Geometric Quantization in Complex and Harmonic Analysis

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These are informal notes, subject to continuous changes and corrections
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Chapter 0

Overview

0.1 $d^+$-Quantization, $d \geq 0$

border: $d$-dimensional manifold $S$, closed (compact) but possibly disconnected (many-particle system)
bordism: $d + 1$ manifold $\Sigma$, connected but non-closed, with boundary $\partial \Sigma = S$
border symplectic manifold $M$
border complex manifold: family of Kähler manifolds $M_\tau$
border complex quantization: family of Hilbert spaces $H^2(M_\tau)$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$
classical bordism: flow of symplectomorphisms
quantum bordism: flow of unitary operators

0.2 $0^+$-Quantization, Quantum Mechanics

border: point $S = \mathbb{S}^0$ or finite number of points
bordism: interval $[0,t]$, 1-manifold with boundary

Example 0.2.1. $Q$ configuration space
border symplectic manifold $T^*Q$
border complex manifold: family of Kähler manifolds $M_\tau$
border complex quantization: family of Hilbert spaces $H^2(M_\tau)$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $L^2(Q)$
classical bordism: geodesic flow
quantum bordism: time evolution $e^{itH}$

Example 0.2.2. $G$ compact Lie group, $T$ maximal torus
border symplectic orbit $G/T$
border complex orbit: family of Kähler manifolds $G^C/G_r^C$
border complex quantization: family of highest weight Hilbert spaces $G^C/G_r^C$
border symplex quantization: projectively flat connexion on bundle of Hilbert spaces
quantum bordism: no time evolution $H = 0$
0.3 1+-Quantization, Conformal Field Theory

border: circle $S = S^1$, or disjoint union of circles=compact 1-manifold without boundary

bordism: cylinder $[0,1] \times S^1$ or connected Riemann surface $\Sigma$ with boundary

Example 0.3.1. $G$ compact Lie group

border symplectic quotient

$C^\infty(S^1, G)/G$

border complex quotient: family of Kähler manifolds

$O(D, G^C)$

border complex quantization: positive energy representations of loop group (G. Segal)

border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$

Example 0.3.2. 1+1 gravity, $G = SL_2(\mathbb{R})$ non-compact

Example 0.3.3. Restricted Grassmannian, 2d QCD (Rajeev-Turgut)

0.4 2+-Quantization, Topological Quantum Field Theory

border: non-connected compact oriented surface $S$ without boundary

bordism: connected non-compact 3-manifold $\Sigma$ with boundary $\partial \Sigma = S$

Example 0.4.1. Chern-Simons theory: $G$=compact Lie group

border symplectic quotient (compact)

$H^1(S, G) = Hom(\pi_1(S), G)$

border complex quotient: family of Kähler manifolds

$H^1(S, G^C)$

border complex quantization: family of Hilbert spaces $H^2(M_\tau)$

border symplex quantization: projectively flat connexion on bundle of Hilbert spaces $H^2(M_\tau)$

Example 0.4.2. 2+1 gravity=Chern-Simons theory for non-compact Lie group $SL_2(\mathbb{R})$ (Verlinde)

0.5 $3 \leq d \leq 8$, Higher gauge theory and special holonomy

- gauge theory in 4 dimensions, $SU(2)$-holonomy
- Calabi-Yau manifolds in 6 dimensions, $SU(3)$-holonomy
- $G_2$-manifolds in 7-dimensions
- $Spin(7)$-manifolds in 8 dimensions

Since spacetime is supposed to have dimension $\leq 11$ ($M$-theory) or $\leq 12$ ($F$-theory), Kaluza-Klein compactification to 4-dimensional Minkowski space yields 'border' manifolds of dimension $\leq 8$. 
Chapter 1

Manifolds, Connexions and Curvature

1.1 Manifolds

Consider smooth manifolds over \( \mathbb{R} \) and complex manifolds over \( \mathbb{C} \). We use the term \( K \)-manifold for \( K = \mathbb{R}, \mathbb{C} \). If not specified otherwise, maps, functions, sections etc. will be smooth for \( K = \mathbb{R} \) or holomorphic for \( K = \mathbb{C} \).

• Jordan manifolds

Jordan manifolds are symmetric manifolds of arbitrary rank, associated with Jordan algebras and Jordan triples. The basic example is projective space (rank 1)

\[
P^* = \{ E \subset K^{1+s} : \dim E = 1 \}.
\]

Let \( Z \) be a \( K \)-vector space, endowed with a ternary composition \( Z \times Z \times Z \to Z \), denoted by

\[
(x, y, z) \mapsto \{x; y; z\},
\]

which is bilinear symmetric in \((x, z)\) and anti-linear in the inner variable. Define

\[
D(x, y)z := \{x; y; z\}.
\]

Then \( Z \) is called a Jordan triple if the Jordan triple identity

\[
\left[ D(x, y), D(u, v) \right] = D(\{x; y; u\}, v) - D(u, \{v; x; y\})
\]

holds. \( Z \) is called hermitian (over \( K \)) if the sesqui-linear form

\[
(x, y) \mapsto \text{tr} \ D(x, y)
\]

is non-degenerate and hermitian

\[
\text{tr} \ D(x, y) = \text{tr} \ D(y, x).
\]

A hermitian Jordan triple is called \( ^0 \)hermitian, if the trace form (??) is positive definite. If there are \( q \) negative eigenvalues, then \( Z \) is called \( ^q \)hermitian. We will mostly be concerned with complex \( ^0 \)hermitian Jordan triples.
The basic example is $Z = K^{r \times s}$ with the ternary composition
\[
\{u; v; w\} := uv^* w + vw^* u
\]
which makes sense for rectangular matrices. More generally, the full classification of irreducible complex $^0$hermitian Jordan triples is

- **matrix triple** $Z = C^{r \times s}$, $\{x; y; z\} = xy^* z + zy^* x$, rank $= r \leq s$, $a = 2$ (complex case), $b = s - r$
- $r = 1$, $Z = C^{1 \times s} = C^s$, $\{x; y; z\} = (x|y)z + (z|y)x$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- **spin factor** $Z = C^a + 2$, $\{x; y; z\} = (x \cdot y)z + (z \cdot y)x$, $r = 2$, $b = 0$
- **exceptional Jordan triples** of dimension 16 ($r = 2$) and 27 ($r = 3$), $a = 8$ (octonion case)

For $(u, v) \in Z^2 := Z \times Z$ the endomorphism
\[
B_{u, v} z := z - \{u; v; z\} + \frac{1}{4}\{u; v; z; v\}; u
\]
of $Z$ is called the **Bergman operator**. For matrices it becomes
\[
B_{u, v} z = (I_r - uv^*) z (I_s - v^* u)
\]
which again makes sense for rectangular matrices.

A pair $(x, y) \in Z^2$ is called **quasi-invertible** if $B_{x,y}$ is invertible. In this case the element
\[
x^y := B_{x,y}^{-1}(x - \{x; y; x\})
\]
in $Z$ is called the **quasi-inverse**. For rectangular matrices the quasi-inverse is given by
\[
x^y := (I_r - xy^*)^{-1} x = x (I_s - y^* x)^{-1}
\]
which is again a rectangular matrix.

By $[, ]$, we have the **addition formulas**
\[
B_{x, y + z} = B_{x, y} B_{x, y + z}
\]
and
\[
x^{y + z} = (x^y)^z.
\]

This implies that
\[
[x, a] = [y, b] \iff (x, a - b) \text{ quasi-invertible and } y = x^{a-b}
\]
defines an equivalence relation on $Z^2$. Informally, $[x, a] = [x^{a-b}, b]$. The compact quotient manifold
\[
\hat{Z} = Z^2 / R = \{[m, a] : \ z, a \in Z\}
\]
is a compact symmetric space called the **conformal hull** of $Z$. Its non-compact dual is the connected 0-component
\[
\hat{Z} := \{m \in Z : \ B_{z, z} \text{ invertible}\}^0,
\]
which is a bounded symmetric domain in its circular and convex Harish-Chandra realization.
Example 1.1.1. For the matrix triple $Z = K^{r \times s}$, $\hat{Z}$ can be identified with the Grassmannian

$$G_r(K^{r+s}) = \{ E \subset K^{r+s} : \dim E = r \}.$$ 

The embedding $\sigma^0 : Z \subset \hat{Z}$ is given by mapping $m \in K^{r \times s}$ to its graph

$$\sigma^0_m := \{(\xi, \xi z) : \xi \in K^{1 \times r} \} \subset K^{1 \times r} \times K^{1 \times s} = K^{1 \times (r+s)}.$$ 

of $m \in K^{r \times s}$. Via this embedding, we have

$$\hat{Z} = \{ m \in Z : I_r - zz^* > 0 \} = \{ m \in Z : I_s - z^* z > 0 \}.$$ 

For $r = 1$, $\hat{Z}$ becomes projective space $P^s$ and $\hat{Z}$ is the unit ball $B^s$.

A basic theorem of M. Koecher characterizes hermitian symmetric spaces in terms of Jordan triples:

**Theorem 1.1.2.** In the complex setting, for every $^+$hermitian Jordan triple $Z$ the conformal hull $\hat{Z}$ is a compact hermitian symmetric space, and every such space arises this way. Similarly, every hermitian bounded symmetric domain can be realized as the spectral unit ball $\hat{Z}$ of a hermitian Jordan triple $Z$.

Thus there is a 1-1 correspondence between $^0$hermitian Jordan triples and $^0$hermitian symmetric spaces of compact/non-compact type. Via this correspondence the two exceptional symmetric spaces can be treated on an equal footing with the classical types. For real Jordan triples and symmetric spaces, the above 1-1 correspondence is ‘almost’ true (some exceptional symmetric spaces are missing).

*Peirce manifolds
*Jordan-Kepler manifolds
*Jordan-Schubert varieties

• **Restricted Grassmannian**

We now describe an infinite-dimensional example.

**Example 1.1.3.** Let $A$ be an associative unital Banach algebra. The set

$$S := \{ s \in A : s^2 = 1 \}$$

of all symmetries in $A$ is a Banach manifold, with tangent space

$$T_sS = \{ \dot{s} \in A : \dot{s} + \dot{s} s = 0 \}.$$ 

The set

$$P := \{ p \in A : p^2 = p \}$$

of all idempotents in $A$ is a manifold, with tangent space

$$T_pP = \{ \dot{p} \in A : \dot{p} + \dot{p} p = \dot{p} \}.$$ 

If $A$ is a $*$-algebra, one obtains real manifolds by restricting to self-adjoint symmetries or projections, resp.

**Lemma 1.1.4.** There is a 1-1 correspondence between (self-adjoint) symmetries $s \in S$ and idempotents $p \in P$ given by $p = \frac{s + 1}{2}$ and $s = 2p - 1$, respectively.

**Proof.** We have

$$\left(\frac{s + 1}{2}\right)^2 = \frac{1}{4}(s^2 + 2s + 1) = \frac{1}{4}(1 + 2s + 1) = \frac{s + 1}{2}$$

and

$$(2p - 1)^2 = 4p + 1 - 4p = 1.$$

$\square$
Identifying a subspace $E$ with its orthogonal projection $p_E$ or the corresponding symmetry $s_E = 2p_E - 1$, the complex Grassmannian becomes a connected component of the manifold of self-adjoint projections (resp. symmetries) for the block-matrix algebra

$$A = C^{(r+s)\times (r+s)} = \begin{pmatrix} C^{r\times r} & C^{r\times s} \\ C^{s\times r} & C^{s\times s} \end{pmatrix}.$$ 

This is more precisely the **symplectic realization** of the complex Grassmannian.

*The infinite-dimensional restricted Grassmannian $G_{res}$ arises by taking symmetries $s$ in $A = \mathcal{L}(H)$, for a complex Hilbert space $H$, such that $s - 1$ is of trace class. In the approach by Rajeev-Turgut, it plays a basic role in 2-dimensional QCD.

**Loop groups**

Let $G$ be a compact connected 1-connected simple Lie group. Let 

$$(\xi|\eta) := -\text{tr}(\text{ad}_\xi \text{ad}_\eta)$$

be the negative Killing form. Let $S := S^1$ be the circle and

$$C^\infty_*(S,G) = \{ m : S \to G : m(1) = e \}$$

be the based loop group. It has the tangent space

$$T_m C^\infty_*(S,G) = C^\infty_*(S,g) = \{ u : S \to g : u(1) = 0 \}.$$ 

**Conformal blocks**

Let $S$ be a compact oriented surface. Let $G$ be a compact Lie group with Lie algebra $g$. Then the set

$$\Omega^1(S,G)$$

of all connexions $A$ on the trivial $G$-bundle $S \times G$ is an affine space of infinite dimension. It has the tangent space

$$T_A(\Omega^1(S,G)) = \Omega^1(S,g)$$

at any $A \in \Omega^1(S,g)$.

In the following, most manifolds will be constructed as **quotient manifolds** under an equivalence relation. Let $N$ be a (not necessarily connected) manifold and $R \subset N \times N$ be a closed submanifold which defines an equivalence relation on $N$. Then $M := N/R$ is a manifold if the *Godement properties* [?, ] hold: The projections $R \to M$ must be submersions. For $u \in N$ let $[u]$ denote the equivalence class in the quotient manifold $M = N/R$.

### 1.1.1 Covered manifolds

A **covered manifold** is a $K$-manifold $M$ endowed with an open covering by local charts $\sigma^a : U_a \to M$, where $U_a$ is a domain in a vector space $L \equiv K^n$. Then the open sets $V_a := \sigma^a(U_a) \subset M$ cover $M$. Denote by

$$\sigma_a := (\sigma^a)^{-1} : V_a \to U_a \subset L$$

the inverse of $\sigma^a$. The charts are related by transition maps

$$\sigma_b^a = \sigma_b \circ \sigma^a$$
satisfying \( \sigma^a = \sigma^b \circ \sigma_a^b \), \( \sigma_b^a \circ \sigma_a = \sigma_b \) and 
\[
\sigma_c^a = \sigma_c^b \circ \sigma_a^b.
\]

We define two (closely related) equivalence relations for a covered manifold. First, consider the disjoint union 
\[
U := \bigcup U_a \times \{a\}
\end{equation}
edowed with the equivalence relation 
\[
(x, a) \approx (y, b) \iff x \in U_a, y \in U_b, \sigma_a x = \sigma_b y.
\end{equation}

Equivalently, \( y = \sigma_b^a(x) \). In the following, we often write argument variables, such as \( x, y \), as a subscript, in order to save brackets. Now consider the disjoint union 
\[
V := \bigcup V_a \times \{a\}
\end{equation}
edowed with the equivalence relation 
\[
(m, a) \sim (m, b) \iff m \in V_a \cap V_b.
\end{equation}

Then \( M = U/ \approx = V/ \sim \).

**Jordan manifolds**

**Example 1.1.5.** Consider the projective space \( M = \mathbb{P}^s \). For \( 0 \leq i \leq s \) let \( U_i = L = \mathbb{C}^s \) and define the charts 
\[
\sigma^i : \mathbb{C}^s \to \mathbb{P}^s, \quad \sigma^i(z^0, \ldots, \hat{z}^i, \ldots, z^s) := [z^0, \ldots, \hat{1}^i, \ldots, z^s].
\end{equation}

Conversely, put 
\[
V_i := \{[\zeta] \in \mathbb{P}^s : \zeta^i \neq 0\}.
\end{equation}

Then 
\[
\sigma_i : V_i \to \mathbb{C}^s, \quad \sigma_i[\zeta] = \left( \frac{\zeta^0}{\zeta^i}, \frac{\hat{1}^i}{\zeta^i}, \ldots, \frac{\zeta^s}{\zeta^i} \right).
\end{equation}

The transition maps (for \( i < j \)) are given by 
\[
\sigma^j_{i}(z^0, \ldots, \hat{1}^i, \ldots, z^j, \ldots, z^s) = \left( \frac{z^0}{z^j}, \ldots, \frac{\hat{1}^i}{z^j}, \ldots, \frac{z^j}{z^j}, \ldots, \frac{z^s}{z^j} \right).
\end{equation}

In this way \( \mathbb{P}^s = U/ \approx \) becomes a covered manifold. In the special case \( s = 1 \) (Riemann sphere) we obtain 
\[
\sigma^0(z^1) := [1, z^1], \quad \sigma_0[\zeta] := \frac{\zeta^1}{\zeta^0},
\end{equation}
\[
\sigma^1(z^0) := [m^0, 1], \quad \sigma_1[\zeta] := \frac{\zeta^0}{\zeta^1},
\end{equation}
\[
\sigma^1_0(z^1) = \frac{1}{z^1}, \quad \sigma^1_0(z^0) = \frac{1}{z^0}.
\end{equation}

*finite charts for Grassmannian*

For the conformal hull \( \hat{Z} \) of a hermitian Jordan triple \( Z \), instead of a finite covering we have a 'continuous' covering by local charts 
\[
\sigma^a : Z \to \hat{Z}, \quad z \mapsto \sigma_a^z := [z, a]
\end{equation}
for any $a \in Z$. Thus $U = Z \times Z =: Z^2$ in this case, so that

$$\hat{Z} = Z^2 / \approx.$$  

In the special case $a = 0$ we write $z^0 = z$ and obtain the affine embedding

$$\sigma^0 : Z \subset \hat{Z}.$$  

If $(z, a)$ is quasi-invertible, then $\sigma^a_z = [z, a] = [z^a, 0] = (z^a)^0 = z^a$. In view of the addition formula (??) the transition map between two local charts $\sigma^a$ and $\sigma^b$ is given by

$$\sigma^a_b(z) = z^{a-b}$$

on the open set $\{ z \in Z : (z, a - b) \text{ quasi-invertible} \}$.

### 1.1.2 Homogeneous manifolds

Another basic type of quotient manifolds are the **homogeneous** manifolds. Let $G$ be a Lie group with a closed subgroup $H \subset G$. Then the equivalence relation $R := \{(g, gh) : g \in G, h \in H\}$ on $G$ is invariant under left $G$-translations and hence

$$M = G/H = G/R$$

becomes a quotient manifold with a left $G$-action.

