_a \cap U_b$ and we obtain a global $(1,1)$-form ω with $-i\omega = \partial\overline{\partial}K_a$ on U_a, satisfying $-id\omega = (\partial + \overline{\partial})\partial\overline{\partial}K_a = \overline{\partial}\partial\overline{\partial}K_a = -\partial\overline{\partial}\overline{\partial}K_a = 0$. For positivity, we compute

$$\overline{\partial}K_a = \overline{\partial}\log(1 + (z'|z')) = \frac{(z'|dz')}{1 + (z'|z')}$$

and hence

$$\partial\overline{\partial}K_a = \frac{\partial(z'|dz')}{1 + (z'|z')} + \Big(\partial\frac{1}{1 + (z'|z')}\Big) \wedge (z'|dz') = \frac{1}{1 + (z'|z')}\sum_{j \neq a} dz^j \wedge d\overline{z}^j - \frac{(dz'|z') \wedge (z'|dz')}{(1 + (z'|z'))^2}$$

$$= \frac{1}{(1 + (z'|z'))^2}\sum_{i,j \neq a} (\delta_i^j(1 + (z'|z')) - \overline{z}^i z^j)\ dz^i \wedge d\overline{z}^j.$$

By Cauchy-Schwarz, the $n \times n$ matrix $(\delta_i^j(1 + (z'|z')) - \overline{z}^i z^j)$ (for indices $0 \leq i, j \leq n$ distinct from a) is positive definite. Hence ω is a Kähler form. □

Definition 3.3.4. *The* **Chern class** *of a cocycle line bundle* $\mathcal{V} \times^{\beta}_{\sim} \mathbf{C}$ *is the integral 2-cocycle*

$$\boldsymbol{c}(\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} \mathbf{C}) \sim \frac{1}{2\pi i}(\log \beta_a^b + \log \beta_b^c + \log \beta_c^a) \in H^2(M, \mathbf{Z}).$$

Lemma 3.3.5. *In terms of a metric* h^a *satisfying* $h^a |\beta_a^b|^2 = h^b$ *the Chern class is cohomologous to the family of closed* $(1,1)$*-forms*

$$\boldsymbol{c}(F) \sim \frac{1}{2\pi i} \overline{\partial}\partial \log h^a.$$

Proof. Identifying the Cech and Dolbeault description, the closed $(1,1)$-forms $\overline{\partial}\partial \log h^a$ correspond to the Cech 2-cocycle $\log \beta_a^b + \log \beta_b^c + \log \beta_c^a$. □

Definition 3.3.6. *A line bundle* L *on* M *is said to be* ^{0}**positive** *on an open subset* $V \subset M$, *if*

$$i\boldsymbol{c}(L) \sim \sum_{i,j} h_{ij} dz^i \wedge d\bar{z}^j,$$

where (h_{ij}) *is* ^{0}*positive on* V.

In general, let $D \subset M$ be a divisor (irreducible subvariety of codimension 1) in a compact complex manifold M. For a coordinate cover (V_a) there exist holomorphic functions $f_a : V_a \to \mathbf{C}$ such that $D \cap V_a = \{m \in V_a : \; f_a(m) = 0\}$. We may choose f_a such that

$$\beta_a^b(m) := \frac{f_a(m)}{f_b(m)}$$

is holomorphic and nowhere zero on $V_a \cap V_b$. Then the cocycle $(\beta_a^b) \in H^1(M, \mathbf{C}^\times)$ defines a line bundle

$$[D] = \mathcal{V} \overset{\beta}{\underset{\sim}{\times}} \mathbf{C}$$

which corresponds to the divisor D.

3.3.2 Blow-up process

Let $L = \mathbf{C}^n$. For projective space $\mathbf{P}(L) = \mathbf{P}^{n-1}$ consider the open subsets

$$V_i := \{[\zeta] \in \mathbf{P}(L) : \; \zeta^i \neq 0\} \subset \mathbf{P}(L)$$

for $1 \leq i \leq n$, with coordinate charts

$$\tau^i : \mathbf{C}^{n \setminus i} \to V_i, \quad \zeta^{n \setminus i} \mapsto [\zeta^{n \setminus i}; 1^i].$$

The set

$$N := \{(z, [\zeta]) \in L \times \mathbf{P}(L) : \; z \in [\zeta]\} = \{(z, [\zeta]) \in L \times \mathbf{P}(L) : \; z_i \zeta_j = z_j \zeta_i \; \forall \, 1 \leq i, j \leq n\}$$

$$= \{(z, [\zeta]) \in L \times \mathbf{P}(L) : \; \mathrm{rank} \begin{pmatrix} z_1, \dots, z_n \\ \zeta_1, \dots, \zeta_n \end{pmatrix} \leq 1\}.$$

is an n-dimensional submanifold of $L \times \mathbf{P}(L)$. The canonical projection

$$\pi : N \to \mathbf{P}(L), \quad (z, [\zeta]) \mapsto [\zeta]$$

is a submersion. Consider the open covering

$$N_i := \{(z, [\zeta]) \in N : \; \zeta^i \neq 0\} = \pi^{-1}(V_i)$$

of N.

Lemma 3.3.7. *We have coordinate charts*

$$\tau^i : \mathbf{C}^n \to N_i, \quad \tau^i(t', t^i) := (t^i(t', 1^i), [t', 1^i]),$$

where $t' \in \mathbf{C}^{n \setminus i}$.

Proof. Let $t'' \in \mathbf{C}^{n \setminus i,j}$. The equality

$$\tau^i(t'', t^i, t^j) = (t^i(t'', 1^i, t^j), [t'', 1^i, t^j]) = \tau^j(s'', s^i, s^j) = (s^j(s'', s^i, 1^j), [s'', s^i, 1^j])$$

for $t^i \neq 0 \neq t^j$ shows

$$\tau^i_j(t'', t^i, t^j) = \Big(\frac{1}{t^j} t'', \frac{1^i}{t^j}, t^i t^j\Big).$$

Note that

$$(s'', s^i, 1^j) = \Big(\frac{1}{t^j} t'', \frac{1^i}{t^j}, 1^j\Big) = \frac{1}{t^j}(t'', 1^i, t^j).$$

□

Proposition 3.3.8. *Let M be a complex n-manifold and $p \in M$. Choose a chart $\acute{\sigma} : \acute{U} \to \acute{V} \subset M$ such that $0 \in \acute{U} \subset L$ and $p = \acute{\sigma}_0 \in \acute{V}$. Put*

$$\acute{N} := \{(z, [\zeta]) \in N : \ z \in \acute{U}\}.$$

Then the disjoint union

$$\tilde{M} := (M \setminus p) \,\dot{\cup}\, \mathbf{P}(L)$$

becomes a manifold such that $M \setminus p \subset \tilde{M}$ is an open subset and the bijective map

$$F : \acute{N} \to (\acute{V} \setminus p) \,\dot{\cup}\, \mathbf{P}(L) \subset \tilde{M},$$

defined by

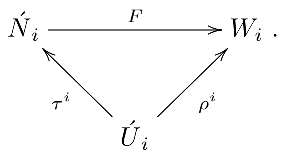
$$F(z, [\zeta]) := \begin{cases} \acute{\sigma}_z \in \acute{V} \setminus p \subset M \setminus p & z \neq 0 \\ [\zeta] \in \mathbf{P}(L) & z = 0 \end{cases},$$

is biholomorphic.

Proof. Put $\acute{U}_i := \tau_i(\acute{N}_i)$ and define charts $\rho^i : \acute{U}_i \to \tilde{M}$ by

$$\rho^i(t) = F(t^i(t', 1^i), [t', 1^i]) = \begin{cases} \acute{\sigma}(t^i(t', 1^i)) & t^i \neq 0 \\ [t', 1^i] & t^i = 0 \end{cases}.$$

Then $W_i := F(\acute{N}_i) = \rho^i(\acute{U}_i) \subset \tilde{M}$ are open subsets and, in view of (**??**) and (**??**), there is a commuting diagram



We also have the charts $\sigma^a : U_a \to V_a$ covering $W_0 := M \setminus p$. Thus

$$\tilde{M} = W_0 \cup W_1 \cup \ldots \cup W_n.$$

We show that the collection σ^a, ρ^i are local charts for $\tilde{M}$ (the chart $\acute{\sigma}$ is not needed any more.) Since $\rho^i = (F \dot{\cup} I) \circ \tau^i$ the transition maps

$$\rho_i^j := \rho_i \circ \rho^j = ((F \dot{\cup} I) \circ \tau^i)^{-1} \circ ((F \dot{\cup} I) \circ \tau^j) = \tau_i \circ \tau^j = \tau_i^j$$

are biholomorphic. Now let $m \in V_a \cap \acute{M}_S^T = V_a \cap \acute{M}_S$. On $V_i' := \tau_i(N_i \setminus T)$, the diagram (**??**) simplifies to

$$\begin{array}{ccc} \acute{N}_T & \xrightarrow{F} & \acute{M}_S \\ \cup\Big\uparrow & & \Big\uparrow\rho^i \\ N_i \setminus T & \xleftarrow[\tau^i]{} & V_i' \end{array}$$

Thus the identity $\sigma^a(m) = \rho^i(w) = F(\tau^i(w))$ implies

$$\sigma_a \circ \rho^i(w) = z = (\sigma_a \circ F^{-1} \circ \tau^i)(w), \quad \rho_i \circ \sigma^a(m) = w = (\rho_i \circ \tau_i \circ F)(m).$$

Since f is biholomorphic, the assertion follows. $\square$

The manifold

$$\tilde{M} = (M \setminus p) \,\dot{\cup}\, \mathbf{P}(L) = \tilde{\mathcal{U}}/\sim$$

is called the **blow-up** of M at the point p.