- **Jordan manifolds**

projective space

$$P^s = SU(1, s)/U(s)$$

Grassmannian: Let $Z = \mathbb{C}^{r \times s}$, endowed with the operator norm $\|z\| = \sup \text{spec}(zz^*)^{1/2}$. Then the matrix unit ball

$$\hat{Z} = \{ m \in \mathbb{C}^{r \times s} : \|z\| < 1 \} = \{ m \in \mathbb{C}^{r \times s} : I - zz^* > 0 \}$$

is a symmetric domain under the **pseudo-unitary group**

$$\hat{G} = U(r, s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(r + s) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

which acts on $\hat{Z}$ via **Moebius transformations**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (m) = (az + b)(cz + d)^{-1}.$$  

Its maximal compact subgroup is

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in U(r), d \in U(s) \right\}.$$  

with the linear action $m \mapsto azd^*$. For the compact dual we have

$$G_r(C^{r+s}) = U(r + s)/U(r) \times U(s)$$

For a general hermitian Jordan triple, let $K = \text{Aut}(Z)$ denote the compact linear Lie group of all Jordan triple automorphisms of $Z$. The **structure group** $\hat{K} \subset GL(Z)$ is generated by all invertible
Bergman operators $B_{a,b}$, where $(a,b) \in \mathbb{Z}^2$ is quasi-invertible. It acts via linear transformations on $\hat{Z}$. On the other hand, the non-linear transformations of $\hat{Z}$ are the translations

$$t_az := z + a$$

and the quasi-inverse maps

$$t^*_az := z^{-a}$$

for $a \in \mathbb{Z}$. The **conformal group** $\hat{G}$ of $\mathbb{Z}$ is an algebraic Lie group generated by these three types of transformations. It acts transitively on $\hat{Z}$, giving a **conformal realization**

$$\hat{Z} = \hat{G}/\hat{G}_0$$

as a **flag manifold**. Here the **parabolic subgroup**

$$\hat{G}_0 := \{ g \in \hat{G} : g(0) = 0 \}$$

is generated by $\hat{K}$ and the quasi-inverse maps (??). Putting $B^*_{a,b} = B_{b,a}$, one can show that $\hat{K}$ carries an involution such that

$$K = \{ k \in \hat{K} : k^* = k^{-1} \}.$$ 

This can be extended to an involution of $\hat{G}$ mapping $t_a$ to $t^*_a$. Then

$$\hat{G} = \{ g \in \hat{G} : g^* = g^{-1} \}$$

is a compact subgroup of $\hat{G}$ which still acts transitively on $\hat{Z}$ and satisfies

$$\hat{G} \cap \hat{G}_0 = K.$$ 

This yields a **metric realization**

$$\hat{Z} = \hat{G}/K$$

of $\hat{Z}$ as a compact hermitian symmetric space. Let

$$s_0z := -z$$

denote the symmetry at the origin $0 \in Z$. Then

$$\hat{G} := \{ g \in \hat{G} : g^{-1} = s_0g^*s_0 \}$$

is a non-compact subgroup of $\hat{G}$ which acts transitively on the spectral unit ball $\tilde{Z}$ and also satisfies

$$\hat{G} \cap \hat{G}_0 = K.$$ 

This gives a **metric realization**

$$\tilde{Z} = \tilde{G}/K$$

of $\tilde{Z}$ as a non-compact hermitian symmetric space. In summary, we have a diagram of Lie groups

For $K = \mathbb{C}$ the structure group $\hat{K}$ is a complexification of $K$, and the conformal group $\hat{G}$ is a complexification of $\hat{G}$ and of $\hat{G}$. Moreover, $\hat{G}$ is the full biholomorphic automorphism group of $\hat{Z}$, and $\hat{G}$ is the full biholomorphic automorphism group of $\tilde{Z}$. (These remarks hold more precisely for the connected components of the identity.)

In terms of the classification of complex hermitian Jordan triples we have
\[ K = U(r) \times U(s) : z \mapsto u z v, \ u \in U(r), \ v \in U(s), \ Z = \mathbb{C}^{r \times s} \]
\[ K = U(s), \ Z = \mathbb{C}^{1 \times s} = \mathbb{C}^s \]
- symmetric matrices \( a = 1 \) (real case)
- anti-symmetric matrices \( a = 4 \) (quaternion case)
\[ K = T \cdot SO(a + 2) \quad \text{spin factor} \ Z = \mathbb{C}^{a+2} \]
\[ K = ?, \ Z = \mathbb{C}^{16} \text{exc and } K = T \cdot E_6, \ Z = \mathbb{C}^{27} \text{exc}. \]

1.2 Bundles

For any fibre bundle \( B \) over a manifold \( M \), let \( \Gamma(B) \) denote the set of all sections (smooth/holomorphic) \( \Phi : M \to B, \) satisfying \( \pi \circ \Phi = I_M \). For the trivial bundle \( B = M \times F \) with fibre \( F \) we write \( \Gamma(M \times F) = \Gamma(M,F) \).

Thus \( \Gamma(M,F) = \mathcal{C}^\infty(M,F) \) for \( K = \mathbb{R} \) and \( \Gamma(M,F) = \mathcal{O}(M,F) \) for \( K = \mathbb{C} \). Denote by \( T^M \) the tangent bundle if \( K = \mathbb{R} \) and the holomorphic tangent bundle if \( K = \mathbb{C} \). Thus in the complex case we have the complexified tangent space \( T^C_m M := T^M_m \oplus T^\mathbb{R}_m M \)
and the real tangent space \( T^R_m M \) is the real subspace \( T^R_m M = \{ v + \pi : v \in T^M_m M \} \).

The complex structure \( J : T^R_m M \to T^R_m M \) is given by \( J(v + \pi) = iv + \bar{\pi} \)
for all \( v \in T^R_m M \).

**Example 1.2.1.** For a domain \( M \subset \mathbb{C} \), with coordinate \( z = x + iy \), we have the holomorphic/antiholomorphic tangent vectors
\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \]
satisfying
\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad i \frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}. \]

The complex structure is
\[ J \frac{\partial}{\partial x} = J \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \right) = i \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}} = -\frac{\partial}{\partial y} \]
\[ J \frac{\partial}{\partial y} = -i J \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}} \right) = -i \left( i \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \right) = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}. \]

Let \( P \) be a principal fibre bundle with Lie structure group \( H \), called an \( H \)-bundle in the following, over \( M = P/H \). Any \( H \)-module \( (E, \pi) \) (i.e., a finite dimensional vector space \( E \) endowed with a representation \( \pi \) of \( H \)), gives rise to an **associated vector bundle**
\[ P^\pi_H = \{ [p, \phi] = [ph, h^{-1}\phi] : p \in P, h \in H, \phi \in E \}. \]
Let \([p] \in M\) denote the equivalence class of \(p \in P\). Writing \(\Phi_{[p]} = [p, \tilde{\Phi}_p]\), for the so-called homogeneous lift \(\Phi\) of a section \(\Phi\), one obtains an isomorphism

\[
\Gamma(P \times^H E) \equiv \{ f \in \Gamma(P, E) : f_{ph} = h^{-\pi} f_p \ \forall p \in P, \ h \in H \}.
\]

In the following, principal bundles and their associated vector bundles will be defined in terms of cocycles.

**Proposition 1.2.2.** Let \(M = N/\sim\) be a quotient manifold for an equivalence relation \(R \subset N\). Let \(\beta : R \to H\) be a smooth map into a Lie group \(H\), denoted by \((u, v) \mapsto \beta_{u}^{v}\), which has the cocycle property

\[
\beta_{w}^{v} \beta_{u}^{w} = \beta_{u}^{v}
\]

for all triples \(u \sim v \sim w\) in \(N\). Then

\[
N/\sim := \{ [u, h] = [v, (\beta_{u}^{v})^{-1} h] : (u, v) \in R, \ h \in H \}
\]

becomes an \(H\)-bundle over \(M = N/R\), with projection \(N \times^H H \to M\), \([u, h] \mapsto [u]\).

As a consequence any \(H\)-module \(E\) gives rise to an induced vector bundle

\[
N/\sim \times^H E := (N/\sim)^{\beta} \times^H E = \{ [u, \phi] = [v, (\beta_{u}^{v})^{-\pi} \phi] : (u, v) \in R, \ \phi \in E \}
\]

over \(M\). Writing \(\Phi_{[u]} = [u, \Phi_u]\) one obtains an isomorphism

\[
\Gamma(N/\sim \times^H E) \equiv \{ f \in \Gamma(N, E) : f_v = (\beta_{u}^{v})^{-\pi} f_u \ \forall (u, v) \in R \}.
\]

We often omit the reference to \(\pi\) if the context is clear.

### 1.2.1 Covered manifolds

For a covered manifold \(M\) consider maps \(\beta_{a}^{b} : V_a \cap V_b \to H\) satisfying the cocycle property

\[
\beta_{c}^{a} (m) \beta_{b}^{c} (m) = \beta_{b}^{a} (m)
\]

for all \(m \in V_a \cap V_b \cap V_c\). Then

\[
\beta_{a,m}^{b} := \beta_{a}^{b} (m)
\]

defines a cocycle \(\beta : R \to H\) in the sense of (??). Hence

\[
\mathcal{V}/\sim := \{ [m, h]_{a} = [m, \beta_{a}^{h} (m)]_{b} : m \in V_a \cap V_b, \ h \in H \}
\]

becomes an \(H\)-bundle over the quotient manifold \(M = \mathcal{V}/\sim\). Any \(H\)-module \(E\) gives rise to an induced vector bundle

\[
\mathcal{V}/\sim \times^H E := (\mathcal{V}/\sim)^{\beta} \times^H E = \{ [m, \phi]_{a} = [m, \beta_{a}^{h} (m) \phi]_{b} : m \in V_a \cap V_b, \ \phi \in E \}
\]

over \(M\). Writing \(\Phi_{[m]} = [m, \Phi^{a}(m)]_{a}\) one obtains an isomorphism

\[
\Gamma(\mathcal{V}/\sim \times^H E) \equiv \{ (\Phi^{a}) \in \prod_{a} \Gamma(V_a, E) : \Phi^{a}(m) = \beta_{a}^{h} (m) \Phi^{h}(m) \ \forall m \in V_a \cap V_b \}.
\]

By local triviality, every principal bundle and every vector bundle can be realized this way.
The **tangent bundle** arises as follows. For a covering family of charts \( \sigma^a \) of \( M \) and a map \( f : V_a \to E \) we write
\[
\frac{\partial f}{\partial \sigma_a}(m) := (f \circ \sigma^a)'(x) \in \text{Hom}(L, E)
\]
if \( m = \sigma^a_x \in V_a \). Applying this notation to \( E = L \) and \( f = \sigma_b \), we obtain
\[
\frac{\partial \sigma_b}{\partial \sigma_a}(m) = (\sigma_b \circ \sigma^a)'(x) = (\sigma^a_b)'(x) \in \mathcal{L}(L) \text{ endomorphisms}
\]
for \( m = \sigma^a_x \in V_a \cap V_b \). Now let \( m = \sigma^a_x = \sigma^b_y \). Since \( \sigma = \sigma^b_y(x) \), the chain rule yields
\[
\frac{\partial \sigma_c}{\partial \sigma_a}(m) = (\sigma^a_c)'(x) = (\sigma^b_c)'(y)(\sigma^a_b)'(x) = \frac{\partial \sigma_c}{\partial \sigma_b}(m) \frac{\partial \sigma_b}{\partial \sigma_a}(m)
\]
for all \( x \), with \( y := \sigma^b_y(x) \). We sometimes write this relation in the opposite order
\[
t \cdot d_x \sigma^a_c = (t \cdot d_x \sigma^a_b) \cdot d_y \sigma^b_c
\]
for all \( t \in L \). It follows that
\[
\hat{\sigma}^a_b(m) := \frac{\partial \sigma_a}{\partial \sigma_b}(m)
\]
defines a \( GL(L) \)-valued cocycle on \( R \). Hence we obtain a \( GL(L) \)-bundle
\[
\mathcal{V} \overset{\hat{\sigma}}{\times} GL(L) = \{[m, h]_a = [m, h \frac{\partial \sigma_b}{\partial \sigma_a}(m)]_b : m \in V_a \cap V_b\}
\]
over \( M \), called the **bein bundle**. Via the defining representation of \( GL(L) \), we obtain the associated vector bundle
\[
\mathcal{V} \overset{\hat{\sigma}}{\times} L = \{[m, t]_a = [m, \frac{\partial \sigma_b}{\partial \sigma_a}(m)t]_b : m \in V_a \cap V_b, \ t \in L\}
\]
which is isomorphic to the tangent bundle \( TM \) by identifying \([m, t]_a\) with \((T_x \sigma^a)t\) for \( m = \sigma^a_x \). In fact, we have
\[
(T_x \sigma^a)t = T_x(\sigma_b \circ \sigma^a)t = (T_y \sigma^b)(\sigma^a_b)t = (T_y \sigma^b)\frac{\partial \sigma_b}{\partial \sigma_a}(m)t
\]
for \( m = \sigma^a_x = \sigma^b_y \). The corresponding sections (vector fields) are
\[
\Gamma(\mathcal{V} \overset{\hat{\sigma}}{\times} L) \equiv \{(T^a) \in \prod_a \Gamma(V_a, L) : T^a_m = \frac{\partial \sigma_b}{\partial \sigma_a}(m)T^a_m \ \forall m \in V_a \cap V_b\}.
\]
Similarly, the cotangent bundle \( T^*M \) is isomorphic to the cocycle bundle
\[
\mathcal{V} \overset{\hat{\sigma}}{\times} L^* = \{[m, \theta]_a = [m, \theta \circ \frac{\partial \sigma_b}{\partial \sigma_a}(m)]_b : m \in V_a \cap V_b, \ \theta \in L^*\}
\]
by identifying \([m, \theta]_a\) with \( \theta \circ (T_m \sigma_a) \) when \( m = \sigma^a_x \). In fact, we have
\[
\theta \circ (T_m \sigma_a) = \theta \circ T_m(\sigma_b \circ \sigma_a) = \theta \circ (\sigma_b)'(x)(T_m \sigma_b) = \theta \circ \frac{\partial \sigma_b}{\partial \sigma_a}(m)(T_m \sigma_b)
\]
for \( m = \sigma^a_x = \sigma^b_y \). The corresponding sections (1-forms) are
\[
\Gamma(\mathcal{V} \overset{\hat{\sigma}}{\times} L^*) \equiv \{(\Theta^a) \in \prod_a \Gamma(V_a, L^*) : \Theta^a_m = \Theta^b_m \circ \frac{\partial \sigma_a}{\partial \sigma_b}(m) \ \forall m \in V_a \cap V_b\}.
\]
These bundles can also be described in the setting \( M = \mathcal{U}/\sim \). The formulas are
\[
\mathcal{U} \overset{\hat{\sigma}}{\times} GL(L) = \{[x, h]_a = [y, h(\sigma^b_y)'(x)]_b : x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y\}.
\]
\[ \mathcal{U}^a_L = \{ [x, t]_a = [y, (\sigma^a_y)_x t]_b : x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y, \ t \in L \} \]
\[ \Gamma(\mathcal{U}^a_L) \equiv \{ (T^a) \in \prod_a \Gamma(U_a, L) : T^a_y = (\sigma^a_y)_x T^a_x \ \forall x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y \} , \]
\[ \mathcal{U}^a \times L^* = \{ [x, \vartheta]_a = [y, \partial \circ (\sigma^b_y)_x \vartheta]_b : x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y, \ \vartheta \in L^* \} \]
\[ \Gamma(\mathcal{U}^a \times L^*) \equiv \{ (\Theta^a) \in \prod_a \Gamma(U_a, L^*) : \Theta^a_y = \Theta^a_x \circ (\sigma^a_y)_x x \in U_a, \ y \in U_b, \ \sigma^a_x = \sigma^b_y \} . \]

In case \( M \) carries an \( H \)-structure, for a closed subgroup \( H \subset GL(L) \), the transition maps \( \sigma^a_x \) can be chosen such that \( \frac{\partial \sigma^a_x}{\partial \sigma^b_y}(m) = (\sigma^a_y)_x \in H \), and we obtain \( H \)-bundles instead.

• Jordan manifolds

**Example 1.2.3.** For the projective space \( M = \mathbb{P}^* \) taking derivatives of (??), we obtain
\[ e_k \cdot (\partial_m \frac{z^i}{z^j}) = \frac{\partial}{\partial z^k} \partial_m \frac{z^i}{z^j} = \frac{\delta^i_j}{(z)^2} \]
for the standard base \( e_k \) of \( \mathbb{C}^* \). For any other \( u = u^k e_k \in \mathbb{C}^* \) we obtain
\[ u \cdot (\partial_m \frac{z^i}{z^j}) = u^k e_k \cdot (\partial_m \frac{z^i}{z^j}) = u^k \frac{\delta^i_j}{(z)^2} = u^i \frac{z^j - z^i}{(z)^2} \]

**Proposition 1.2.4.** For a hermitian Jordan triple \( Z \) the conformal hull carries a \( \hat{K} \)-structure. More precisely, the map \( \beta : Z^2 \rightarrow \hat{K} \) defined by
\[ \beta_{z,a}^{w,b} := B_{z,a-b} \]
is a cocycle, and the induced \( \hat{K} \)-bundle \( Z^2 \times \beta^\sim \hat{K} \) over \( \hat{Z} \) is the bein (tangent frame) bundle.

**Proof.** The cocycle property follows from the addition formula (??). The well-known identity
\[ \partial_z t_a^* = B_{z,-a}^{-1} \]
implies that the transition map \( \sigma^a_{z,a} = t_a^* \) has the derivative
\[ \partial_z \sigma^a_{z,a} = \partial_z t_a^{*}_{-a} = B_{z,a-b}^{-1} \cdot \]
This gives the bein bundle. \( \square \)

As a consequence, any \( \hat{K} \)-module \( E \) yields an induced vector bundle
\[ Z^2^\sim \times E \equiv (Z^2 \times \hat{K})^\sim \times E \equiv \{ [z, \phi]_a = [z^{a-b}, B_{z,a-b}^{-1} \phi]_b : (z, a - b) \text{ quasi-invertible} \} \]
over \( \hat{Z} = Z^2 / \approx \). Writing \( \Phi_{[z,a]} = [z, \Phi^a_z]_a \) the sections \( \Phi \) are described by
\[ \Gamma(Z^2 \times E) \equiv \{ (\Phi^a) \in \Pi_a \Gamma(Z, E) : \Phi^a_{z,a-b} = B_{z,a-b}^{-1} \Phi^a_z \} . \]
Since \( Z \subset \hat{Z} \) is a dense open subset via the embedding \( z \mapsto z^0 = [z, 0] \), a section \( \Phi \) is uniquely determined by its trivialization \( \Phi^\sim \equiv \Phi^0 \). Thus the mapping \( \Phi \mapsto \Phi^\sim \) identifies \( \Gamma(Z^2 \times \beta^\sim E) \) with a vector space of maps from \( Z \) to \( E \). For the defining representation \( \hat{K} \subset GL(Z) \) we obtain the **tangent bundle**
\[ Z^2^\sim \times Z \equiv (Z^2 \times \hat{K}) \times Z \equiv \{ [z, t]_a = [z^{a-b}, B_{z,a-b}^{-1} t]_b : (z, a - b) \text{ quasi-invertible, } t \in Z \} \equiv T\hat{Z} \]
and the cotangent bundle
\[ Z^2 \times Z^* = (Z \times \hat{K}) \times Z^* = \{ [z, \vartheta]_a = [z^{a-b}, \vartheta \circ B_{z,a-b}] : (z, a - b) \text{ quasi-invertible} \} \equiv T^* \hat{Z}. \]

**Example 1.2.5.** Riemann sphere

### 1.2.2 Homogeneous manifolds

For a Lie group \( G \) with a closed subgroup \( H \subset G \), we may regard
\[ G = G \times H \]
as an \( H \)-bundle over \( M := G/H \). The **homogeneous vector bundle** associated to an \( H \)-module \( E \) of \( H \) is given by
\[ G \times_H E := \{ [g, \phi] = [gh, h^{-1}\phi] : g \in G, h \in H, \phi \in E \}. \]

It is \( G \)-equivariant under the action
\[ g' H [g', \phi] := [gg', \phi] \]

**• Jordan manifolds**

The derivative \( \partial_0 q \) of \( q \in \hat{G}_0 \) belongs to \( \hat{K} \), and
\[ \partial_0 : \hat{G}_0 \to \hat{K}, \quad q \mapsto \partial_0 q. \]
is a homomorphism.