Lemma 3.3.9. *The collection of holomorphic functions*

$$\tilde{\beta}_0^i : W_i \cap W_0 \to \mathbf{C}^*, \quad \tilde{\beta}_0^i(F(z,[\zeta])) := z^i,$$

$$\tilde{\beta}_i^j : W_i \cap W_j \to \mathbf{C}^*, \quad \tilde{\beta}_i^j(F(z,[\zeta])) := \frac{\zeta^j}{\zeta^i}$$

form a cocycle on $\tilde{M} = (M \setminus p) \,\dot{\cup}\, \mathbf{P}(L)$.

Proof. Note that $\tilde{\beta}_0^i(F(z,[\zeta])) = z^i$ is non-zero since $V_a \cap \mathbf{P}(L) = \emptyset$. On $W_0 \cap W_i \cap W_j$ we have

$$\tilde{\beta}_0^i(m)\ \tilde{\beta}_i^j(m) = \beta^i(F^{-1}(m))\ \beta_i^j(F^{-1}(m)) = \beta^j(F^{-1}(m)) = \tilde{\beta}_0^j(m)$$

and

$$\tilde{\beta}_0^i(m)\ \tilde{\beta}_j^0(m) = \beta^i(F^{-1}(m))\ \frac{1}{\beta^j(F^{-1}(m))} = \tilde{\beta}_i^j(m).$$

$\square$

Lemma 3.3.10. *The line bundle* $\tilde{\mathcal{U}} \times_{\sim}^{\tilde{\beta}} \mathbf{C}$ *over* $\tilde{M} = \tilde{\mathcal{U}}/\sim$ *associated with the cocycle* (**??**) *corresponds to the divisor* $\mathbf{P}(L) \subset \tilde{M}$. *In formulas*

$$[\mathbf{P}(L)] = \tilde{\mathcal{U}} \overset{\tilde{\beta}}{\underset{\sim}{\times}} \mathbf{C}$$

Proof. We have $W_0 \cap \mathbf{P}(L) = \emptyset$. If $i > 0$, then every point in W_i has the form

$$m = F(t^i(t', 1^i), [t', 1^i]) = \begin{cases} \acute{\sigma}(t^i(t', 1^i)) & t^i \neq 0 \\ [t', 1^i] & t^i = 0 \end{cases}.$$

Thus the intersection $W_i \cap \mathbf{P}(L)$ on the coordinate chart W_i correspond to $t^i = 0$. Therefore, on $W_i \cap W_j$, the cocycle associated with $\mathbf{P}(L)$ is given by $\frac{t^i}{t^j} = \tilde{\beta}_j^i(m)$. $\square$

Our next goal is to determine the Chern class of this line bundle in terms of a metric. Choose a smooth function $h : M \to \mathbf{R}^+$ satisfying

$$h(m) = 1$$

for $m \in M \setminus \acute{V}$, and

$$h(\acute{\sigma}_z) = (z|z)$$

for all $z \in \acute{U}$ with $(z|z) < \epsilon$.

Lemma 3.3.11. *The smooth functions*

$$\tilde{h}^0 : W_0 \to \mathbf{R}^+, \ \tilde{h}^0(m) := h(m),$$

$$\tilde{h}^i : W_i \to \mathbf{R}^+, \ \tilde{h}^i(\rho^i(t', t^i)) := \frac{h(\acute{\sigma}(t^i(t', 1^i)))}{|t^i|^2}$$

define a ^{0}*metric on the line bundle* $[\mathbf{P}(L)] = \tilde{\mathcal{U}} \times_{\sim}^{\tilde{\beta}} \mathbf{C}$.

Proof. If $0 < |t| < \epsilon$ then (**??**) implies

$$\frac{h(\acute{\sigma}(t^i(t', 1^i)))}{|t^i|^2} = \frac{\|t^i(t', 1^i)\|^2}{|t^i|^2} = \|(t', 1^i)\|^2.$$

Therefore (**??**) defines a smooth function on W_i. By Proposition **??** we need to verify the property

$$\tilde{h}^i_m = |\tilde{\beta}^j_i(m)|^2 \ \tilde{h}^j_m$$

for $0 \leq i, j \leq n$ and $m \in W_i \cap W_j$. Assume first $i, j > 0$. Let $m = F(z, [\zeta]) = \rho^i(t'', t^i, t^j) = \rho^j(s'', s^i, s^j) \in W_i \cap W_j$. Then $z = t^i(t'', 1^i, t^j) = s^j(s'', s^i, 1^j)$ and $[\zeta] = [t'', 1^i, t^j] = [s'', s^i, 1^j]$, Therefore $t^i = s^j s^i, \ s^j = t^i t^j$ and $\zeta = \zeta^i(t'', 1^i, t^j) = \zeta^j(s'', s^i, 1^j)$. This implies $t^i \zeta^j = s^j s^i \zeta^j = s^j \zeta^i$ and hence

$$|\tilde{\beta}^j_i(m)|^2 \ \tilde{h}^j(m) = |\frac{\zeta^j}{\zeta^i}|^2 \ \frac{h(\acute{\sigma}_m)}{|s^j|^2} = \frac{h(\acute{\sigma}_m)}{|t^i|^2} = \tilde{h}^i(m).$$

On the other hand, if $m \in W_0 \cap W_i$, for $i > 0$, then $m = \acute{\sigma}_m = \rho^i(t', t^i)$ for $z = t^i(t', 1^i) \in \acute{U} \setminus 0$. Hence $z^i = t^i$ and

$$|\tilde{\beta}^i_0(m)|^2 \ \tilde{h}^i(m) = |z^i|^2 \ \frac{h(\acute{\sigma}_m)}{|t^i|^2} = h(\acute{\sigma}_m) = \tilde{h}^0(m).$$

□

Corollary 3.3.12. *The Chern class is given by the family of* $(1,1)$*-forms*

$$\boldsymbol{c}([\mathbf{P}(L)]) = \boldsymbol{c}(\tilde{\mathcal{U}} \overset{\tilde{\beta}}{\underset{\sim}{\times}} \mathbf{C}) \sim \Big(\frac{1}{2\pi i} \overline{\partial}\partial \log \tilde{h}^\ell\Big)_{\ell=0}^n.$$

Lemma 3.3.13. *Let* $\pi : \tilde{M} \to M$ *be the canonical projection, mapping* $\mathbf{P}(L)$ *to* p*. Then the* $(1,1)$*-form*

$$\overline{\partial}\partial(h \circ \pi + \log \tilde{h}^\ell) = \pi^*(\overline{\partial}\partial h) + \overline{\partial}\partial \log \tilde{h}^\ell)$$

on W_ℓ *is* ^{0}*positive on a neighborhood of* $\mathbf{P}(L) \subset \tilde{M}$.

Proof. For fixed $\ell > 0$ and $(t|t) < \epsilon$ the condition (**??**) implies

$$\tilde{h}^\ell(\rho^\ell(t', t^\ell)) = \|(t', 1^\ell)\|^2 = 1 + (t'|t').$$

Putting $(t'|dt') := \sum_{j \neq \ell} t^j \ d\overline{t}^j, \ (dt'|t') := \sum_{i \neq \ell} \overline{t}^i \ dt^i$, we have

$$\overline{\partial} \log(1 + (t'|t')) = \frac{\overline{\partial}(t'|t')}{1 + (t'|t')} = \frac{(t'|dt')}{1 + (t'|t')}$$

and hence

$$\begin{aligned} \partial\overline{\partial} \log(1 + (t'|t')) &= \frac{\partial(t'|dt')}{1 + (t'|t')} - \frac{(dt'|t')}{1 + (t'|t')} \wedge \frac{(t'|dt')}{1 + (t'|t')} \\ &= \frac{1}{1 + (t'|t')} \sum_{i,j \neq \ell} \delta^j_i \ dt^i \wedge d\overline{t}^j - \frac{(dt'|t') \wedge (t'|dt')}{(1 + (t'|t'))^2} \end{aligned}$$

$$= \frac{1}{(1+(t'|t'))^2} \sum_{i,j\neq\ell} dt^i \wedge\ d\bar{t}^j \Big(\delta_i^j(1+(t'|t')) - \bar{t}^i\ t^j\Big).$$

The matrix $A_i^j := \delta_i^j(1+(t'|t')) - \bar{t}^i\ t^j$ corresponds to the hermitian form

$$(\xi,\eta) \mapsto (\xi'|\eta')(1+(t'|t')) - (\xi'|t')(t'|\eta') = (\xi'|\eta') + \Big((\xi'|\eta')(t'|t') - (\xi'|t')(t'|\eta')\Big).$$

By Cauchy-Schwarz, this is semi-positive but vanishes on the hyperplane $t^\ell = 0$. We need the extra h-term for positivity: Near $\mathbf{P}(L)$ we have $\|t\|^2 < \epsilon$ and hence $P^*(\overline{\partial}\partial h) = P^*(\overline{\partial}\partial(z|z))$, with $z = t^\ell(t', 1^\ell)$ and $(z|z) = |t^\ell|^2(1+(t'|t'))$. Therefore

$$\overline{\partial}(z|z) = \overline{\partial}\Big(|t^\ell|^2(1+(t'|t'))\Big) = (\overline{\partial}|t^\ell|^2)(1+(t'|t')) + |t^\ell|^2\ \overline{\partial}(t'|t') = (t^\ell d\bar{t}^\ell)(1+t'|t') + |t^\ell|^2(t'|dt')$$

and hence

$$\begin{aligned}\partial\overline{\partial}(z|z) &= \partial(t^\ell d\bar{t}^\ell)(1+t'|t') - (t^\ell d\bar{t}^\ell)\wedge\partial(1+t'|t') + (\partial|t^\ell|^2)\wedge(t'|dt') + |t^\ell|^2\ \partial(t'|dt') \\ &= dt^\ell \wedge d\bar{t}^\ell(1+t'|t') - (t^\ell d\bar{t}^\ell)\wedge(dt'|t') + (\bar{t}^\ell dt^\ell)\wedge(t'|dt') + |t^\ell|^2 \sum_{j\neq\ell} dt^j \wedge d\bar{t}^j.\end{aligned}$$