**Proposition 1.2.6.** The mapping
\[ [z, h]_a \mapsto [t^*_{z,a} t_z, h] \]
induces an isomorphism
\[ Z^2 \times \hat{K} \equiv \hat{G} \times \hat{K} \]
of \( \hat{K} \)-bundles over \( \hat{Z} \).

**Proof.** The transformation \( g := t^*_{z,a} t_z \in \hat{G} \) has the derivative
\[ \partial_0 g = (\partial_z t^*_{z,a})(\partial_0 t_z) = \partial_z t^*_{z,a} = B^{-1}_{z,a} \]
Now let \([z, \phi]_a = [w, B^{-1}_{z,a-b} \phi]_b\). Then \([t^*_{z,a} t_z(0)] = t^*_{z,a}(z) = z^a = w^b = t^*_{z,b}(w) = t^*_{z,b} t_w(0)\). Hence there exists \( q \in \hat{G}_0 \) such that \([t^*_{z,b} t_w = t^*_{z,a} t_z q] \). Then
\[ B^{-1}_{w,b} = \partial_0(t^*_{z,b} t_w) = \partial_0(t^*_{z,a} t_z q) = \partial_0(t^*_{z,a} t_z) \partial_0 q = B^{-1}_{z,a} \partial_0 q. \]

Therefore \( \partial_0 q = B_{z,a} B^{-1}_{w,b} = B_{z,a-b} \) by the addition formula (??). This implies
\[ [t^*_{z,a} t_z, h] = [t^*_{z,b} t_w q^{-1}, h] = [t^*_{z,b} t_w, (\partial_0 q)^{-1} h] = [t^*_{z,b} t_w, B^{-1}_{z,a-b} h]. \]

Hence the assignment (??) is a well-defined map \( Z^2 \times \hat{K} \to \hat{G} \times \hat{G}_0 \hat{K} \), which is a bijection.

Thus for any \( \hat{G}_0 \)-module \( E \) the mapping \([z, \phi]_a \mapsto [t^*_{z,a} t_z, \phi] \) induces a vector bundle isomorphism
\[ Z^2 \times \hat{E} \equiv \hat{G} \times \hat{G}_0 \hat{E} \]
As a consequence, the vector bundle \( Z^2 \times \hat{E} \) carries a \( \hat{G} \)-action. This is not obvious in the coordinate chart picture.

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1.3 0-Geometry: Hermitian Metrics

A 0-geometry on a vector bundle is a hermitian metric. We can also allow pseudo-metrics of indefinite signature and speak generally of metric vector spaces and vector bundles. The positive definite case will be called 0-metric (0 negative eigenvalues).

Let $P$ be an $H$-bundle over $M = P/H$. Then any metric $H$-module $E$ defines a metric $([p, φ] | [p, η]) := (ξ|η)$ on the associated vector bundle $P \times_H E$. This is well-defined since

$$([ph, h^{-1} φ] | [ph, h^{-1} φ]) = (h^{-1} ξ|h^{-1} ξ) = (ξ|η).$$

1.3.1 Covered manifolds

Consider an $H$-valued cocycle $β^a_b$ and a metric $H$-module $E$.

**Lemma 1.3.1.** Let $E$ be a metric vector space, with inner product $(ξ|η)$. A family of smooth maps $h^a : V_a \to H^a(E)$ (self-adjoint invertible), satisfying the compatibility condition

$$h^a_m = β^b_a(m)^* h^b_m β^a_b(m)$$

for all $m \in V_a \cap V_b$ defines a metric on $V \times_H E$ via

$$([m, φ]_a | [m, η]_a)_m := (ξ|h^a_m η)_m.$$  

For $E = \mathbb{C}$, a family of smooth functions $h^a : V_a \to \mathbb{R}^>$, satisfying the compatibility condition

$$h^a_m = |β^b_a(m)|^2 h^b_m$$

for all $m \in V_a \cap V_b$ defines a 0-metric on the line bundle $V \times_H \mathbb{C}$ via

$$([m, φ]_a | [m, η]_a)_m := (ξ|h^a_m η)_m.$$  

**Proof.** The identification $[m, φ]_a = [m, β^b_a(m)φ]_b$ yields

$$(m, β^b_a(m)φ)_a | [m, β^b_a(m)η]_a = (β^b_a(m)ξ|h^b_m β^a_b(m)η) = (ξ|β^b_a(m)^*h^b_m β^a_b(m)η) = (ξ|h^a_m η) = ([m, φ]_a | [m, η]_a)_m.$$

□

**Proposition 1.3.2.** Let $E$ be a 0-metric vector space. Let $(χ_a)$ be a partition of unity subordinate to $V$. Then the family

$$h^a := \sum c β^c_a^* χ_c β^c_a, \quad h^a_m := \sum c β^c_a(m)^* χ_c(m) β^c_a(m)$$

defines a 0-metric on $V \times_H E$. For $E?\mathbb{C}$, the family

$$h^a := \sum c |β^c_a|^2 χ_c, \quad h^a_m := \sum c |β^c_a(m)|^2 χ_c(m)$$

defines a 0-metric on the line bundle $V \times_H \mathbb{C}$.  

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Let Proposition 1.3.3. Proof. Since the sum (??) is locally finite and the $\chi_c(m)$ add up to 1, (??) defines a smooth map from $V_a$ to the positive definite matrices. For $m \in V_a \cap V_b$ the cocycle property (??) implies

$$\beta^b(m)^* h^b_m \beta^b(m) = \beta^b(m)^* \sum_c \beta^b_c(m)^* \chi_c(m) \beta^b_c(m) \beta^b(m)$$

$$= \sum_c (\beta^b_c(m)^*) \chi_c(m) \beta^b_c(m) \beta^b(m) = \sum_c \beta^b_c(m)^* \chi_c(m) \beta^b_c(m) = h^a_m$$

Proof. The cocycle property follows from $\kappa^a \kappa^b = (h^a)^{1/2} \beta^a_b (h^b)^{-1/2} (h^b)^{1/2} \beta^b_c (h^c)^{-1/2} = (h^a)^{1/2} \beta^a_b \beta^b_c (h^c)^{-1/2} = (h^a)^{1/2} \beta^a (h^c)^{-1/2} = \gamma_c.$ Moreover, we have

$$\kappa^a \kappa^b = ((h^a)^{1/2} \beta^a_b (h^b)^{-1/2})^* (h^a)^{1/2} \beta^a_b (h^b)^{-1/2} = (h^a)^{1/2} \beta^a_b^* h^a_b (h^b)^{-1/2} = (h^a)^{-1/2} h^a_b (h^b)^{-1/2} = I.$$

As a consequence we may form the $^0$-metric vector bundle

$$\mathcal{V} \sim E = \{ (m, \xi) \mapsto (m, \gamma^a_b (m) \xi)_b : m \in V_a \cap V_b \}.$$ It carries the $^0$-metric

$$((m, \xi)_a | (m, \eta)_a) = (\xi | \eta).$$

since the condition (??) is trivially satisfied by $h^a_m = I_E$.

For the tangent bundle, a family of smooth maps $h^a : V_a / U_a \to \mathcal{H}^* (L)$, satisfying the compatibility condition

$$h^a_m = \frac{\partial}{\partial \sigma_a} (m)^* h^b_m \frac{\partial}{\partial \sigma_b} (m), \quad h^a_x = (\sigma^a (y)^* h^b (\sigma^b (y)^*$$
on $V_a \cap V_b / U_a \cap U_b$ defines a tangent metric on $\mathcal{V} / \mathcal{U} \times \mathcal{L} \equiv TM$ via the assignment

$$([m, u]_{a} [m, v]_{a})_m := (u | h^a_m v), \quad ([x, u]_{a} [x, v]_{a}) := (u | h^a_v v),$$

Similar for the cotangent bundle. In the positive case the family

$$h^a_m := \sum_c \frac{\partial}{\partial \sigma_a} (m)^* \chi_c(m) \frac{\partial}{\partial \sigma_c} (m)$$

induces a tangent $^0$-metric on $M$. The associated unitary cocycle is

$$\kappa^a_b (m) := (h^a_m)^{1/2} \frac{\partial}{\partial \sigma_b} (m) (h^b_m)^{-1/2}$$

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• Jordan manifolds

**Example 1.3.4.** For \( K = \mathbb{C} \) consider the holomorphic tangent bundle \( TP \) on the Riemann sphere \( P^1 \), endowed with the tangent metric
\[
h^0_z = (1 + z \bar{z})^{-2}.
\]
The coordinate change \( w := \frac{1}{z} \) yields
\[
h^1_w(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}) = (1 + w \bar{w})^{-2}.
\]*The metric is invariant under \( SU(2) \).

### 1.3.2 Homogeneous manifolds

If \( H \subset G \) is a closed subgroup and \( E \) is a metric \( H \)-module, then \( G \times_H E \) becomes a \( G \)-equivariant metric vector bundle with respect to the fibre metric
\[
([g, \phi] | [g, \psi])_{gH} := (\phi | \psi).
\]
For line bundles, with \( \phi, \psi \in \mathbb{C} \), the \( 0 \)-metric is
\[
([g, \phi] | [g, \psi])_{gH} := \overline{\phi} \psi.
\]

• Jordan manifolds

Any unitary \( K \)-representation \( (E, \pi) \) has a holomorphic extension to \( \hat{K} \). Then the mapping \([z, \phi] \mapsto [t^*_{-a} t_z, \phi]\) induces an isomorphism
\[
Z^2 \kappa \times E \equiv \hat{G} \pi \times K E
\]
of hermitian holomorphic vector bundles. As a consequence, the restricted \( \hat{G} \)-action on the vector bundle \( Z^2 \times \beta \cup E \) is isometric.

### 1.4 1-Geometry: Connexions

The Lie algebra \( \Gamma(TM) \) of vector fields on \( M \) is endowed with the commutator
\[
[X, Y]_m = X_m \cdot d_m Y - Y_m \cdot d_m X.
\]
The infinitesimal action of vector fields on maps \( \Phi : G \to E \) is given by
\[
(d_X \Phi)_g := X_g \cdot T_g \Phi.
\]
**Proposition 1.4.1.**
\[
d_X (d_Y \Phi) - d_Y (d_X \Phi) = d_{[X,Y]} \Phi.
\]
For a real manifold \( M \) let
\[
\Omega^r(M, \mathbb{R}) = \Gamma(T^r M)
\]
denote the space of all smooth real \( r \)-forms over \( M \). Thus
\[
\Omega^0(M) = \Gamma(M \times \mathbb{R}) = \Gamma(M, \mathbb{R})
\]
consists of all smooth functions on \( M \). The exterior derivative
\[
d : \Omega^r(M) \to \Omega^{r+1}(M)
\]

is defined by the Palais formula: Given vector fields $\xi^0, \ldots, \xi^p$ then
\[
(d\omega)(X^0, \ldots, X^p) = \sum_{i=0}^{p} (-1)^i X^i_\xi \omega(X^0, \ldots, \hat{X}^i, X^i, \ldots, X^p) + \sum_{i<j} (-1)^{i-j} \omega([X^i, X^j], X^0, \ldots, \hat{X}^i, \hat{X}^j, \ldots, X^p).
\]
This definition differs from [\text{Proposition 3.11}] by a factor of $\frac{1}{p!}$, but makes sense in any characteristic. For a 2-form $\omega$ we obtain
\[
(X, Y, Z) d\omega_m = X \cdot (Y, Z) \omega - Y \cdot (X, Z) \omega + Z \cdot (X, Y) \omega - ([X, Y], Z) \omega - ([Y, Z], X) \omega + ([X, Z], Y) \omega
\]
If $B$ is a smooth vector bundle over a real manifold $M$, one can still define differential forms $\Omega^r(B)$, but the exterior differential $d$ makes sense only if $B = M \times E$ is trivial. In this case we write $\Omega^r(M \times E) = \Omega^r(M, E)$.

For a complex manifold $M$, the complexified tangent space splits into the holomorphic and anti-holomorphic tangent space. The complexified smooth differential forms have a splitting
\[
\Omega^r(M, \mathbb{C}) = \sum_{p+q=r} \Omega^{p,q}(M)
\]
into $(p, q)$-forms. Accordingly, the differential
\[
d : \Omega^r(M, \mathbb{C}) \rightarrow \Omega^{r+1}(M, \mathbb{C})
\]
splits as $d = \partial + \overline{\partial}$, with
\[
\partial : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q}(M), \quad \overline{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M).
\]
If $B$ is a holomorphic vector bundle over a complex manifold $M$, the anti-linear part $\overline{\partial}$ of the exterior differential is still well-defined.

In general, for a $G$-bundle $P$ over $M = P/G$ let $P \times^G \mathfrak{g}$ denote the adjoint $\mathfrak{g}$-bundle of $P$. The space $\Omega^1(P)$ of all $G$-connexions on $P$ over $M$ is an affine space, with tangent space
\[
T_A \Omega^1(P) = \Omega^1(P, \mathfrak{g})
\]
at any $A \in \Omega^1(P)$. We write $\Omega^1(M, G)$ for the space of connexions on the trivial $G$-bundle $M \times G$, with tangent spaces $\Omega^1(M, G)$.

Let $P \times^G \mathbb{C}$ be an associated vector bundle. Let $m \in M$ and $u \in T_m M$. Choose $p \in P$ with $m = [p] = \pi(p)$. For any connexion $A \in \Gamma^1(P)$ the horizontal subspace $T^A_p P$ yields an isomorphism
\[
T_p \pi : T^A_p P \rightarrow T_m M.
\]
Hence there exists a unique horizontal tangent vector $u^A \in T^A_p P$ such that $T_p(\pi)u^A = u$. Given a section $\Phi$, apply $u^A$ to the smooth function $\Phi : P \rightarrow \mathbb{C}$ we obtain $u^A \cdot d_p \Phi \in \mathbb{C}$. Then
\[
u \cdot d_m^A \Phi = [p, u^A \cdot d_p \Phi]
\]
is independent of the choice of $p$ [\text{Section III.1, Lemma on p. 115}], and we obtain the covariant differential as a 1-form $d^A \Phi$. The map
\[
d^A : \Omega^0(P, \mathbb{C}) \rightarrow \Omega^1(P, \mathbb{C}), \quad \Phi \mapsto d^A \Phi
\]

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satisfies the Leibniz rule

\[ d^A(f \Phi) = df \wedge \Phi + f \cdot d^A \Phi \]

for all sections \( \Phi \in \Omega^0(P \times_G E) \) and functions \( f \in \Omega^0(M, K) \). On the other hand, given a vector field \( X \in \Gamma(TM) \) we define the \textbf{covariant derivative} \( d^A_X \) acting on sections. The two notions are related by

\[ X \cdot d^A \Phi = d^A_X \Phi. \]

The value at a given point \( m \in M \) is denoted by

\[ (X \cdot d^A \Phi)_m = (d^A_X \Phi)_m = X_m \cdot (d^A_m \Phi) \]

Thus there is a canonical mapping

\[ \Omega^1(P) \times \Omega^0(P \times_G E) \to \Omega^1(P \times_G E), \quad (A, \Phi) \mapsto d_A \Phi. \]

If \( P \times HE \to M \) is a holomorphic vector bundle one can also consider the anti-linear part

\[ \bar{d}^A : \Omega^0(P \times E) \to \Omega^{0,1}(P \times E). \]

**Proposition 1.4.2.** For any tangent metric \( g \) there is a unique \textbf{Levi-Civita connexion} \( \bar{\partial}g \) which satisfies

\[ d_Xg(Y, Z) = g(d^g_XY, Z) + g(Y, d^g_XZ) \]

and is torsion-free, i.e.,

\[ d^g_XY - d^g_YX = [X, Y]. \]

It is given by

\[ 2g(d^g_XY, Z) = d_Xg(Y, Z) + d_Yg(Z, X) - d_Zg(Y, X) + g(Z, [X, Y]) - g(Y, [X, Z]) - g([Y, Z], X) \]

**Proof.** Combining the two properties yields

\[ d_Xg(Y, Z) + d_Yg(Z, X) - d_Zg(Y, X) = g(d^g_XY, Z) + g(Y, d^g_XZ) + g(Y, d^g_XZ) - g(Y, d^g_XZ) - g(Y, d^g_XZ) = g(d^g_XY + d^g_YX, Z) + g(Y, d^g_XZ - d^g_ZX) + g(Y, d^g_ZX - d^g_XZ) = 2g(d^g_XY + d^g_YZ, [X, Z]) + g(Y, [X, Z]) + g([Y, Z], X) \]

The definition of the Levi-Civita connexion \( \bar{\partial}g \) is analogous to the exterior derivative

\[ d\omega(X, Y, Z) = d_X\omega(Y, Z) - d_Y\omega(X, Z) + d_Z\omega(X, Y) - \omega([X, Y], Z) - \omega([X, Z], Y) - \omega([Y, Z], X) \]

of a 2-form \( \omega \).

Now let \( P = N \times_- H \) be a cocycle \( H \)-bundle on \( M = N/R \). Then the adjoint bundle is

\[ \left( N \times H \right)_{\text{ad}} \cong \left( N \times_R \mathfrak{h} \right) \]

Hence the affine space \( \Omega^1(N \times_- H) \) of all \( H \)-connexions on \( N \times_- H \) has the tangent space

\[ T_A(\Omega^1(N \times_- H)) = \Omega^1(N \times_R \mathfrak{h}) \]

at any \( A \in \Omega^1(N \times_- H) \).
1.4.1 Covered manifolds

For covered manifolds, connexions are constructed as follows. A connexion $A$ on $\mathcal{V} \times^\beta E$ is given by the covariant differential

$$d_A : \Omega^0(\mathcal{V} \times^\beta E) \to \Omega^1(\mathcal{V} \times^\beta E).$$

Given $v \in T_a M$ and a section $\Phi \in \Omega^0(\mathcal{V} \times^\beta E)$ we have local representatives $(v \cdot d^A_m \Phi)^a \in E$ for $m \in V_a$.