By (**??**), the divisor $\mathbf{P}(L)$ corresponds to $z = 0$. On W_ℓ this is equivalent to $t^\ell = 0$. If $t^\ell = 0$ then

$$\partial\overline{\partial}(z|z) = dt^\ell \wedge d\bar{t}^\ell(1+t'|t').$$

By continuity, it follows that the sum

$$\partial\overline{\partial}(h\circ\pi + \log\tilde{h}^\ell) = \partial\overline{\partial}((z|z) + \log(1+(t'|t')))$$

is positive definite on a neighborhood of $\mathbf{P}(L)$ in W_ℓ. □

Proposition 3.3.14. *The Chern class of the divisor $\mathbf{P}(L) \subset \tilde{M}$ is negative:*

$$\boldsymbol{c}[\mathbf{P}(L)] < 0$$

near $\mathbf{P}(L)$.

Proof. By Lemma **??** $P^*(\overline{\partial}\partial h) + \overline{\partial}\partial(\log\tilde{h})$ is strictly positive near $\mathbf{P}(L)$. Now

$$\partial\overline{\partial}(h\circ\pi) = \partial\overline{\partial}(\pi^* h) = d\overline{\partial}(\pi^* h) \sim 0$$

is null-cohomologous in $\tilde{M}$. Hence $\mathbf{c}[-\mathbf{P}(L)]$ is cohomologous to a positive line bundle. □

3.3.3 Proof of the Kodaira embedding theorem

Proposition 3.3.15. *The canonical line bundles $K(\tilde{M})$ and $K(M)$ are related by*

$$K(\tilde{M}) = \pi^* K(M) + (n-1)[\mathbf{P}(L)],$$

where $\pi : \tilde{M} \to M$. is the canonical projection.

Proof. Relative to the coordinate charts $\sigma^a : U_a \to V_a \subset M \setminus p$ and $\acute{\sigma} : \acute{U} \to \acute{V} \ni p$, the canonical line bundle of M is given by the cocycle

$$\begin{cases} J_b^a = \det \frac{\partial\sigma_b}{\partial\sigma_a} & \text{on } V_a \cap V_b \\ J_a = \det \frac{\partial\sigma_a}{\partial\acute{\sigma}} & \text{on } V_a \cap \acute{V} \end{cases}.$$

Passing to $\tilde{M}$, with coordinate charts $\sigma^a : U_a \to V_a \subset W_0$ and $\rho^i : V_i \to W_i$, the cocycle (??) is supplemented by

$$\begin{cases} J_i^j := \det \frac{\partial \rho_i}{\partial \rho_j} & \text{on } W_i \cap W_j \\ J_a^i := \det \frac{\partial \sigma_a}{\partial \rho_i} & \text{on } \tilde{M}_i \cap V_a \end{cases}.$$

Now let $s = \rho_j^i(t) = \rho^j \circ \rho^i(t)$. Then $\acute{\sigma}(\tau^i(t)) = \acute{\sigma}(\tau^j(s))$ and hence

$$(t^t t'', t^i, t^i t^j) = (s^j s'', s^j s^i, s^j).$$

Therefore

$$s^j = t^i t^j,\ s^i = \frac{t^i}{s^j} = \frac{1}{t^j},\ s'' = \frac{t^i}{s^j} t'' = \frac{1}{t^j} t''$$

The partial derivatives $\frac{\partial}{\partial t^i} s^i = 0,\ \frac{\partial}{\partial t^i} s^j = t^j,\ \frac{\partial}{\partial t^i} s^k = 0,\ \frac{\partial}{\partial t^j} s^i = \frac{-1}{(t^j)^2},\ \frac{\partial}{\partial t^j} s^j = t^i,\ \frac{\partial}{\partial t^j} s^k = \frac{-t^k}{(t^j)^2},\ \frac{\partial}{\partial t^k} s^i = 0,\ \frac{\partial}{\partial t^k} s^j = 0,\ \frac{\partial}{\partial t^k} s^\ell = \frac{\delta_k^\ell}{t^j}$ yield the Jacobi matrix

$$\frac{\partial \rho_i}{\partial \rho_j} = \begin{pmatrix} 0 & t^j & 0 \\ -1/(t^j)^2 & t^i & -t''/(t^j)^2 \\ 0 & 0 & I''/t^j \end{pmatrix}$$

with determinant

$$J_i^j = \det \frac{\partial \rho_i}{\partial \rho_j} = \frac{1}{(t^j)^{n-2}} \det \begin{pmatrix} 0 & t^j \\ -1/(t^j)^2 & t^i \end{pmatrix} = \frac{1}{(t^j)^{n-1}}.$$

Finally, let $\sigma^a(w) = \rho^i(t) = \acute{\sigma}(m)$, where $z = (t^i t', t^i)$. Since $z^i = t^i,\ z^k = t^i t^k$ we obtain $\frac{\partial z^i}{\partial t^i} = 1,\ \frac{\partial z^k}{\partial t^i} = t^k,\ \frac{\partial z^i}{\partial t^k} = 0,\ \frac{\partial z^\ell}{\partial t^k} = t^i \delta_k^\ell$. Therefore

$$\frac{\partial z}{\partial t} = \begin{pmatrix} 1 & t' \\ 0 & t^i\ I' \end{pmatrix}$$

has the determinant

$$\det \frac{\partial z}{\partial t} = (t^i)^{n-1}.$$

It follows that

$$\det \frac{\partial w}{\partial t} = \det \frac{\partial w}{\partial z}\ \det \frac{\partial z}{\partial t} = (t^i)^{n-1}\ \det \frac{\partial w}{\partial z} = (t^i)^{n-1} \det \frac{\partial \sigma_a}{\partial \grave{\sigma}}.$$

□

Proposition 3.3.16. *Let E be a strictly positive line bundle on M. For distinct $p, q \in M$ consider $\tilde{M} = (M_{\{p\}}^{\mathbf{P}(L)})_{\{q\}}^{\mathbf{P}(L)}$, with canonical projection $\pi : \tilde{M} \to M$. Then for k sufficiently large, the bundle $k\pi^* E - K_{\tilde{M}} - [\mathbf{P}_p] - [\mathbf{P}_q]$ on $\tilde{M}$ is strictly positive.*

Proof. By Proposition **??** we have

$$F := k\pi^* E - K_{\tilde{M}} - [\mathbf{P}_p] - [\mathbf{P}_q] = \pi^*(kE - K_M) - n\ [\mathbf{P}_p] - n\ [\mathbf{P}_q].$$

It follows that

$$\boldsymbol{c}(F) = \pi^* \boldsymbol{c}(kE - K_M) - n\ \boldsymbol{c}[\mathbf{P}_p] - n\ \boldsymbol{c}[\mathbf{P}_q].$$

Since $E > 0$, there exists k so large that $kE - K_M > 0$. Then $\pi^*(kE - K_M) \geq 0$ on $\tilde{M}$ and $\pi^*(kE - K_M) > 0$ on $\tilde{M} \setminus (\mathbf{P}_p \cup \mathbf{P}_q)$, where π is biholomorphic. By Proposition 3.3.13, we have $\boldsymbol{c}[\mathbf{P}_p] < 0$ near $\mathbf{P}_p$, and similarly, $\boldsymbol{c}[\mathbf{P}_q] < 0$ near $\mathbf{P}_q$. Therefore (**??**) is strictly positive on $\tilde{M}$. □

Lemma 3.3.17. *Let $\pi : \tilde{M} = M_p^P \to M$ where $P \subset M_p^P$ is a divisor isomorphic to $\mathbf{P}(L)$. For a line bundle $\mathcal{L}$ over M let $\mathcal{O}_p \otimes \mathcal{L}$ denote the sheaf of holomorphic sections $M \to \mathcal{L}$ which vanish at p. Let $\mathcal{O}_P \otimes \pi^* \mathcal{L}$ denote the sheaf of holomorphic sections $\tilde{M} \to \pi^* \mathcal{L}$ which vanish on P. Then $H^1(\tilde{M}, \mathcal{O}_P \otimes \pi^* \mathcal{L}) = 0$ implies $H^1(M, \mathcal{O}_p \otimes \mathcal{L}) = 0$.*

Proof. Let $\mathcal{V} = (V_a)$ be an open covering of M such that $\mathcal{L}|_{V_a}$ is trivial. Then $\tilde{V}_a := \pi^{-1}(V_a)$ form an open covering $\tilde{\mathcal{V}}$ of $\tilde{M}$. Let $\Phi_{ab} \in Z^1(\mathcal{V}, \mathcal{O}_p \otimes \mathcal{L})$ be a 1-cocycle. Thus $\Phi_{ab} : V_a \cap V_b \to \mathcal{L}$ are holomorphic sections vanishing on $V_{ab} \cap \{p\}$. Therefore $\Phi_{ab} \circ \pi : \tilde{V}_a \cap \tilde{V}_b \to \pi^*\mathcal{L}$ are holomorphic sections vanishing on $\tilde{V}_{ab} \cap P$. For any sheaf $\mathcal{S}$, the canonical map