**Proposition 1.4.3.** A family $m \mapsto A^a_m$ of $\mathfrak{gl}(E)$-valued 1-forms on $V_a$ such that

$$A^a = \beta^a(m) \left( d^b_m \beta^b_a + \Lambda^b_m \beta^b(m) \right),$$

for $m \in V_a \cap V_b$, as an identity of linear functionals $T_m M \to \mathfrak{gl}(E)$, defines a (global) connexion $A$ on $\mathcal{V} \times^\beta E$ with covariant derivative

$$(v \cdot d^A)_{\Phi}^a = v \cdot d_m \Phi^a + (v \cdot A^a_m) \Phi^a_m.$$  

Here $v \cdot d_m \Phi^a \in E$ and $v \cdot A^a_m \in \mathcal{L}(E)$. For $E = \mathbb{C}$, a family of 1-forms $A^a_m$ on $V_a$, satisfying

$$A^a_m - A^b_m = \frac{d_m \beta^b}{\beta^b_a(m)} = d_m \log \beta^b_a$$

for all $m \in V_a \cap V_b$, yields a global connexion $A$ on $\mathcal{V} \times^\beta \mathbb{C}$ with covariant derivative (??).

**Proof.** In order to define a global connexion, we need to check the compatibility relation

$$(v \cdot d^A)_{\Phi}^a = \beta^a(m)(v \cdot d^A)_{\Phi}^b$$

for $m \in V_a \cap V_b$ and $v \in T_m M$. The condition (??) becomes

$$v \cdot A^a_m = \beta^a(m) \left( v \cdot d_m \beta^a + (v \cdot A^a_m) \beta^a(m) \right)$$

with $v \cdot d_m \beta^a \in \mathcal{L}(E)$. Since $v \cdot d_m \Phi^a = v \cdot d_m (\beta^a \Phi^a) = (v \cdot d_m \beta^a) \Phi^a(m) + \beta^a(m)(v \cdot d_m \Phi^a)$ by the product rule, we have

$$\beta^a(m)(v \cdot d_m \Phi^a) = \beta^a(m)(v \cdot d_m \beta^a \Phi^a(m) + v \cdot d_m \Phi^a).$$

Hence (??) implies

$$(v \cdot d^A)^a(m) = v \cdot d_m \Phi^a + (v \cdot A^a_m) \Phi^a(m) = v \cdot d_m \Phi^a + \beta^a(m) \left( v \cdot d_m \beta^a + (v \cdot A^a_m) \beta^a(m) \right) \Phi^a(m)$$

$$= v \cdot d_m \Phi^a + \beta^a(m) \cdot (v \cdot d_m \beta^a) \Phi^a(m) + \beta^a(m) \cdot (v \cdot A^a_m) \Phi^a(m)$$

$$= \beta^a(m) \left( v \cdot d_m \Phi^a + (v \cdot A^a_m) \Phi^a(m) \right) = \beta^a(m)(v \cdot d^A)^b(m).$$

For $E = \mathbb{C}$, we have

$$\beta^a(m) \cdot A^a_m \beta^a(m) = A^a_m \beta^a(m) = A^a_m.$$

Thus (??) simplifies to (??). $\square$

The space $\Omega^1(\mathcal{V} \times^\beta \mathbb{GL}(E))$ of all connexions on $\mathcal{V} \times^\beta E$ is an affine space, with tangent space

$$T_A(\Omega^1(\mathcal{V} \times^\beta \mathbb{GL}(E))) = \Omega^1(\mathcal{V} \times^\beta \mathfrak{gl}(E)).$$

In fact, let $(A^a_1)$ and $(A^a_2)$ be two connexions on $\mathcal{V} \times^\beta E$. Then

$$A^a := A^a_1 - A^a_2$$

is a smooth mapping $V_a \cap V_b \to \mathfrak{gl}(E)$ such that

$$A^a = \beta^a \Lambda^b \beta^b_a$$

on $V_a \cap V_b$. Thus $(A^a)$ defines a global $\mathfrak{gl}(E)$-valued 1-form on $M$.  

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Proposition 1.4.4. Let \((\chi_a)\) be a partition of unity subordinate to \((V_a)\). Then the family
\[
A^a = \sum_c c \beta_c^a (d\beta_c^a)
\]
defines a global connexion on \(\mathbb{V} \times \beta E\).

Proof. On \(V_a \cap V_b \cap V_c\) we have
\[
\beta_c^a (d\beta_c^a) = -(d\beta_c^a) \beta_c^a = -(d(\beta_c^a \beta_c^b)) \beta_c^a = - (d\beta_c^a \beta_c^b \beta_c^a - \beta_c^a (d\beta_c^b)) \beta_c^a
\]
\[
= - (d\beta_c^b) \beta_c^a + \beta_c^a (d\beta_c^b) \beta_c^b = \beta_c^a (d\beta_c^b) + \beta_c^a \beta_c^b (d\beta_c^b) \beta_c^a = \beta_c^a (d\beta_c^b + \beta_c^b (d\beta_c^b) \beta_c^a).
\]
it follows that
\[
A^a = \sum_c c \beta_c^a (d\beta_c^a) = \sum_c \chi_c c \beta_c^a (d\beta_c^a + \beta_c^a (d\beta_c^b) \beta_c^b) = \beta_c^a (d\beta_c^a + \sum_c \chi_c c \beta_c^a (d\beta_c^b) \beta_c^b)
\]
\[
= \beta_c^a (d\beta_c^b + A^a \beta_c^b).
\]
\[
\square
\]

For the tangent bundle, a family \(m \mapsto A_m^a\) of \(\text{gl}(L)\)-valued 1-forms on \(V_a\) such that
\[
A^a = \frac{\partial \sigma_a}{\partial \sigma_a} \left( \frac{\partial \sigma_a}{\partial \sigma_b} + A^b \frac{\partial \sigma_a}{\partial \sigma_b} \right)
\]
on \(V_a \cap V_b\), defines a global tangent connexion \(A\) on \(M = \mathbb{V} / R\), with covariant derivative
\[
(v \cdot d^A X)^a_m = v \cdot d_m X^a + (v \cdot A^a_m) X^a_m.
\]
Here \(v \cdot d_m X^a \in L\) and \(v \cdot A^a_m \in L(L)\).

If \(M\) is a complex manifold, we consider holomorphic vector bundles over \(M\) defined by holomorphic cocycles \(\beta^a_b\).

Theorem 1.4.5. Let \(M\) be a complex manifold, with a metric on \((h^a)\) on \(\mathbb{V} \times \beta E\). Then the family
\[
(\partial h)^a_m := (h^a_m)^{-1} \partial_m h^a
\]
of \((1,0)\)-forms induces a (unique) connexion \(\partial h\) on \(\mathbb{V} \times \beta E\) which satisfies
\[
d_X (\xi | \eta) = (d^X \xi | \eta) + (\xi | d^X \eta)
\]
for all vector fields \(X \in \Gamma_1(M_R)\), and the (‘torsion-free’) condition
\[
\bar{\partial}^h \Phi = 0
\]
for all holomorphic sections \(\Phi \in \Gamma(\mathbb{V} \times \beta E)\). For \(E = \mathbb{C}\), given a 0-metric \((h^a)\) on the line bundle \(\mathbb{V} \times \beta E\), the family
\[
(\partial h)^a_m := \frac{\partial_m h^a}{h^a_m} = \partial_m \log h^a
\]
of \((1,0)\)-forms induces the Chern connexion \(\partial h\) on \(\mathbb{V} \times \beta \mathbb{C}\).

Proof. Since we take the \(\mathbb{C}\)-linear Wirtinger derivative \(\partial_m h^a\), it follows that \((\partial h)^a\) is a \((1,0)\)-form with values in \(\text{gl}(E)\). Since \(\beta^a_b\) is holomorphic in \(m\) we have \(\partial_m \beta^a_b = 0\) and \(\partial_m \beta^b_a = d_m \beta^b_a\). Applying the product rule to \(h^a_m = \beta^b_a(m)^* h^b_m \beta^a_b(m)\) we obtain
\[
\partial_m h^a = \beta^b_a(m)^* \left( (\partial_m h^b) \beta^b_a(m) + h^b_m (\partial_m \beta^b_a) \right) = \beta^b_a(m)^* \left( (\partial_m h^b) \beta^b_a(m) + h^b_m (d_m \beta^b_a) \right).
\]
It follows that
\[(\partial h)_m^a = (h_m^a)^{-1} \partial_m h^a = \left( \beta_a^b(m)(h_m^b)^{-1} \beta_m^a(m)^* \right) \beta_m^b(m)^* \left( (\partial_m h^b)(h_m^b)^{-1} \beta_m^a(m) + h_m^b (d_m \beta_m^a) \right) \]
\[= \beta_a^b(m) \left( (h_m^b)^{-1} (\partial_m h^b) \beta_m^a(m) + d_m \beta_m^a \right) = \beta_m^a(m) \left( (\partial h)^a_m h^b \beta_m^b(m) + d_m \beta_m^a \right).\]
Thus (??) is satisfied. For the second assertion, let \( \Phi = (\Phi^a) \) be a holomorphic section. Then the \( \Phi^a \) are holomorphic and hence \( \partial_m \Phi^a = 0 \). It follows that
\[(v \cdot d_m^0 \Phi)^a = v \cdot d_m \Phi^a + (h_m^a)^{-1} (v \cdot \partial_m h^a) \Phi^a(m) = v \cdot \partial_m \Phi^a + (h_m^a)^{-1} (v \cdot \partial_m h^a) \Phi^a(m)\]
is \( \mathbb{C} \)-linear in \( v \). Therefore the anti-linear part \( (\partial_m^0 \Phi)^a \) vanishes for all \( a \) and hence \( \overline{\partial}^0 h \Phi = 0 \). \( \square \)

For a tangent metric \( (h^a) \) the family
\[(\partial h)_m^a := \frac{\partial_m h^a}{h_m^a}\]
of \((1,0)\)-forms induces the Chern connexion \( \partial h \) on the tangent bundle \( V \times_{\mathbb{C}} L \equiv TM \). A complex manifold \( M \) endowed with a tangent metric is called a **hermitian manifold**. For a cocycle description, endow \( L \) with an inner product \( (\xi|\eta) \).

- **Jordan manifolds**

**Example 1.4.6.** For the holomorphic tangent bundle of \( P^1 \), with metric (??), the general formula (??) yields the connexion 1-form
\[(\partial h)_z^0 = \frac{\partial h^0}{h_z^0} = (1 + z \overline{z})^2 \frac{\partial}{\partial z} (1 + z \overline{z})^{-2} dz = -2(1 + z \overline{z})^2 (1 + z \overline{z})^{-3} \overline{z} dz = \frac{-2\overline{z}}{1 + z \overline{z}} dz\]
of type \( (1,0) \).

### 1.4.2 Homogeneous manifolds

We first construct some vector fields on \( M = G/H \). Consider the left translation action
\[g^L g' := gg'\]
of \( G \) on itself. For \( \gamma \in g \) define a vector field \( \gamma^L \in \Gamma(TG) \) by
\[\gamma^L := (T_e g^L) \gamma = \gamma \cdot (T_e g^L) \in T_g G\]

**Lemma 1.4.7.** The vector field \( \gamma^L \) on \( G \) is left-invariant, i.e. for each \( g \in G \) the left translation \( g^L \) on \( G \) satisfies
\[g^L \gamma^L = \gamma^L.\]

**Proof.** This follows from
\[(g^L \gamma^L)_{gg'} = (T_g \gamma^L)(T_{gg'} g^L) \gamma = T_e (g^L \circ g^L) \gamma = T_e ((gg')^L) \gamma = \gamma^L_{gg'}.\]
\[\square\]
Consider the left translation action $g \mapsto g^\lambda$ of $G$ on $G/H$ given by

$$g^\lambda(g'H) := gg'H.$$ 

Then the canonical projection $\pi : G \to G/H$ satisfies

$$\pi \circ g^\lambda = g^\lambda \circ \pi$$

for all $g \in G$. For $\gamma \in g$ define a vector field $\gamma^\lambda \in \Gamma(G/H)$ by

$$\gamma^\lambda_{g'H} := (Te g^\lambda)(Te \pi) \gamma$$

**Lemma 1.4.8.** The vector field $\gamma^\lambda$ on $G/H$ is left-invariant, i.e. for each $g \in G$ the left translation $g^\lambda$ on $G/H$ satisfies

$$g^\lambda \gamma^\lambda = \gamma^\lambda.$$ 

**Proof.** This follows from

$$(g^\lambda \gamma^\lambda)_{gg'H} = (Te g^\lambda)(Te \pi)(g \gamma) = T_H (g^\lambda \gamma)(Te \pi) \gamma = T_H ((gg')^\lambda)(Te \pi) \gamma = \gamma_{gg'}^\lambda.$$

**Lemma 1.4.9.** For all $\gamma \in g$ we have

$$\pi_* \gamma^L = \gamma^\lambda,$$

i.e.,

$$\gamma^\lambda_{g'H} := (Te g^\lambda)(Te \pi) \gamma = (T_g \pi)(T_e g^L \gamma)$$

**Proof.**

**Lemma 1.4.10.** The left-invariant vector field $\tilde{\gamma}$ satisfies

$$(d_{\gamma^L} f)_g = \partial^0_t f_{g \exp(t \gamma)}$$

**Proof.**

Let $M = G/H$. Then we have a commuting diagram

![Commuting Diagram](https://via.placeholder.com/150)

**Lemma 1.4.11.** For $\eta \in \mathfrak{h}$ we have

$$d_{\eta^L} \Phi = -\eta^\pi \Phi$$
Proof. It follows from (??) and Lemma (??) that

$$(d_{g^L} \dot{\Phi})_g = \partial_t^0 \dot{\Phi}_{g \exp(t\eta)} = \partial_t^0 \exp(t\eta)^{-1} \dot{\Phi}_g = -\eta^\pi \dot{\Phi}_g.$$ \hfill \Box$

Consider a vector space splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

which is $Ad_H$-invariant. Thus $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, but not necessarily $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. For $\gamma \in \mathfrak{g}$ we write $\gamma_h$ and $\gamma_m$ for the projections. Given an $H$-module $(E, \pi)$ we consider the corresponding infinitesimal action

$$\eta^\pi := \partial_t^0 \exp(t\eta)^{-1}$$

for all $\eta \in \mathfrak{h}$.

**Proposition 1.4.12.** The left-invariant connexion $A$ associated with a splitting (??) has the covariant derivative

$$(d_{\gamma_h}^A \Phi)^\pi = d_{\gamma_h} \Phi + \gamma_h^\pi \Phi, \quad (\gamma^\lambda \cdot d_{\lambda}^A \Phi)^\pi = \gamma^L_g \cdot d_{\gamma_h} \Phi + \gamma_h^\pi \Phi$$

for all $\gamma \in \mathfrak{g}$.

**Proof.** For $M = G/H$ every tangent vector in $T_{gH}M$ can be written as

$$\gamma_h^L = (\gamma \cdot T_e \pi) \cdot (T_H g^\lambda)$$

for a uniquely determined $\gamma \in \mathfrak{m}$. Then $\gamma^L_g = \gamma \cdot (T_e g^L)$ belongs to $(T_e g^L)\mathfrak{m} = T_q^A G$, since the connexion is left-invariant, and the projection is

$$(T_H \pi) \gamma^L_g = (T_H \pi)(T_e g^L) \gamma = (T_H g^\lambda)(T_e \pi) \gamma = \gamma^L_H.$$

Therefore $\gamma^L_g = (\gamma^\lambda_h)^A$ is the horizontal lift of $\gamma^\lambda$. Now (??) implies

$$(d_{\gamma_h}^A \Phi)_g = \gamma_h^\lambda \cdot d_{\gamma_h g} \Phi = [g, (\gamma^\lambda_h)^A \cdot d_{\gamma_h} \Phi] = [g, \gamma^L_g \cdot d_{\gamma_h} \Phi] = [g, (d_{\gamma_h} \Phi)_g].$$

Equivalently, we have $(d_{\gamma_h}^A \Phi)^\pi = d_{\gamma_h} \Phi$ for all $\gamma \in \mathfrak{m}$. This implies

$$(d_{\gamma_h}^A \Phi)^\pi = d_{\gamma_h} \Phi = d_{\gamma_h} \Phi - d_{\gamma_h}^L \Phi$$

for all $\gamma \in \mathfrak{g}$, since, by (??), we have $\gamma^\lambda = 0$ on $M$ for all $\gamma \in \mathfrak{h} = \text{Ker} T_e \pi$ and hence both sides of (??) vanish. Applying (??) to $\eta := \gamma_h$, the assertion follows. \hfill \Box

### 1.5 2-Geometry: Curvature

For every $G$-connexion $A \in \Omega^1(P)$ the covariant derivative (??) has a canonical extension

$$d^A : \Omega^j(P \times E) \to \Omega^{j+1}(P \times E)$$

for $j \geq 0$, satisfying a graded Leibniz rule

$$d^A(\partial \wedge \Phi) = d\partial \wedge \Phi + (-1)^j \partial \wedge d^A \Phi$$

for all $\Phi \in \Omega^j(P \times E)$ and $\partial \in \Omega^1(M, K)$. Thus there is a canonical mapping

$$\Omega^1(P) \times \Omega^j(P \times E) \to \Omega^{j+1}(P \times E), \quad (A, \Phi) \mapsto dA \Phi.$$

If $P \times E \to M$ is a holomorphic vector bundle one can also consider the anti-linear part

$$\overline{d}^A : \Omega^{p,q}(P \times E) \to \Omega^{p,q+1}(P \times E).$$

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Proposition 1.5.1. The square
\[ d^A d^A : \Omega^0(P \times_\mathfrak{h} E) \to \Omega^2(P \times_\mathfrak{h} E) \]
can be written as
\[ d^A (d^A \Phi) = (d^A A) \wedge \Phi. \]
for the curvature 2-form \( d^A A \in \Omega^2(P \times_\mathfrak{h} \mathfrak{h}) \). More generally, the square
\[ d^A d^A : \Omega^0(P \times E) \to \Omega^2(P \times E) \]
is given by
\[ d^A d^A (\vartheta \otimes \Phi) = (d^A A) \wedge (\vartheta \otimes \Phi) \]

Proof. Using \( \otimes \) also for multiplication by functions, we have
\[ d^A (d^A (f \otimes \Phi)) = d^A (df \otimes f + f \otimes d^A \Phi) = d(df) \otimes f - df \wedge d^A \Phi + df \wedge d^A \Phi + f \otimes (d^A d^A \Phi) = f \otimes (d^A A) \Phi \]
since \( ddf = 0 \). Thus \( d^A d^A \) commutes with multiplication by functions \( f \) and is therefore a multiplication by a 2-form with values in the bundle \( P \times_\mathfrak{h} \mathfrak{h} \). \( \square \)

For a matrix group the curvature is given by
\[ d^A A = dA + [A \wedge A]. \]
Thus the curvature depends in a non-linear, quadratic manner on \( A \). For abelian groups, we have \( d^A A = dA \).

For a holomorphic vector bundle with metric \( h \) we have the Chern connexion \( \partial h \), with covariant derivative \( d^\partial h \) and curvature \( d^\partial h \partial h \).