$$H^1(\tilde{\mathcal{V}}, \mathcal{S}) \to H^1(\tilde{M}, \mathcal{S})$$

is injective. Hence the assumption implies $H^1(\tilde{\mathcal{V}}, \mathcal{O}_P \otimes \pi^*\mathcal{L}) = 0$. It follows that there exist holomorphic sections $\psi_a : \tilde{V}_a \to \pi^*\mathcal{L}$ vanishing on $\tilde{V}_a \cap P$ such that

$$\Phi_{ab} \circ \pi = \psi_a - \psi_b.$$

Suppose first that $p \notin V_a$. Then $\tilde{V}_a \subset \tilde{M} \setminus P$ and $\pi : \tilde{V}_a \to V_a$ is biholomorphic. Therefore

$$\Phi_a := \psi_a \circ \pi^{-1} : V_a \to \mathcal{L}$$

is a holomorphic section vanishing on $V_a \cap \{p\} = \emptyset$. Suppose now that $p \in V_a$. Then the restriction $\pi : \tilde{V}_a \setminus P \to V_a \setminus p$ is biholomorphic. Thus $\psi_a \circ \pi^{-1} : V_a \setminus p \to \mathcal{L}$ is a holomorphic section. Since $\mathcal{L}|_{V_a}$ is trivial, we may apply Hartogs' extension theorem (for $n > 2$) to obtain a holomorphic section $\Phi_a : V_a \to \mathcal{L}$ satisfying

$$\Phi_a|_{V_a \setminus p} = \psi_a \circ \pi^{-1}.$$

Therefore $\Phi_a \circ \pi = \psi_a$ on $\tilde{V}_a \setminus P$. By continuity (or analytic continuation) it follows that

$$\Phi_a \circ \pi = \psi_a$$

on $\tilde{V}_a$. This implies $\Phi_a(p) = \psi_a(P) = 0$. Thus we obtain a family $(\Phi_a) \in H^0(\mathcal{V}, \mathcal{O}_p \otimes \mathcal{L})$ such that $\Phi_{ab} = \Phi_a - \Phi_b$. Therefore $(\Phi_{ab}) = 0 \in H^1(\mathcal{V}, \mathcal{O}_p \otimes \mathcal{L})$. Since $\mathcal{V}$ is arbitrary and, in general,

$$H^q(M, \mathcal{S}) = \lim_{\mathcal{V}} H^q(\mathcal{V}, \mathcal{S}),$$

the assertion follows. □

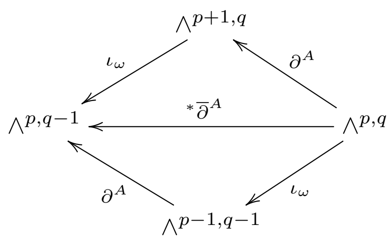
Theorem 3.3.18. *(Kodaira Vanishing Theorem) Let* $L > 0$. *Then*

$$H^q(X, \mathcal{O}^p \otimes L) = 0 \quad \forall\, p + q > n.$$

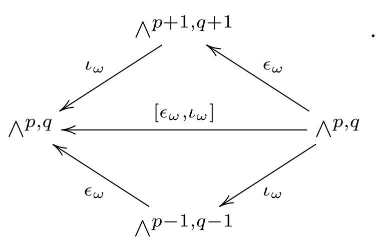
Proof. Since $L > 0$ the square $(d^A)^2$ of its Chern connexion is the exterior multiplication ϵ_ω by a positive $(1,1)$-form ω. which is therefore a Kähler metric. Consider the operators

$$\square^A := \partial^A\, {}^*\partial^A + {}^*\partial^A\, \partial^A, \quad \overline{\square}^A := \overline{\partial}^A\, {}^*\overline{\partial}^A + {}^*\overline{\partial}^A\, \overline{\partial}^A$$

(Hodge-Laplacian). Thus



and



For Kähler manifolds we have the **Bochner-Kodaira-Nakano Identity**

$$\overline{\Box}^A - \Box^A = [\epsilon_\omega, \iota_\omega].$$

By Dolbeault and Hodge theory we have

$$H^p(M, \Omega^q \otimes L) = \mathcal{H}^{p,q}_{\overline{\Box}^A}(M, L)$$

Now let $u \in \mathcal{H}^{p,q}_{\overline{\Box}^A}(M, L) = H^p(M, \Omega^q \otimes L)$. Then

$$\int\limits_M dm(\Box^A u|u)_m = \int\limits_M dm(\partial^{A\ *}\partial^A u + {}^*\partial^A\ \partial^A u|u)_m = \int\limits_M dm\Big(({}^*\partial^A u|{}^*\partial^A u) + (\partial^A u|\partial^A u)_m\Big) \geq 0$$

and hence

$$\int\limits_M (\overline{\Box}^A u|u) = \int\limits_M (\Box^A u + [\epsilon_\omega, \iota_\omega]u|u) = \int\limits_M (\Box^A u|u) + \int\limits_M ([\epsilon_\omega, \iota_\omega]u|u) \geq \int\limits_M ([\epsilon_\omega, \iota_\omega]u|u).$$