1.5.1 Covered manifolds

For a covered manifold \( M \) this looks as follows. For any vector field \( X \in \Gamma(TM) \) we put
\[ (X \cdot d^A \Phi)^a_m := (X_m \cdot d^A \Phi)^a. \]
Then the family \( (X \cdot d^A \Phi)^a \) of smooth maps \( V_a \to E \) is a localized section. The curvature of \( A \) is defined by
\[ d^A_X (d^A_X \Phi) - d^A_{[X,Y]} \Phi = d^A A(X,Y) \cdot \Phi \]

Proposition 1.5.2. The curvature of \( (A^a) \) is given by the family
\[ (u,v) \cdot (d^A A)^a_m := v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot d_m A^a) + [u \cdot A^a_m, v \cdot A^a_m], \]
of \( \mathfrak{gl}(E) \)-valued 2-forms. Here \( [S,T] = ST - TS \) is the commutator in \( \mathfrak{gl}(E) \). For \( E = \mathbb{C} \) the curvature of \( (A^a) \) simplifies to
\[ (u,v) \cdot (d^A A)^a_m := v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot d_m A^a) = (dA^a)_m(u,v). \]

Proof. Let \( X,Y \) be smooth vector fields on \( M \). Then
\[ (Y_m \cdot d^A_m \Phi)^a = Y_m \cdot d_m A^a + (Y_m \cdot A^a_m) \Phi^m \]
and hence
\[ (X_m \cdot d^A_m (Y \cdot d^A \Phi))^a = X_m \cdot d_m (Y_m \cdot d^A_m \Phi)^a + (X_m \cdot A^a_m) (Y_m \cdot d^A_m \Phi)^a \]

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For a line bundle, the commutator part 

\( \text{the last two summands,} \)

For the commutator we obtain, using symmetry of the second derivative
\( d \) In other words, the exterior differential satisfies
\( \text{Consider first the wedge product. Since} \)
\( \text{Proof.} \) Therefore \( (X,Y) = ([X,Y]_m \cdot d_m h^a) \)
\( \text{it follows that} \)
\( \text{For a line bundle, the commutator part} \ [u \cdot A^a_m, v \cdot A^a_m] \text{vanishes.} \)

**Proposition 1.5.3.** For a holomorphic metric vector bundle \( V \times E \) the Chern connexion \( ((\partial h)^a) \) satisfies
\( \partial (\partial h)^a = (\partial h)^a \wedge (\partial h)^a \)
and
\( \overline{\partial}(\partial h)^a = (\overline{\partial}\partial h)^a. \)

In other words, the exterior differential \( d^A = \partial^A + \overline{\partial}^A \) has the \((2,0)-part\) \((\partial h)^a \wedge (\partial h)^a \) and the \((1,1)-part\) is given by the curvature \((d^{\partial h}(\partial h)^a)\).

**Proof.** Consider first the wedge product. Since
\( (Y \cdot (\partial h)^a)_m = (h^a_m)^{-1}(Y_m \cdot \partial_m h^a) \)
we have
\( (\partial h)^a_m \wedge (\partial h)^a_m = [X_m \cdot (\partial h)^a_m, Y_m \cdot (\partial h)^a_m] = [(h^a_m)^{-1}(X_m \cdot \partial_m h^a), (h^a_m)^{-1}(Y_m \cdot \partial_m h^a)] \)
\( = (h^a_m)^{-1}(X_m \cdot \partial_m h^a)(h^a_m)^{-1}(Y_m \cdot \partial_m h^a) - (h^a_m)^{-1}(Y_m \cdot \partial_m h^a)(h^a_m)^{-1}(X_m \cdot \partial_m h^a). \)
and hence
\( h^a_m ((\partial h) \wedge (\partial h))^a_m = (X_m \cdot \partial_m h^a)(h^a_m)^{-1}(Y_m \cdot \partial_m h^a) - (Y_m \cdot \partial_m h^a)(h^a_m)^{-1}(X_m \cdot \partial_m h^a). \)
Therefore \((\partial h)^a \wedge (\partial h)^a\) is a differential form of type \((2,0)\) since both \(X\) and \(Y\) involve holomorphic Wirtinger derivatives. Consider now the exterior differential
\( (X,Y) d\Theta = d_X (Y \cdot (\partial h)) - d_Y (X \cdot (\partial h)) - [X,Y] \cdot (\partial h) \)
for vector fields \(X, Y\). The product and quotient rules imply
\( d_X (Y \cdot (\partial h)^a)_m = (h^a_m)^{-1}(d_X (Y_m \cdot \partial_m h^a)) - (h^a_m)^{-1}(d_X h^a)(h^a_m)^{-1}(Y_m \cdot \partial_m h^a). \)
Therefore
\( h^a_m d_X (Y \cdot (\partial h)^a)_m = d_X (Y_m \cdot \partial_m h^a) - (d_X h^a)(h^a_m)^{-1}(Y_m \cdot \partial_m h^a) = (X_m d_m Y)(\partial_m h^a) + Y_m (X_m d_m \partial_m h^a) - (d_X h^a)(h^a_m)^{-1}(Y_m \cdot \partial_m h^a). \)
It follows that
\( h^a_m ((X,Y) d\Theta) = (X_m d_m Y - Y_m d_m X) \cdot \partial_m h^a + Y_m (X_m d_m \partial_m h^a) - X_m (Y_m d_m \partial_m h^a) \)
\[-(d_X h^a)(h_m^a)^{-1}(Y_m \partial_m h^a) + (d_Y h^a)(h_m^a)^{-1}(X_m \partial_m h^a) - [X, Y]_m \partial_m h^a = Y_m(X_m \partial_m h^a) - X_m(Y_m \partial_m h^a) - (X_m \partial_m h^a)(h_m^a)^{-1}(Y_m \partial_m h^a) + (Y_m \partial_m h^a)(h_m^a)^{-1}(X_m \partial_m h^a),\]

since the first and last terms cancel. Subtracting (??) we obtain the curvature
\[ h_m^a ((X,Y)\Omega) = h_m^a ((X,Y)d\theta - (\partial h)^a_m \wedge (\partial h)^a_m) \]
\[ = Y_m(X_m \partial_m h^a) - X_m(Y_m \partial_m h^a) - (X_m \partial_m h^a)(h_m^a)^{-1}(Y_m \partial_m h^a) + (Y_m \partial_m h^a)(h_m^a)^{-1}(X_m \partial_m h^a). \]

Finally, the second holomorphic derivatives \( Y_m(X_m \partial_m \partial_m h^a) = X_m(Y_m \partial_m \partial_m h^a) \) vanish by Schwarz' theorem. Therefore
\[ h_m^a ((X,Y)\Omega^a) = Y_m(X_m \partial_m h^a) - X_m(Y_m \partial_m h^a) - (X_m \partial_m h^a)(h_m^a)^{-1}(Y_m \partial_m h^a) + (Y_m \partial_m h^a)(h_m^a)^{-1}(X_m \partial_m h^a). \]

It follows that \( \Omega \) is a differential form of type (1,1), involving only mixed derivatives. In summary, \( d(\partial h) \) has the (1,1)-part \( d^{0b} \partial h \) and the (2,0)-part \( (\partial h) \wedge (\partial h) \). Since \( (\partial h) \) is of type (1,0), \( d(\partial h)^a \) has no (0,2)-part, and the assertion follows.

**Proposition 1.5.4.** For hermitian holomorphic line bundles the curvature \((1,1)\)-form \((d^{0b} \partial h)^a \) is closed.

**Proof.** The curvature is given by
\[ (d^{0b} \partial h)^a = \overline{\partial}(\partial h)^a = \overline{\partial} \partial \log h^a. \]

Since \( \overline{\partial} = \partial^2 = \partial \overline{\partial} + \overline{\partial} \partial = 0 \), it follows that
\[ d(\overline{\partial} \partial \log h^a) = (\overline{\partial} + \partial)(\overline{\partial} \partial \log h^a) = \overline{\partial} \partial(\partial \log h^a) + \partial \overline{\partial}(\partial \log h^a) = -\overline{\partial} \partial \partial(\partial \log h^a) = 0. \]

1.5.2 Homogeneous manifolds

Consider the invariant connexions on homogeneous vector bundles over \( G/H \) given by a splitting (??) of the Lie algebra \( \mathfrak{g} \).

**Proposition 1.5.5.** For \( \gamma, \delta \in \mathfrak{m} \) the curvature is given by the 'multiplication operator'
\[ (d^A A(\gamma, \delta) \Phi) = -[\gamma, \delta]_H^a \Phi. \]

**Proof.** For \( \gamma, \delta \in \mathfrak{m} \) we have vanishing \( \mathfrak{h} \)-projection. Hence (??) implies
\[ (d^A(d^A \Phi) - d^A(d^A \Phi) - d^A_{(\gamma, \delta)} \Phi)^{\sim} = d_\gamma d_\delta \Phi - d_\delta d_\gamma \Phi - d_{[\gamma, \delta]} \Phi = -[\gamma, \delta]_H^a \Phi = -[\gamma, \delta]_H^a \Phi. \]

Here we used that \( \pi_* \gamma = \gamma \) and \( \pi_* \delta = \delta \) implies \( \pi_* [\gamma, \delta] = [\gamma, \delta] \), so that \([\gamma, \delta] \) is a horizontal lift of \([\gamma, \delta] \).
Chapter 2

Classical Phase Spaces

2.1 Symplectic Manifolds and Kähler Manifolds

A 2-form \( \omega \in \Omega^2(M) \) on a smooth manifold \( M \) is called symplectic if \( d\omega = 0 \) and \( \omega \) is non-degenerate, i.e. for each \( m \in M \) the canonical map

\[
\omega_m : T_m M \rightarrow T^*_m M,
\]

arising as a special case of (??), is a linear isomorphism. Alternatively (in the finite-dimensional case), \( \omega_m(u, v) = 0 \) for all \( v \in T_m M \) implies \( u = 0 \). A symplectic manifold is a manifold \( M \) endowed with a symplectic 2-form \( \omega \). Then \( \dim M = 2n \) is even. The Liouville measure is defined by the \( 2n \)-form

\[
\frac{1}{n!} \omega^n.
\]

Proposition 2.1.1. Let \( Q \) be a real manifold (configuration space). Then the cotangent bundle

\[
M = T^* Q
\]

is a symplectic manifold (phase space), with symplectic form

\[
\omega_{x, \xi}(\dot{x}, \dot{\xi}, \dot{y}, \dot{\eta}) = \dot{x} \dot{\eta} - \dot{y} \dot{\xi}
\]

for all \( \dot{x}, \dot{y} \in T_x Q \) and \( \dot{\xi}, \dot{\eta} \in T^*_x Q \)

Proof. Let \( \pi : T^* Q \rightarrow Q \) denote the canonical projection. Then \( T_{x, \xi} \pi : T_{x, \xi}(T^* Q) \rightarrow T_x Q \). Define a global 1-form \( \vartheta \in \Omega^1(T^* Q) \) by

\[
\vartheta_{x, \xi} v := \xi((T_{x, \xi} \pi)v).
\]

for all \( v \in T_{x, \xi}(T^* Q) \). Thus we apply \( \xi \in T^*_x Q \) to \( (T_{x, \xi} \pi)v \in T_x Q \). Then

\[
\omega := d\vartheta
\]

is closed, since \( d^2 = 0 \), and non-degenerate.

The symplectic manifold \( T^* Q \) is given in its real polarization. We will work instead with complex polarizations. This is crucial for harmonic analysis but also quantum field theory.

Lemma 2.1.2. Let \((M, J, h)\) be a hermitian complex manifold. Then

\[
\omega_m(u + \overline{u}, v + \overline{v}) := \frac{h_m(u, v) - h_m(v, u)}{2i}
\]

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is a (not necessarily closed) non-degenerate 2-form $\omega \in \Omega^2(M, \mathbb{R})$, satisfying
\[
\omega_m(J_mX, J_mY) = \omega_m(X, Y)
\]
for all $X, Y \in T^R_m M$.

**Proof.** Since $J(u + \pi) = iu + i\pi$ we have
\[
\omega_m(J_m(u + \pi), J_m(v + \pi)) = \omega_m(iu + i\pi, iv + i\pi) = \frac{h_m(iu + iv, iv + i\pi)}{2i}
\]
\[
= \frac{h_m(u, v) - h_m(v, u)}{2} = \omega_m(u + \pi, v + \pi)
\]
\[\square\]

Define a Riemannian metric $g$ on $M$ by
\[
g_m(X, Y) := \omega_m(J_mX, Y).
\]

Then
\[
g_m(u + \pi, v + \pi) = \omega_m(J_m(u + \pi), v + \pi) = \omega_m(iu + i\pi, v + \pi) = \frac{h_m(iu, v) + h_m(v, iu)}{2i} = \frac{h_m(u, v) + h_m(v, u)}{2}.
\]

Then
\[
g_m(u + \pi, u + \pi) = h_m(u, u).
\]

If $h$ is a $0$-metric (positive definite), it follows that $g_m(X, X) > 0$ for all $0 \neq X \in T^R_m M$. The hermitian metric $h$ can be recovered from $\omega$ and $g$ via
\[
h_m(u, v) = g_m(u + \pi, v + \pi) + i\omega_m(u + \pi, v + \pi).
\]

Thus on a symplectic manifold $(M, \omega)$ the formula (1) yields a 1-1 correspondence between almost complex structures $J$ and Riemannian (pseudo)-metrics $g$. An almost complex structure $J$ on $(M, \omega)$ is called **compatible** if (1) is a positive-definite Riemannian metric. By Proposition 1 every symplectic manifold has a compatible almost complex structure, which however may not be integrable. This leads to the important

**Definition 2.1.3.** The following equivalent definitions define a 0**Kähler manifold**:

- A symplectic manifold $(M, \omega)$ with a compatible almost complex structure $J$ which is **integrable** (vanishing Nijenhuis tensor) and hence, by the Newlander-Nirenberg theorem, is a complex structure.

- A 0**hermitian manifold** $M$ such that the resulting 2-form $\omega$ is **closed**

- A 0**-hermitian manifold** such that the tangent Chern connexion $\partial h$ and the Levi-Civita connexion $\partial g$ coincide (after proper identification)

**Example 2.1.4.** For $Q = \mathbb{R}^n$ we take $M = \mathbb{C}^n$ with it standard complex structure $J$. The hermitian metric
\[
h_z(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 1
\]
introduced in (1) leads to
\[
\omega_z(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = \omega_z(i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} + \frac{\partial}{\partial z}) = \frac{1}{2i}h_z(i\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) - h_z\left(\frac{\partial}{\partial z}, i\frac{\partial}{\partial z}\right)) = \frac{1}{2i}(2i) = 1.
\]

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It follows that $h$ induces the symplectic form
$$\omega = dp_j \wedge dq^j,$$
when we identify $q = x$ and $p = y$. In differential form language we have
$$\omega = dy \wedge dx = \frac{dz - d\bar{z}}{2i} \wedge \frac{dz \wedge d\bar{z}}{2} = \frac{1}{4i} (dz \wedge d\bar{z} - d\bar{z} \wedge dz) = \frac{1}{2i} dz \wedge d\bar{z}.$$
Thus $h_z$ corresponds to the $(1,1)$-form $dz \wedge d\bar{z}$.

- **Jordan manifolds**

The Bergman metric $h_m(u,v) = \text{tr} D(B^{-1}_z u, v)$ is positive definite and we obtain a symplectic form
$$\omega_m(u + \bar{u}, v + \bar{v}) := \frac{h_m(u,v) - h_m(v,u)}{2i}$$
which is closed, as will be shown later. Hence the Jordan manifolds
$$Z \subset \hat{Z} \subset \check{Z}$$
are Kähler manifolds.

- **Restricted Grassmannian**

**Proposition 2.1.5.** On the space $S$ of symmetries the imaginary symplectic form
$$\omega = \text{tr} s ds ds = s_i^j ds_k^i \wedge ds_k i$$
is closed.

- **Loop groups**

For the (parallelizable) loop space $\Gamma(S,G)$ the tangent space $\Gamma(S,g)$ has a class of hermitian Sobolev type metrics
$$(u|v)^k = \int_S ds \; ((\Delta^k u)_s|v_s).$$
For $k = 0$ this gives the basic $L^2$-metric
$$(u|v)^0 = \int_S ds \; (u_s|v_s).$$
For $k = 1/2$ one obtains a Kähler metric
$$(u|v)^{1/2} = \int_S ds \; ((|D|u)_s|v_s)$$
with Kähler form
$$\omega_k(u,v) = \frac{1}{2\pi} \int_S ds \; (u'_s|v_s).$$
There is also a 1-metric
$$(u|v)^1 = \int_S ds \; ((\Delta u)_s|v_s) = \int_S ds \; (u'_s|v'_s)$$
• Conformal blocks

**Proposition 2.1.6.** Let \( S \) be a compact oriented surface. Let \( G \) be a compact Lie group with Lie algebra \( \mathfrak{g} \). Then the affine space \( \Omega^1(S, G) \) of all connexions \( A \) on the trivial \( G \)-bundle \( S \times G \) carries the symplectic form

\[
\omega_A(\dot{A}_1, \dot{A}_2) = \int_S \text{tr}[\dot{A}_1 \wedge \dot{A}_2]
\]

where \( \dot{A} \in T_A(\Omega^1(S, G)) = \Omega^1(S, \mathfrak{g}) \).

**Proof.** We write

\[
\Lambda = \lambda_i \otimes \gamma^i
\]

for scalar 1-forms \( \lambda_i \in \Omega^1(S) \) and a basis \( \gamma^i \in \mathfrak{g} \). Choose a \( U \)-invariant inner product \( \text{tr}[\gamma, \gamma'] \) on \( \mathfrak{g} \), for example the negative Killing form in the semi-simple case. Then the scalar 2-form

\[
\text{tr}[\Lambda \wedge \Lambda'] := \lambda_i \wedge \lambda'_j \text{tr}[\gamma^i, \gamma^j] \in \Omega^2(S)
\]

is independent of the choice of basis \( \gamma^i \) and can be integrated over \( S \). For \( U = SU_n(\mathbb{C}) \) we use \( -\text{tr}\gamma\gamma' \). Is this of complex type?  

---

**2.1.1 Homogeneous manifolds**

Let \( G \) be a Lie group with a (right) action

\[
M \times G \to M, \quad (m, g) \mapsto mg
\]

on a manifold \( M \). The corresponding infinitesimal action

\[
M \times \mathfrak{g} \to TM, \quad (m, u) \mapsto u_\sim m
\]

of the Lie algebra \( \mathfrak{g} \) is defined by

\[
u_m = \partial^0_t (m \cdot \exp(tu)).
\]

Here \( \exp : \mathfrak{g} \to G \) is the exponential map. For any \( m \in M \) the stabilizer subgroup

\[G_m := \{g \in G : m \cdot g = m\}\]

has the Lie algebra

\[\mathfrak{g}_m := \{u \in \mathfrak{g} : \nu_m = 0\} \}

The quotient manifold

\[G^m := G_m \backslash G
\]

has the tangent space

\[T_m(G^m) = \{\nu_m : u \in \mathfrak{g}\} = \mathfrak{g}_m \backslash \mathfrak{g}
\]

**Lemma 2.1.7.**

\[\nu_m(T_m g) = \underline{A_d^{-1}u}_{m \cdot g}
\]

**Proof.**

\[
u_m(T_m g) = \partial^0_t (m \cdot \exp(tu))(T_m g) = \partial^0_t \left( (m \cdot \exp(tu)) \cdot g \right) = \partial^0_t \left( m \cdot (\exp(tu) \cdot g) \right) = \partial^0_t \left( m \cdot g \cdot (g^{-1} \exp(tu) \cdot g) \right) = \partial^0_t \left( m \cdot g \cdot \exp(t \cdot Ad_{g^{-1}} u) \right) = \underline{Ad_{g^{-1}}u}_{m \cdot g}
\]

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Put $\text{Int}_g(g') := gg'g^{-1}$. Then
\[
Ad_g := T_e(\text{Int}_g)
\]
defines an action of $G$ on the Lie algebra $\mathfrak{g} = T_e G$. The \textbf{co-adjoint action} $\mathfrak{g}^* \times G \to \mathfrak{g}^*$ on the linear dual space $M := \mathfrak{g}^*$ is defined by
\[
(m \circ Ad_g)u := m(Ad_gu).
\]
for all $m \in \mathfrak{g}^*$ and $u \in \mathfrak{g}$. Put
\[
ad_u v = [u, v].
\]

\textbf{Lemma 2.1.8.} For any $m \in \mathfrak{g}^*$ the stabilizer subgroup
\[
G_m := \{ g \in G : m \circ Ad_g = m \}
\]
has the Lie algebra
\[
\mathfrak{g}_m := \{ u \in \mathfrak{g} : m \circ ad_u = 0 \}.
\]

\textbf{Proof.} Let $u \in \mathfrak{g}$ and let $g_t \in G$ be a smooth curve with $g_0 = e$ and $\partial_t g_t = u$. Then
\[
\partial_t (m \circ Ad_{g_t})(v) = \partial_t m((Ad_{g_t})v) = m(\partial_t (Ad_{g_t})v) = m[u, v] = (m \circ ad_u)v
\]
Since $v \in \mathfrak{g}$ is arbitrary, it follows that
\[
\partial_t m \circ Ad_{g_t} = m \circ ad_u.
\]
Regarding $m \circ Ad_{g_t}$ as a curve in the orbit $G^m = G_m \setminus G$ it follows that
\[
T_m(G^m) = \{ m \circ ad_u : u \in \mathfrak{g} \}.
\]

For $u \in \mathfrak{g}$ let
\[
u_m := u + \mathfrak{g}_m \in T_u G^m
\]
denote the equivalence class. For each $\xi \in \mathfrak{g}^*$ we have the action
\[
(\xi \circ (Adg)|v) := (\xi)(Adg)v
\]
for all $v \in \mathfrak{g}$. Now we define
\[
\omega_\xi(\xi \circ (adu)|\xi \circ (adv)) := [\xi, v]
\]

\textbf{Theorem 2.1.9.} For $m \in \mathfrak{g}^*$ define a bilinear form $\omega_m$ on $T_m G^m$ by
\[
\omega_m(\nu_m, \nu_m) := m[u, v]
\]
for all $u, v \in \mathfrak{g}$. This is well-defined and yields a $G$-invariant symplectic form on the coadjoint orbit $G^m$.

\textbf{Proof.} Suppose $u, u' \in \mathfrak{g}$ and $v, v' \in \mathfrak{g}$ satisfy $u - u' \in \mathfrak{g}^m$ and $v - v' \in \mathfrak{g}^m$. Then $m \circ ad(u - u') = 0 = m \circ ad(v - v')$ and
\[
m[u, v] - m[u', v'] = m[u - u', v] + m[u', v - v'] = (m \circ ad_{u - u'})v - (m \circ ad_{v - v'})u' = 0.
\]
This shows that $m[u, v]$ depends only on the equivalence class $\nu_m$ of $\nu_m$. Hence (??) is well-defined.

The tangent space $T_m G_m$ consists of all linear functionals $\nu_m = m \circ ad_u$, for $u \in \mathfrak{g}$. Suppose that $\nu_m \in T_m G_m$ belongs to the radical of $\omega_m$. Then
\[
(m \circ ad)(u)(v) = m(ad_u)v = m[u, v] = \omega_m(\nu_m, \nu_m) = 0
\]

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for all \( v \in \mathfrak{g} \). Thus \( m \circ \text{ad}_u = 0 \) as a tangent vector to \( G_m \). Therefore \( \omega \) is non-degenerate.

To show that \( \omega \) is \( G \)-invariant, we apply (15) and obtain

\[
(y_m, \omega_m)(g^* \omega)_m = (y_m(T_m g), \omega_m(T_m g)) \omega_m = (\text{Ad}^{-1}_g u, \text{Ad}^{-1}_g v) \omega_m = (m \circ \text{Ad}_g^{-1} u, \text{Ad}_g^{-1} v) \omega_m = m[u, v] = (y_m, \omega_m)(g^* \omega)_m.
\]

Thus we have

\[
g^* \omega = \omega
\]

for all \( g \in G \).

Every \( u \in \mathfrak{g} \) induces a vector field \( y \) on \( G_m \) by

\[
m \cdot \exp(t y) = m \circ \text{Ad}_{\exp(t u)}.
\]

For fixed \( v \in \mathfrak{g} \) consider the smooth function

\[
f^v_m := |v|.
\]

Then

\[
(y_m f^v)_m = \partial^0 f^v m \circ \text{Ad}_{\exp(t u)} = \partial^0 f^v_{m \circ \text{Ad}_{\exp(t u)}} = m(\partial^0 f^v_{\text{Ad}_{\exp(t u)}} v) = m[u, v].
\]

Since

\[
\omega_m(y_m \cdot, \omega_m) = f^v_m
\]

it follows that \( (y_m \omega(y, w))_m = m[u[v, w]] \) and the Jacobi identity implies

\[
y_m \omega(y, w) + y_m \omega(y, u) + y_m \omega(u, v) = 0.
\]

On the other hand, we have

\[
[y, y]_m = \omega_m([u, v]_m, \omega_m) = m[[u, v]w].
\]

Using the Jacobi identity again, we obtain

\[
\omega_m([y, y]_m, \omega_m) + \omega_m([y, y]_m, \omega_m) + \omega_m([w, y]_m, \omega_m) = m([[u, v]w] + [v, w]u + [w, u]v) = 0.
\]

In summary, \( d\omega(y, y, y) = 0 \). Thus \( d\omega = 0 \).

For \( u_0, u_1, u_2 \in \mathfrak{g} \) we consider the right invariant vector fields

\[
y_g := u \cdot T_e(R_g)
\]

acting on \( G \) and also on \( G^\xi = G_\xi \backslash G \). Consider the function

\[
f(m) := m[u_1, u_2]
\]

on the orbit. Then

\[
f(o \cdot g^0_u) = (o \cdot g^0_u)[u_1, u_2] = o[g^0_u \cdot [u_1, u_2]]
\]

and therefore

\[
(y_0 \cdot f)(o) = \partial^0 f(y_0 \cdot g^0_u) = o[\partial^0 g^0_u \cdot [u_1, u_2]] = o[[u_0[u_1, u_2]].
\]

Thus the first three terms sum up to zero by the Jacobi identity. For the second type we have

\[
\omega_m([y, y]_m, \omega_m) = \omega_m([u, v]_m, \omega_m) = m[[u, v], w].
\]

Thus the last three terms sum up to zero by the Jacobi identity.
**Jordan manifolds**

- projective space
- Grassmannian

**Proposition 2.1.10.** Let $G$ be a real semi-simple Lie group of hermitian type, with maximal compact subgroup $K$. Then the 'symmetric domain' $\hat{Z} = G/K$ is a coadjoint orbit, whose (Kostant-Kirillov)-symplectic structure agrees with $\omega$. Moreover, the compact dual space (conformal hull)

$$\hat{Z} = G^C/K^C \cdot Z$$

is a compact Kähler manifold, and $\omega$ we have

$$(\hat{Z}, \omega) = (G^C/G^C_T, \text{Im}h)$$

**Proof.** Define $m : \hat{g} \rightarrow i\mathbb{R}$ by

$$m(\gamma) = (iz \frac{\partial}{\partial z} |_{\gamma_0})$$

**Proposition 2.1.11.** Let $\hat{G}$ be a simply-connected compact Lie group, with maximal torus $\hat{T}$. Then the full flag manifold $\hat{G}/\hat{T} = \hat{T} = \hat{G}_m$ is a coadjoint orbit for the linear functional $m : \hat{g} \rightarrow i\mathbb{R}$ given by

$$mY := \rho Y_0.$$ 

Here $Y \mapsto Y_0$ is the projection onto $\hat{t}$.

**Proof.** With respect to the root decomposition

$$g = t \oplus \sum_{\beta \in \Delta} \mathfrak{g}_\beta$$

we write elements in $g$ as

$$Y = Y_0 + \sum_{\beta \in \Delta} Y_\beta.$$ 

Define a linear form $m : g \rightarrow \mathbb{C}$ by

$$mY := \rho Y_0.$$ 

Let $X = X_0 + \sum_{\alpha > 0} (X_\alpha - X^*_\alpha) \in \hat{g}$ satisfy $m \circ ad_X = 0$. Let $\beta > 0$ and $Y \in \mathfrak{g}_{-\beta}$ be arbitrary. Then

$$[X, Y] = [X_0, Y] + \sum_{\alpha > 0} [X_\alpha - X^*_\alpha, Y] = -(\beta X_0)Y + \sum_{\alpha > 0} ([X_\alpha, Y] - [X^*_\alpha, Y])$$

has the $t$-projection

$$[X, Y]_0 = [X_0, Y] = c \cdot H_\beta.$$ 

Since $\beta > 0$ we have $\rho H_\beta > 0$. Therefore $0 = (m \circ ad_X)Y = \rho [X, Y]_0 = c \cdot \rho H_\beta$ implies $c = 0$. Hence $[X_\beta, Y] = 0$ for all $Y \in \mathfrak{g}_{-\beta}$, showing that $X_\beta = 0$ for $\beta > 0$. Thus $X = X_0 \in t$. 

**Lemma 2.1.12.** For each $w \in W$

$$G^w \cdot T^C \cdot G^w \subset G^C$$

is open.
Moreover
\[ G/T = G^C/G^C_+ \]
is a compact Kähler manifold, and we have
\[ (G/T, \omega) = (G^C/G^C_+, \text{Im} h) \]

*Peirce manifolds as coadjoint orbits

- **Restricted Grassmannian**
- **Loop groups**

Let \( G \) be a simply-connected and simply laced (ADE) Lie group. Put \( S := S^1 \) and let
\[ L = C^\infty(S, G) \]
with Lie algebra
\[ \Lambda := C^\infty(S, \mathfrak{g}). \]
Then the (smooth) dual is
\[ \Lambda^+ := C^\infty(S, \mathfrak{g}^*) \]
under the pairing
\[ (m|\gamma) := \int_S ds \, m_s \gamma_s \]
for all \( m \in \Lambda, m \in \mathfrak{g}^* \). The coadjoint action is

Its orbit of 0 is the loop space
\[ \Omega(S) = \{ m \in C^\infty(S, \mathfrak{g}^*) : m_0 = m_{2\pi} \} \]

It carries the symplectic form
\[ \omega(\xi, \eta) := \frac{1}{2\pi} \int_0^{2\pi} ds (\xi'(s), \eta(s)) \]

- **Conformal blocks**

**Proposition 2.1.13.** For the affine symplectic space \( \Omega^1(S, G) \) of \( G \)-connexions on a compact oriented surface \( S \) we consider the group
\[ \Omega^0(S, G) \]
acting by gauge transformations
\[ g \cdot A := gAg^{-1} + g^{-1}dg. \]
This action preserves the symplectic structure (??).

Thus \( \Omega^1(S, G) \) becomes a \( \Omega^0(S, G) \)-equivariant symplectic manifold. The Lie algebra of \( \Omega^0(S, G) \) is identified with \( \Omega^0(S, \mathfrak{g}) \) under the pointwise Lie bracket. Define a pairing \( \Omega^0(S, \mathfrak{g}) \otimes \Omega^2(S, \mathfrak{g}) \to \mathbb{R} \) by
\[ (\gamma, \Theta) \mapsto \int_\Sigma \text{tr}[\gamma \cdot \Theta]. \]
Here we write
\[ \Theta = \vartheta \otimes \eta \]
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for some 2-form $\vartheta \in \Omega^2(S, \mathbb{R})$ and $\eta \in \mathfrak{g}$. Then the $\mathfrak{g}$-valued 2-form

$$[\Theta \cdot \gamma] = \vartheta [\eta, \gamma]$$

gives rise to a scalar 2-form

$$\text{tr} [\Theta \cdot \gamma] = \vartheta \text{tr}[\eta, \gamma]$$

which can be integrated over $S$. Via this pairing we identify $\Omega^2(S, \mathfrak{g})$ with a subspace of the dual space $\Omega^0(S, \mathfrak{g})^*$. The full continuous dual should be of distribution type.

### 2.2 Hamiltonian vector fields, Poisson bracket

$C^\infty(M, \mathbb{R})$

Hamiltonian vector fields: For any function $f \in C^\infty(M, \mathbb{R})$ define a vector field $\tilde{f}$ on $M$ by

$$\omega_m(\tilde{f}_m, Y_m) := d_m(f) Y_m$$

for all $Y \in \Gamma(M, T^*M)$. Then the Poisson bracket is defined by

$$\{\tilde{f}_1, \tilde{f}_2\} = [\tilde{f}_1, \tilde{f}_2]$$

We say that $f_1, f_2$ are in involution if $\{f_1, f_2\} = 0$. Completely integrable classical systems $f_1, \ldots, f_n$ in pairwise involution. classical dynamics: Geodesic flow on $T^*Q$ multi-flow: $A$-action, $G = KAN$ Iwasawa decomposition

Prequantization $f \mapsto f + i\nabla f$ quantum dynamics: $e^{i\Delta}$ on $L^2(Q)$, quantization of geodesic flow multi-dynamics: Berezin transform for real Jordan manifolds

### 2.3 Moment Map and Classical Reduction

Coadjoint orbits, moment map and symplectic quotient

$$T^*Q \times \text{Diff}(X) \to T^*Q, \quad \sigma \mapsto T^*\sigma$$

is a symplectic action with moment map

$$\mu : T^*X \to \Gamma_1(X)^+$$

$$\mu_x, \xi v = \xi v_x$$

for all $v \in \Gamma_1(X)$

A symplectic manifold $(M, \omega)$ endowed with a smooth (right) action $M \times G \to M, g \mapsto R_g$ preserving $\omega$

$$R_g^*\omega = \omega$$

is called a $G$-equivariant symplectic manifold. Let $M$ be a symplectic manifold endowed with a symplectic $G$-action. The associated infinitesimal action of the Lie algebra $\mathfrak{g}$ defines a tangent vector

$$\gamma_m := \partial^0_t (g_t \cdot m) \in T_m M$$

for every $m \in M$. Here $g_t \in G$ is a smooth curve with $g_0 = I$ and $\partial^0_t g_t = \gamma$. 39
Definition 2.3.1. A smooth map
\[ \mu : M \to \mathfrak{g}^* \]
is called a moment map if for each \( v \in \mathfrak{g} \) the smooth function \( \mu^v : M \to \mathbb{R} \), defined by
\[ \mu^v(m) := \mu(m)v \]
for the standard pairing \( \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \), has the differential
\[ d_m(\mu^v) = \omega_{m v_m} \]
for all \( m \in M \). Here \( \omega_{m v_m} \in T^1_m M \) since \( v_m \in T_m M \) and \( \omega_m \in T^2_m M \). Thus
\[ t \cdot d_m\mu^v = \omega_m(t, v_m) \]
for all \( u \in T_m M \).

Formally, we have \( d\mu = \omega \). This 'explains' that a moment map is unique up to a constant if \( M \) is connected

Theorem 2.3.2. Let \( \mu : M \to \mathfrak{g}^* \) be a Hamiltonian \( G \)-action. Suppose that \( G \) acts freely and properly on \( \mu^{-1}(0) \). Then
\[ M//G := \mu^{-1}(0)/G \]
is a smooth manifold of dimension \( \dim M - 2 \dim G \), which carries a unique symplectic form \( \omega_0 \in \Omega^2(M//G) \) satisfying
\[ p^*\omega = t^*\omega, \]
where \( p : \mu^{-1}(0) \to M \) is the inclusion map and \( p : \mu^{-1}(0) \to M//G \) is the canonical projection.

Proposition 2.3.3. Suppose that \( \mu \) is a moment map for the symplectic \( G \)-action on \( M \). Then, for each \( \gamma \in \mathfrak{g} \) the smooth function \( \mu^\gamma \) has the Hamiltonian vector field \( \gamma_M \) acting on \( M \). (One says that the \( G \)-action is hamiltonian)

***Symplectic quotient

• Jordan manifolds

Example 2.3.4. The torus action \( \mathbb{C}^n \times \mathbb{T}^n \to \mathbb{C}^n \) is a hamiltonian action. We have the Lie algebra \( \mathfrak{t} = i\mathbb{R}^n \) has the dual space
\[ \mathfrak{t}^+ = n\mathbb{R} \]
and the moment map \( \mu : \mathbb{C}^n \to n\mathbb{R} \) has the form
\[ \mu(m) = (|z|^2, \ldots, |z|^2) \]

• Restricted Grassmannian

Example 2.3.5. \( Gr_{res}(H) = GL_{res}(H)/B_{res} \) has the symplectic structure
\[ \omega = \frac{i}{4} \text{tr} \Phi \ d\Phi \ d\Phi \]
where \( \Phi^2 = 1 \). Hence
\[ \Phi \ d\Phi = -d\Phi \ F \]
and therefore
\[ d\omega = d\Phi^3 = \text{tr}\Phi^2(d\Phi^3) = -\text{tr}(d\Phi)^3\Phi = -\text{tr}(d\Phi)^2\Phi^2 = -d\omega \]
so \( \omega \) is closed.

The moment map is
\[ \mu(\Phi)u = -\text{tr}(\Phi u) \]
where \( u \in g(H) \) satisfies \( u^* = e\). This follows from the computation
\[ 2\text{tr}\Phi[u,\Phi]d\Phi = -d\text{tr}u\Phi. \]

**Example 2.3.6.** Let \( Q \) be a Riemannian manifold. Then \( T^*Q \) is a symplectic quotient. In particular, for \( Q = \mathbb{R}^n \), it follows that \( T^*\mathbb{R}^n \) is a symplectic quotient.