If $[\epsilon_\omega, \iota_\omega]$ is positive definite on each fibre, then $\overline{\Box}^A u = 0$ implies $u = 0$, i.e., $H^p(M, \Omega^q \otimes L) = 0$. Since $(\epsilon_\omega, \iota_\omega, (deg - n)I)$ is a so-called $\mathfrak{sl}_2$-triple, we have

$$([\epsilon_\omega], \iota_\omega]u|u) = (p + q - n)\|u\|^2$$

which is positive for $p + q > n$. □

Corollary 3.3.19. *If $F - K_M > 0$ then*

$$H^q(M, \mathcal{O} \otimes F) = 0 \quad \forall\ q > 0.$$

The **Kodaira map** is defined as follows: Let M be a Kähler manifold such that for all $m \in M$ there exists $\Phi \in \mathcal{O}(\mathcal{V}_\sim \times^\beta \mathbf{C})$ with $\Phi_m \neq 0$. Then $\mathcal{O}(\mathcal{V} \times^\beta_\sim \mathbf{C})$ has finite dimension. We define a holomorphic map

$$\mathcal{K} : M \to \mathbf{P}(\mathcal{O}(\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} \mathbf{C})^*)$$

by the hyperplane

$$\mathcal{K}^z := \{\Phi \in \mathcal{O}(\mathcal{V} \overset{\beta}{\underset{\sim}{\times}} \mathbf{C}) : \ \Phi_m = 0\} = \mathrm{Ker}\mathcal{K}^*_m$$

as the kernel of the evaluation map.

The **Kodaira embedding theorem** (first half) is the following:

Theorem 3.3.20. *Let $E > 0$ be a positive line bundle on a compact Kähler manifold M. Then, for k large enough, the Kodaira map (**??**) for $F = kE$ is injective.*

Proof. For $p \neq q$ in M consider the subsheaf $\mathcal{O}_{p,q} \otimes F \subset \mathcal{O} \otimes F$ of germs vanishing at p and q. Then the so-called 'skyscraper sheaf' $\mathcal{S} = \mathcal{O} \otimes F/\mathcal{O}_{p,q} \otimes F$ has stalks $\mathcal{S}_p \equiv \mathbf{C} \equiv \mathcal{S}_q$, whereas $\mathcal{S}_m = 0$ for $m \in M \setminus \{p, q\}$. The exact sheaf sequence

$$0 \to \mathcal{O}_{p,q} \otimes F \to \mathcal{O} \otimes F \to \mathcal{S} \to 0$$

induces an exact cohomology sequence

$$0 \to H^0(M, \mathcal{O}_{p,q} \otimes F) \to H^0(M, \mathcal{O} \otimes F) \xrightarrow{\kappa^*_{p,q}} H^0(M, \mathcal{S}) = \mathbf{C} \oplus \mathbf{C} \to H^1(M, \mathcal{O}_{p,q} \otimes F) \to H^1(M, \mathcal{O} \otimes F) \to H^1(M, \mathcal{S}) \to \ldots,$$

where $\kappa^*_{p,q}(\Phi) = (\Phi^a_p, \Phi^b_q)$, for $p \in V_a,\ q \in V_b$, is the 'double' evaluation map. In order to show that the Kodaira map (**??**) is injective, it thus suffices to show that $H^1(M, \mathcal{O}_{p,q} \otimes F) = 0$, since then $\kappa^*_{p,q}$ is surjective for every pair $p \neq q$. Let $\pi : \tilde{M} \to M$ be the canonical projection, with $P := \pi^{-1}(p) = \mathbf{P}_p$ and $Q := \pi^{-1}(q) = \mathbf{P}_q$. By Lemma **??** it suffices to show that $H^1(\tilde{M}, \mathcal{O}_{P\cup Q} \otimes F) = 0$. Now $\tilde{F} := \pi^*(kE) - [P] - [Q]$ satisfies $\tilde{F} - K_{\tilde{M}} > 0$ for k large enough, and hence, by corollary **??**, we have $H^1(\tilde{M}, \mathcal{O} \otimes \tilde{F}) = 0$. Since

$$\mathcal{O} \otimes \tilde{F} = \mathcal{O} \otimes (\pi^*(kE) - [P] - [Q]) = \mathcal{O}_{P\cup Q} \otimes \pi^*(kE)$$

we finally obtain $H^1(\tilde{M}, \mathcal{O}_{P\cup Q} \otimes \pi^*(kE)) = 0$ and hence $H^1(M, \mathcal{O}_{p,q} \otimes (kE)) = 0$. It follows that the sheaf $\mathcal{O} \otimes (k\pi^*E - [\mathbf{P}_p] - [\mathbf{P}_q])$ on $\tilde{M}$ satisfies

$$H^1(\tilde{M}, \mathcal{O} \otimes (k\pi^*E - [\mathbf{P}_p] - [\mathbf{P}_q])) = 0.$$

□