**Remark 2.3.7.** Every classical physical system is a symplectic quotient.

- **Conformal blocks**

**Theorem 2.3.8.** The action of \( \Gamma^0(S,G) \) on \( \Gamma^1(S,G) \) by gauge transformations has a moment map
\[ \Gamma^1(S,G) \to \Gamma^2(S,g) \subset \Gamma^0(S,g)^* \]
given by the curvature
\[ \mu_A = \partial\Theta = d\Theta + [\Theta \wedge \Theta] \in \Gamma^2(S,g) \]

**Proof.** We have to show that for each \( \gamma \in \Gamma^0(S,g) \) the smooth function
\[ (\mu_\gamma)(A) := (\mu(A)|\gamma) = (\partial\Theta|\gamma) = \int_S \text{tr}[\gamma \cdot \partial\Theta] \]
of the argument \( A \in \Gamma^1(S,G) \) has the differential
\[ d_A(\mu_\gamma) = \omega_A(\gamma_A, \hat{A}) = \int_S \text{tr}[\gamma_A \wedge \hat{A}] \]
where \( \hat{A} \) and the value \( \gamma_A \) of the vector field at \( A \) belong to \( T_A(\Gamma^1(S,G)) = \Gamma^1(S,g) \). For the left hand side consider a curve \( A_t \) with \( A_0 = A \) and \( \partial_t A_t = \hat{A} \). Since
\[ \partial_t (d^\Lambda A_t) = \partial_t (dA_t + [A_t \wedge A_t]) = d\hat{A} + [\hat{A} \wedge A] = \partial\hat{A} \]
it follows that
\[ d_A(\mu_\gamma) = \partial_t \int_S \text{tr}[\gamma \cdot \partial\Theta] = \int_S \text{tr}[\gamma \cdot \partial_t \partial\Theta] = \int_S \text{tr}[\gamma \cdot \partial\hat{A}]. \]

For the right hand side, consider a curve \( g_t \in \Gamma^0(S,G) \) with \( g_0 = I \) and \( \partial_t g_t = \gamma \). Differentiating
\[ g_t \cdot A = g_t A g_t^{-1} - g_t^{-1}dg_t \]
at \( t = 0 \) we obtain, using Schwarz rule to exchange the differentiation in \( t \) and on \( S \),
\[ \gamma_A = \partial_t (g_t \cdot A) = \gamma A - A\gamma - d\gamma = -\partial\gamma \in \Omega^1(S,g) \equiv T_A(\Omega^1(S,G)). \]

Since \( \partial \) is a graded derivation after applying the trace, it follows that
\[ -\int_S \text{tr}[\gamma_A \cdot \hat{A}] = -\int_S \text{tr}[\partial\gamma \wedge \hat{A}] = \int_S \text{tr}[\gamma \wedge \partial\hat{A}]. \]
\[ \square \]
Corollary 2.3.9. \( \mu^{-1}(0) = \{ A : d^A A = 0 \} \) consists of all flat \( C \)-connexions on \( S \), and the symplectic quotient \( \Omega^1(S,G)/\Omega^0(S,G) \) agrees with the non-abelian 1-cohomology

\[
H_1^c(S,G) = \mu^{-1}(0)/\Omega^0(S,G)
\]

which can be identified with \( \text{Hom}(\pi_1(S),C)/C \). This is a compact symplectic orbifold.

Note that in the non-abelian case higher order cohomology cannot be defined directly (higher categories).

Theorem 2.3.10. (Narasimhan-Seshadri) The symplectic quotient

\[
H^1(S,C) := \Omega^1(S,C)/\Gamma^0(S,C) = \Omega^1_{\text{flat}}(S,C)/C = \text{Hom}(\pi_1(S),C)/C
\]

is the space of all flat \( C \)-connexions on \( S \), modulo conjugation by \( C \). It is an orbifold with smooth part consisting of all irreducible connexions.

Theorem 2.3.11. Fix a complex structure \( \tau \) on \( S \). Then the complex-analytic quotient \( H^1(S,\tau,C,C) \) consists of all semi-stable holomorphic vector bundles over \( S, \tau \). It is an complex-analytic space, with regular part consisting of all stable vector bundles.

2.3.1 Homogeneous Manifolds

For a coadjoint orbit \( G^m \) we have

Proposition 2.3.12. For any \( m \in \mathfrak{g}^* \), the inclusion \( \iota : G^m \to \mathfrak{g}^* \) is a moment map for the co-adjoint action.

Proof. We have to show that for each \( v \in \mathfrak{g} \) the mapping \( \iota^v_m := \iota_m v = mv \) has the differential

\[
\iota_m(d_m \iota^v) = \omega_m(\iota_m v, v).
\]

This follows from

\[
\iota_m(d_m \iota^v) = (\partial_t^0 m \cdot \exp(tu))(d_m \iota^v) = \partial_t^0 \iota_m \exp(tu) = \partial_t^0 (m \cdot \exp(tu))v = \partial_t^0 m(\text{Ad}_{\exp(tu)}v) = \partial_t^0 m((\exp(t \text{ad} u) v)) = m(\text{ad} u v) = m[u, v] = \omega_m(\iota_m v, v).
\]

\( \square \)

2.4 Quantum line bundles

We call a symplectic form \( \omega \) integral, if

\[
\frac{1}{2\pi i} \int_S \omega \in \mathbb{Z}
\]

for all 2-cycles \( S \subset M \). This means that \( \frac{\omega}{2\pi i} \in H^2(M,\mathbb{Z}) \).

Theorem 2.4.1. Let \( \frac{\omega}{2\pi i} \in H^2(M,\mathbb{Z}) \) be an integral symplectic form. Then there exists a complex prequantum line bundle endowed with a hermitian metric \( h \) and a metric connection \( A \) with curvature \( dA = \omega \) (called the first Chern class). Conversely, the integrality condition is also necessary for the existence of a prequantum line bundle.
Proof. Choose a Leray open cover $V_a$ of $M$, meaning that all finite intersections are contractible. Since $d\omega = 0$, the Poincaré Lemma implies that for each $a$ there exists a potential $A^a \in \Omega^1(V_a, i\mathbb{R})$ such that

$$dA^a = \omega|_{V_a}.$$  

Then $d(A^a - A^b)|_{V_a \cap V_b} = \omega - \omega = 0$. Applying the Poincaré Lemma again there exist functions $\ell^a_b \in \Omega^0(V_a \cap V_b, i\mathbb{R})$ such that

$$(A^a - A^b)|_{V_a \cap V_b} = d\ell^a_b.$$  

Put

$$\kappa^a_b := \exp(\ell^a_b) \in T$$  

Since $\omega \in H^1(M, 2\pi i\mathbb{Z})$ is integral, it follows that

$$(\ell^a_b + \ell^b_c + \ell^c_a)|_{V_a \cap V_b \cap V_c} \in 2\pi i\mathbb{Z}.$$  

Therefore the cocycle property

$$(\kappa^a_b \kappa^b_c \kappa^c_a)|_{V_a \cap V_b \cap V_c} = \exp(\ell^a_b + \ell^b_c + \ell^c_a) = 1$$

holds. Hence we obtain a $T$-bundle $\mathcal{V} \times^\kappa T$ and the associated line bundle

$$\mathcal{V} \times^\kappa \mathbb{C} = (\mathcal{V} \times^\kappa T) \times \mathbb{C} = \{ (m, \phi) \mapsto (m, \beta^a_b(m)\phi) : m \in V_a \cap V_b, \phi \in \mathbb{C} \}$$

for the standard $T$-representation $\mathbb{C}$. By corollary 3.2.11 it carries the hermitian metric

$$(\langle m, \phi \rangle | \langle m, \psi \rangle) = \overline{\psi}.$$

Since

$$\kappa^a_b(d\kappa^a_b) = d\ell^a - d\ell^b = A^a - A^b,$$

the family $A^a$ defines a connexion $A$. By (??) the curvature of $(A^a)$ is given by

$$(dA^a)_m(u,v) = v \cdot (u \cdot d_m A^a) - u \cdot (v \cdot A^a) = (dA^a)_m(u,v) = \omega_m(u,v).$$

Hence $dA = \omega$. \hfill \square

For a hermitian holomorphic line bundle, the curvature

$$\omega^a = \overline{\partial}(\mathbb{h}^a)^{-1}\partial \mathbb{h}^a$$

defines an integer cohomology class

$$c_1(L) := \frac{1}{2\pi i} \omega \in H^2(M, \mathbb{Z})$$

called the (first) Chern class. This is a conformal invariant: If the hermitian metric $(\mathbb{h}^a)$ is changed by a conformal factor $\tilde{\mathbb{h}}^a := e^f \mathbb{h}^a$, where $f \in \Gamma(M, \mathbb{R})$, then $\tilde{A}^a := (\tilde{\mathbb{h}}^a)^{-1}\partial \tilde{h}^a = A^a + \partial f$ and therefore

$$\omega^\tilde{a} = \overline{\partial}\tilde{A}^\tilde{a} = \omega^a + \overline{\partial} f.$$  

Since $\overline{\partial} f = d^2 \frac{f}{2\pi i}$, it follows that $\frac{1}{2\pi i} \omega = \frac{1}{2\pi i} \omega$ in $H^2(M, \mathbb{Z})$.

On a Kähler manifold $M$ a **quantum line bundle** is a holomorphic hermitian line bundle whose Chern connexion has curvature $\omega$. We may also consider the scale of all $k$-th powers, with the inverse $\frac{1}{k}$ being interpreted as Planck’s constant.

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Lemma 2.4.2. If a hermitian holomorphic line bundle \((L, h)\) on a Kähler manifold satisfies

\[ h_m(u, v) = \partial_v \partial_u \log h \]

then \(\omega_m\) is the curvature of the Chern connexion \(\partial h\). Thus \((\mathcal{L}, h, \partial h)\) becomes a (pre)-quantum line bundle.

On a complex manifold \(M\) a smooth function \(\ell : M \to \mathbb{R}\) is called **plurisubharmonic** (in short, **plush**) if the Levi form

\[
(\partial \partial^{\ast} \ell)(m)) = \left( \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(m) \right)
\]

is positive (semi-definite). If \((\ell)\) is strictly positive, the \(\ell\) is called strictly plurisubharmonic. In this case the \((1,1)\)-form

\[
\partial \partial^{\ast} \ell = \sum_{i,j} \partial_i \partial_j \ell
\]

on \(M\) is a strictly positive (imaginary) symplectic form on \(M\). Consider the hermitian metric \(h_m := \exp \ell(m)\) on the holomorphic line bundle.

- **Jordan manifolds**

**Example 2.4.3.** For the holomorphic tangent bundle on \(\mathbb{P}^1\) Proposition ?? and (??) yield the curvature \((1,1)\)-form

\[
\bar{\partial} A^0 = -2 \frac{\partial}{\partial \xi} \frac{\overline{\xi}}{(1 + \overline{\xi}^2)^2} \, d\xi \wedge d\overline{\xi} = -2 \frac{1}{(1+z\overline{z})^2} \, d\xi \wedge d\overline{z} = -2 \frac{d\xi \wedge d\overline{z}}{(1+z\overline{z})^2}.
\]

**Lemma 2.4.4.**

\[
\int_{\mathbb{S}^2} \frac{d\xi \wedge d\overline{\xi}}{(1 + \overline{\xi}^2)^2} = 2\pi i.
\]

**Proof.** We have

\[
d\xi \wedge d\overline{\xi} = (dx - i dy) \wedge (dx + i dy) = 2i dx \wedge dy.
\]

Using polar coordinates \(z = r e^{i\phi}\) and putting \(u := r^2\) we obtain

\[
\frac{1}{2\pi i} \int_{\mathbb{S}^2} \frac{d\xi \wedge d\overline{\xi}}{(1 + \overline{\xi}^2)^2} = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{dx dy}{(1 + x^2 + y^2)^2} = \frac{2}{\pi} \int_0^{2\pi} ds \int_0^\infty \frac{r \, dr}{(1 + r^2)^2} = \frac{2}{\pi} \int_0^\infty \frac{r \, dr}{(1 + r^2)^2} = \frac{\infty}{\pi} \int_0^1 \frac{du}{1 + u^2} = \frac{\pi}{2} = 1.
\]

As a consequence we have

\[
\int_{\mathbb{S}^2} \omega = -4\pi i.
\]

Therefore the symplectic form \(\omega\) of the holomorphic tangent bundle is integral, but this bundle is not the ‘minimal’ line bundle associated with an integral symplectic form.
• Loop groups

The corresponding prequantum line bundle is the central extension viewed as a circle bundle over $\Omega(S)$.

Passing to Jordan manifolds, the quasi-determinant $\Delta_{z,w}$ of an irreducible metric Jordan triple $Z$ satisfies
\[
\det B_{z,w} = \Delta_{z,w}^p,
\]
where $p$ is a numerical invariant called the genus. For matrices $Z = K^{r \times s}$ we have
\[
\Delta_{z,w} = \det(I_r - zw^*) = \det(I_s - w^*z).
\]
We also have the addition formula
\[
\Delta_{z,u} \Delta_{z,v} = \Delta_{z,u+v},
\]
which is not trivial since a $p$-th root is involved. It follows that the map $\delta : R \to \mathbb{C}^*$ defined by
\[
\delta_{z,a}^b := \Delta_{z,a} - b
\]
is a cocycle with values in $\mathbb{C}^*$. This cocycle and its integer power $\delta^n$ induces a line bundle
\[
Z^2 \times \delta_n \sim \mathbb{C} := (Z^2 \times \delta_n \sim C \times C) := \{[m, \phi]_a = [m^{a-b}, \Delta_{z,a-b}^n] : \phi \in C, \Delta_{z,a-b} \neq 0\}
\]
over $\hat{Z}$. A holomorphic section $\Phi \in \mathcal{O}(Z^2 \times \delta_n \sim \mathbb{C})$ has the local trivializations
\[
\Phi_{[m,a]} = [m, a, \Phi^a(m)]
\]
for $a \in Z$, where $\Phi^a : Z \to C$ are holomorphic functions satisfying the compatibility condition
\[
\Phi^b(z^{a-b}) = \Delta_{z,a-b}^n \Phi^a(m)
\]
whenever $a, b \in Z$ satisfy $\Delta_{z,a-b} \neq 0$. Since $Z \subset \hat{Z}$ is a dense open subset via the embedding $z \mapsto z^0 = [z, 0]$, a section $\Phi \in \mathcal{O}(Z^2 \times \delta_n \sim C)$ is uniquely determined by its trivialization $\hat{\Phi} := \Phi^0$. Thus via the mapping $\Phi \mapsto \hat{\Phi}$ we may identify $\mathcal{O}(Z^2 \times \delta_n \sim C)$ with a vector space of entire functions on $Z$. Later, this will be determined explicitly.

**Proposition 2.4.5.** For an irreducible $0$ metric Jordan triple $Z$, the $\hat{G}$-invariant Bergman metric on $\hat{Z} \subset Z$ is given by
\[
(u|v) = \text{tr } D(B_{z,-z}^{-1}u, v) = \partial_u \overline{\partial}_v \log \Delta_{z,-z}^p.
\]
It follows that $\hat{Z} \times \delta^p \mathbb{C}$, endowed with the $0$ metric
\[
([u,v][z,v]) := \Delta_{z,-z}^p (u|v)
\]
is the (pre)-quantum bundle for the $0$ Kähler manifold $\hat{Z}$. Similarly, the $\hat{G}$-invariant Bergman metric on $\hat{Z} \supset Z$ is given by
\[
(u|v) = \text{tr } D(B_{z,-z}^{-1}u, v) = \partial_u \overline{\partial}_v \log \Delta_{z,-z}^p.
\]
It follows that $Z^2 \times \delta^p \mathbb{C}$, endowed with the $0$ metric
\[
([u,v][z,v]) := \Delta_{z,-z}^p (u|v)
\]
is the (pre)-quantum bundle for the $0$ Kähler manifold $\hat{Z}$.

**Proof.** We carry out the proof for the matrix case $Z = K^{r \times s}$, where $\Delta_{z,w} = \det(I_r - zw^*)$ and $p = r + s$.
We have
\[
\det a \det z = \det(az) = (\det \circ L_a)(z)
\]
Therefore
\[ \det a \det' u = (\det \circ L_a)'u = \det'(au). \]

It follows that
\[ \det' v = \det a \det'(a^{-1}v) = \det a \tr(a^{-1}v). \]

Therefore
\[ \partial_v \log \det(a) = \tr(a^{-1}v). \]

In the non-compact setting we obtain
\[ \partial_v \log \det(1 - zz^*) = \tr(1 - zz^*)^{-1} \partial_v (1 - zz^*) = -\tr(1 - zz^*)^{-1} v^*. \]

Therefore
\[
-\partial_u \partial_v \log \det(1 - zz^*) = \tr(1 - zz^*)^{-1} \partial_u (1 - zz^*)^{-1} v^* + (1 - zz^*)^{-1} u v^*
\]
\[
= \tr(1 - zz^*)^{-1} \left( u z^* (1 - z^* z)^{-1} v^* + u (1 - z^* z) (1 - z^* z)^{-1} v^* \right)
\]
\[
= \tr(1 - zz^*)^{-1} u (1 - z^* z)^{-1} v^* = \tr(B_{z,z}^{-1} u)v^* = \frac{1}{p} \tr D(B_{z,z}^{-1} u, v)
\]

It follows that
\[
\tr D(B_{z,z}^{-1} u, v) = -p \partial_v \partial_u \log \det(1 - zz^*) = dl_u \partial_v \log \det(1 - zz^*)^{-p}.
\]

In the compact setting we have \( \Delta_{z,-w} = \det(1 + zw^* ) \) and obtain
\[ \partial_v \log \det(1 + zz^*) = \tr((1 + zz^*)^{-1} (\partial_v (1 + zz^* ))) = \tr((1 + zz^*)^{-1} v^*). \]

Therefore
\[ \partial_v \log \det(1 + zz^*) = \tr(1 + zz^*)^{-1} z v^* = \tr(1 + zz^*)^{-1} \left( -u z^* (1 + zz^*)^{-1} v^* + u (1 + zz^*)^{-1} v^* \right)
\]
\[
= \tr(1 + zz^*)^{-1} \left( -u z^* z (1 + z^* z)^{-1} v^* + u (1 + z^* z) (1 + z^* z)^{-1} v^* \right)
\]
\[
= \tr(1 + zz^*)^{-1} u z^* z (1 + z^* z)^{-1} v^* = \tr(B_{z,-z}^{-1} u)v^* = \frac{1}{p} \tr D(B_{z,-z}^{-1} u, v)
\]

It follows that
\[
\tr D(B_{z,-z}^{-1} u, v) = p \partial_v \partial_u \log \det(1 + zz^*) = \partial_u \partial_v \log \det(1 + zz^*)^p.
\]

For \( Z = C \) the calculation simplifies to
\[
\partial_v \partial_u \log(1 - z\overline{v}) = \partial_u \frac{-z\overline{v}}{1 - z\overline{v}} = -u\overline{v}(1 - z\overline{v}) + z\overline{v}(-u\overline{v}) = -\frac{u\overline{v}}{(1 - z\overline{v})^2} = -h_z(u|v)
\]
• Conformal blocks

Determinant line bundle

Fix a complex structure \( S_\tau \). Then \( \Omega^1(S, G) \) acquires a complex structure \( J \) and the covariant derivative \( d^A \) of \( A \) has a \((0, 1)\)-part \( \bar{\partial}^A \).

**Lemma 2.4.6.** \((\Omega^1(S, G), J)\) can be identified with the space

\[
H^1(S_\tau, G^\mathbb{C})
\]

of all holomorphic \( G^\mathbb{C} \)-bundles over \( S_\tau \).

Holomorphic Quillen determinant line bundle over \( H^1(S_\tau, G^\mathbb{C}) \) with connexion whose curvature is the Kähler form.

\[
L_A = \det H^1(S_\tau, E_A) \otimes \det H^0(S_\tau, E_A)
\]

metric defined by regularized determinants of Laplacians.
Chapter 3

Quantum State Spaces

3.1 Reproducing kernels

On the other hand, we obtain an anti-holomorphic map

\[ K : M \to P(\mathcal{O}(M \times \mathbb{C})) \]

by

\[ w \mapsto K_w \in \mathcal{O}(M \times \mathbb{C}) \]

• Jordan manifolds

Example 3.1.1. Consider the projective space \( \mathbb{P}^d \). For \( 0 \leq i \leq d \) put

\[ V_i := \{ [\zeta] \in \mathbb{P}^d : \zeta^i \neq 0 \} \]

where \( [\zeta] := C\zeta \) for \( 0 \neq \mu \in \mathbb{C}^{d+1} \). Define \( \beta^i_j : V_i \cap \mathbb{P}^d \to \hat{G} \) by

\[ \beta^i_j[\zeta] := \frac{\zeta^i}{\zeta^j}. \]

Note that the fraction depends only on \( [\zeta] \). Since the cocycle identity is satisfied, we obtain a \( \mathbb{C}^\times \)-bundle

\[ \mathcal{V}^\beta \sim \mathbb{C}^\times = \{ [[\zeta], h_i] = [[\zeta], h\beta^i_j[\zeta]] \} = \{ [[\zeta], h_i] = [[\zeta], \frac{\zeta^i}{\zeta^j}h_j] \} \]

over \( \mathbb{P}^d = \mathcal{V}/R \). For each \( m \in \mathbb{N} \), let \( \mathcal{C}_m[\zeta] \) be the space of all \( m \)-homogeneous polynomials \( \psi(\zeta) \) in \( \zeta = (\zeta^0, \ldots, \zeta^d) \in \mathbb{C}^{d+1} \). For \( \psi \in \mathcal{C}_m[\zeta] \), define a holomorphic function \( \psi^i : V_i \to \mathbb{C} \) by

\[ \psi^i([\zeta]) := \frac{1}{(\zeta^i)^m} \psi(\zeta). \]

This depends only on \( [\zeta] \) since \( \psi \) is \( m \)-homogeneous. For \( [\zeta] \in V_i \cap V_j \) we have

\[ \psi^i([\zeta]) := \frac{(\zeta^i)^m}{(\zeta^i)^m} \psi^j(\zeta) \]

by definition. Hence the finite family \( (\psi^i) \) defines a holomorphic section of

\[ \mathcal{V}^\beta \sim m\mathbb{C} = (\mathcal{V}^\beta \sim \mathbb{C}^\times)^m \mathbb{C}. \]
Thus we obtain a linear map

$$C_m[\zeta] \to O(P^d \times^B C_m), \quad \psi \mapsto (\alpha^i)$$

which is a $GL_{d+1}(\mathbb{C})$-equivariant isomorphism. After 'symmetry breaking,' $C^{d+1}$, is isomorphic to the space of all polynomials of degree $\leq d$ on $C^d$. For $0 \leq a \leq d$ we define a polynomial $\psi^a$ in $d$ variables by

$$\psi^a(z^0, \ldots, z^d) := \psi(z^0, 1^a, \ldots, z^d)$$

If $\zeta^a \neq 0$ then

$$\psi(\zeta) = \frac{1}{(\zeta^a)^m} \psi^a\left(\frac{\zeta^0}{\zeta^a}, \ldots, \frac{\zeta^d}{\zeta^a}\right)$$

It follows that

$$\psi^a = \psi^b \circ \sigma^a_b.$$ Conversely, let $\psi^a$, $0 \leq a \leq d$ be polynomials in $d$ variables of degree $\leq m$ such that (??) holds. Then there is a unique section $\psi \in O(U \times_{C^m} C)$ satisfying (??). It follows that $O(U \times_{C^m} C)$ can be identified with the space of all $m$-homogeneous polynomials in $\zeta = (\zeta^0, \ldots, \zeta^d)$. This space is irreducible under the natural action of $SL_{d+1}(\mathbb{C})$. For $d = 2$ we obtain the tangent bundle and

$$O(U \times_{C^2} C) = O_1(P^d).$$

**Example 3.1.2.** The tautological bundle $T$ over the Grassmannian $M = G_r(K^{\infty})$ has the fibre $U$ over $U \in M$. Consider the dual bundle $T^*$ and the line bundle $\wedge^r T^*$, whose fibre over $U$ consists of all alternating $r$-multilinear maps from $U$ to $K$. For any index chain $1 \leq i_1 < i_2 < \ldots < i_r \leq r + s$ of length $r$ there is a section $\sigma^{(i_1; \ldots; i_r)}$ of $\wedge^r T^*$, defined by

$$U \mapsto \sigma^{(i_1; \ldots; i_r)}(v_1 \wedge \ldots \wedge v_r) := \det(v_j|\beta_{i_k})_{j,k=1}$$

for all $v_1, \ldots, v_r \in U$. For another $w \in C^{r \times s}$ we put $v_j := (\beta_j, \beta_j z) \in U$ and $\beta_{i_k}^k := \beta_k$ for $1 \leq k \leq r$, and $\beta_{i_k}^n := \beta_k w$. Then

$$(v_j|\beta_{i_k}^n, \beta_{i_k}^m) = (\beta_j|\beta_k) + (\beta_j z|\beta_k w) = (\beta_j(1+z w^*)|\beta_k)$$

showing that the trivialization $\varphi_{z,w} := \sigma^{(i_n)}_{\varphi_{(m)}} = \det(1+z w^*)$. Comparing with (??) we see that

$$Z \times R C \equiv \wedge^r T^*$$

and hence $Z \times^R C$ is the $n$-th power of $\wedge^r T^*$. The action of $\hat{G}$ on $H^2_\Sigma(Z, E)$ is given by

$$(U^{-1}g)(\zeta) := (\partial g)^{-z} \Phi(g\zeta)$$

for all $\Phi \in H^2_\Sigma(\hat{Z}, E)$. Here we use the fact that $\partial g \in \hat{K}$.

**Example 3.1.3.** In the rank 1 case $Z = C^{1 \times d}$ the homogeneous line bundle $Z^2 \times^a C$ over $\hat{Z} = P^d$ has holomorphic sections $\mathcal{K}_w$ with affine trivialization

$$\mathcal{K}_w(m) = (1 + (z|w))^n,$$

where $w \in C$ is arbitrary and $(z|w)$ denotes the inner product. Thus $O(Z^2 \times^a C) \equiv H_n(Z, C)$ consists of all polynomials in $z$ of degree $\leq n$, or equivalently, of all $n$-homogeneous polynomials in $d+1$ variables, under the natural action of $\hat{G} = SU(d + 1)$. For $d = 1$ this space is also described by entire functions $f_0, f_\infty$ on $C$ satisfying the compatibility condition

$$f_\infty(-\frac{1}{z}) = z^n f_0(m)$$

for all $m \in C^*$. 49
More explicitly, for any $w \in Z$ there exists a global holomorphic section $K_{z,w} \in \mathcal{O}(Z^2 \times_R^n \mathbb{C})$ with local trivializations

$$m \mapsto K_{z,w}^m = D^m_{z,a-w},$$

since the relation (??) implies

$$D^n_{z,a-b} K_{z,a-b}^b = D^n_{z,a-b} D^n_{z,b-w} = D^n_{z,a-w} = K_{z,w}^a.$$

**Proposition 3.1.4.** There is a natural $\hat{G}$-equivariant isomorphism

$$\mathcal{O}(Z^2 \times_R^n \mathbb{C}) \cong P^n(Z) := \sum_{m \leq n} P_m(Z).$$

**Proof.** Using the Faraut-Korányi formula we obtain

$$K_{z,w} = K_{0,z,w} = D^n_{z,-w} = \sum_{m \leq n} (-n)_m K^m_{z,-w} = \sum_{m \leq n} (-1)^{|m|} (-n)_m K^m_{z,w}. \square$$

As a special case of (??) the action of $\hat{G}$ on $H^2_{\mathbb{Z}}(Z, \mathbb{C})$ is given by

$$(U^{-1}_g \Phi)(\zeta) := \det(\partial_{\zeta} g)^{-n/p} \Phi(g\zeta)$$

for all $\Phi \in H^2_{\mathbb{R}}(Z, \mathbb{C})$. Since

$$\det B_{z,w} = D^n_{z,-w},$$

where $p$ is the genus of $Z$, the cocycles (??) and (??) are related by

$$\det \beta_{z,a-b} = (\delta_{z,a-b})^p.$$ 

On the level of principal bundles this implies

$$Z^2 \times_\sigma \mathbb{C}^* = Z^2 \times_\sigma \mathbb{C}^* = \hat{G}_u \times_\sigma \mathbb{C}^*$$

for the $p$-th power cocycle $\delta^p$. As a special case consider the determinant character $\delta_k := \det_Z k$ of $K$. Then (??) implies

$$\hat{G}_k \times_\sigma \mathbb{C} = Z^2 \times_\sigma \mathbb{C}. \square$$

In this sense, the line bundle $Z^2 \times_\sigma \mathbb{C}$ is more fundamental.

**Proposition 3.1.5.** Let $(E, \pi)$ be a holomorphic representation of $\hat{K}$. Then, for any $w \in Z$ there exists a global holomorphic section $K_{z,w} \in \mathcal{O}(Z^2 \times_\sigma^n E)$ with local trivializations

$$m \mapsto K_{z,w}^m = B^m_{z,a-w} \xi.$$

In particular, we have

$$K_{z,w} = K_{0,z,w} = B^n_{z,-w} \xi.$$

**Proof.** This follows from (??) which implies

$$B^n_{z,a-b} K_{z,a-b}^b \xi = B^n_{z,a-b} B^n_{z,b-w} \xi = B^n_{z,a-w} \xi = K_{z,w}^a \xi. \square$$

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3.2 Compact Lie Groups and Borel-Weil-Bott Theorem

Theorem 3.2.1. For a metric Jordan triple $Z$, the associated Jordan manifolds $\hat{Z} \subset Z \subset \dot{Z}$

$$\dot{Z} = \hat{G}/K = \hat{G}/\hat{G}_m,$$

are coadjoint orbits, for the linear functional $m : \hat{g} \to i\mathbb{R}$ defined by

$$mX = \text{tr}\partial_bX$$

where $\partial_b X = X'(0) \in \hat{t} \subset \mathfrak{gl}(Z)$.

Proof. The complexified Lie algebra $\hat{\mathfrak{g}} = \hat{g} \otimes \mathbb{C}$ consists of all vector fields

$$X = a + \ell + b^*$$

where $a \in Z$ is the constant vector field,

$$b^*(z) = \frac{1}{2}\{z; b; z\}$$

is a quadratic vector field given by the Jordan triple product, and $\ell = \ell(z) \in \hat{t}$ is a linear vector field. Define an invariant inner product on $\hat{\mathfrak{g}}$ by

$$\langle a + \ell + b^*|\alpha + \lambda + \beta^* \rangle := \text{tr}(D(a, \beta) + \ell\lambda + D(\alpha, b))$$

for all $a, b, \alpha, \beta \in Z$ and $\ell, \lambda \in \hat{t}$. Via the inner product we may identify $\hat{\mathfrak{g}}$ and its dual $\hat{\mathfrak{g}}^*$. We have to find an element $m \in \mathfrak{g}^* \approx \hat{\mathfrak{g}}$ such that

$$\mathfrak{t} = \hat{\mathfrak{g}}_m = \{X = a + \ell + b^* : m \circ \text{ad}_X = 0\}.$$

Since $\dot{Z}$ is circular, we have

$$I := \frac{\partial}{\partial z^*} \in \mathfrak{t}.$$

This element generates the center of $\mathfrak{t}$ and corresponds to the identity on $Z$. Now define

$$mX = m(a + \ell + b^*) = \langle I|X\rangle = \text{tr}\ell.$$

The commutator

$$[X, Y] := d_XY - d_YX$$

yields $[a, a] = 0$ and $[b^*, b^*] = 0$. Moreover $[\ell, a] = \ell a$ as a constant vector field. Moreover,

$$[\ell, \beta^*] = \frac{1}{2}\{\ell z, \{z; b; z\}\} = \{\ell z; \beta, z\} - \frac{1}{2}\ell\{z; \beta, z\} = -\frac{1}{2}\{z; \ell^*\beta, z\} = -(\ell^*\beta)^*$$

as a quadratic vector field. Finally,

$$[a, b^*] = \{[a, \frac{1}{2}\{z; b; z\}] = \{a; b; z\} = D(a, b)z = D(a, b)$$

viewed as a linear vector field. Therefore

$$\text{ad}_X(a + \ell + b^*, a + \lambda + \beta^*) = ([\ell, a] - [\lambda, a]) + ([a, \beta^*] + [\ell, \lambda] - [\alpha, b^*]) + ([\ell, \beta^*] - [\lambda, b^*])$$

$$= (\ell\alpha - \lambda a) + (D(a, \beta) + [\ell, \lambda] - D(\alpha, b^*)) + (\lambda b^* - (\ell^*\beta)^*).$$

Therefore

$$(m \circ \text{ad}_X)(a + \lambda + \beta^*) = \text{tr}(D(a, \beta) + [\ell, \lambda] - D(\alpha, b)) = \text{tr}(D(a, \beta) - D(\alpha, b))$$

since $[\ell, \lambda]$ is a commutator in $\mathfrak{t}$. For $X \in \hat{\mathfrak{g}}$ we need $b = ea, \beta = e\alpha$ where $e = \pm$. Thus

$$(m \circ \text{ad}_X)(a + \lambda + e\beta^*) = e\text{tr}(D(a, \alpha) - D(\alpha, a)).$$

By polarization, it follows that $m \circ \text{ad}_X = 0$ if and only if $x\text{tr}D(a, \alpha) = 0$ for all $\alpha \in Z$. Since $Z$ is non-degenerate, this means $a = 0$ and therefore $X \in \mathfrak{t}$. □
Consider a compact complex projective manifold \( M \), with structure sheaf \( \mathcal{O} \). Let \( \mathcal{O}^q \) denote the sheaf of germs of holomorphic sections of the \( q \)-th exterior power \( T^q M \). Then \( \mathcal{O}^n \) belongs to the canonical bundle of \( n \)-forms.

For any holomorphic vector bundle \( V \) over \( M \), let \( \mathcal{O} \otimes V \) denote the sheaf of germs of holomorphic sections of \( V \), and let \( \mathcal{O}^p \otimes V \) denote the sheaf of germs of holomorphic \( p \)-form sections of \( V \). Since \( M \) is compact, the sheaf cohomology groups \( H^q(M, \mathcal{O}^n \otimes V) \) are finite-dimensional complex vector spaces.

The **Serre duality theorem** states that \( H^q(M, \mathcal{O} \otimes V) \) is in duality with \( H^{n-q}(M, \mathcal{O}^n \otimes V^*) \), where \( V^* \) is the dual vector bundle of \( V \). Thus

\[
H^q(M, \mathcal{O} \otimes V)^* = H^{n-q}(M, \mathcal{O}^n \otimes V^*).
\]

**Jordan manifolds**

**Example 3.2.2.** The group \( \hat{G} = SL_{1+n}(\mathbb{C}) \) contains the parabolic subgroup

\[
\hat{G}_- = \{ p = \begin{pmatrix} p_0 & b \\ 0 & d \end{pmatrix} \}
\]

fixing the line \( \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Then \( \hat{G}/\hat{G}_- = \mathbb{P}^n \). For integers \( m \), consider the homogeneous line bundle

\[
\mathcal{O}(m) = \hat{G}_- \times \mathbb{C} = \{ [g, \phi] = [gp, (p_0)^m \phi] \}
\]

associated with the character \( p \mapsto (p_0)^m \) of \( \hat{G}_- \). Its holomorphic sections

\[
H^0(\hat{G}_- \times \mathbb{C}) = \begin{cases} \mathbb{C}_m[\zeta^0, \ldots, \zeta^n] & m \geq 0 \\ 0 & m < 0 \end{cases}
\]

Then \( \hat{G}_- \times \mathbb{C} = \wedge^n(TM) \) and

\[
\hat{G}_-^{-n-1} \times \mathbb{C} = \wedge^n(T^*M)
\]

is the canonical bundle. By a theorem of Serre, we have

\[
H^q(\hat{G}_- \times \mathbb{C}) = 0
\]

for \( 0 < q < n \). For the \( n \)-th cohomology, we apply Serre duality:

\[
H^n(\hat{G}_- \times \mathbb{C})^* = H^0((\hat{G}_-^{-n-1} \times \mathbb{C}) \otimes (\hat{G}_- \times \mathbb{C})^*) = H^0((\hat{G}_-^{-n-1} \times \mathbb{C}) \otimes (\hat{G}_-^{-m} \times \mathbb{C})) = H^0(\hat{G}_-^{-n-m-1} \times \mathbb{C}).
\]

Thus

\[
H^n(\hat{G}_- \times \mathbb{C}) = \begin{cases} 0 & n + m \geq 0 \\ \mathbb{C}_{-n-m-1}[^*] & n + m < 0 \end{cases}
\]

where \( \mathbb{C}_k[^*] \) carries the contragredient representation.

**3.2.1 Borel Subgroups and full Flag Manifolds**

A semisimple complex Lie algebra \( \mathfrak{g} \) with maximal torus \( \mathfrak{t} \subset \mathfrak{g} \) has a **root decomposition**

\[
\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.
\]

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For every root \( \alpha \in \Delta \) there exists a unique 'coroot' \( H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \) satisfying \( \alpha H_\alpha = 2 \). The weight lattice
\[
\hat{T}^* := \{ \lambda \in \hat{t}^* : \lambda H_\alpha \in \mathbb{Z} \forall \alpha \in \Delta \}
\]
is a free abelian group of rank \( \dim \hat{t} \), containing the roots. The elements of \( \hat{T}^* \) correspond to characters of the group \( \hat{T} \) under taking 'logarithms,' whence the notation. The Weyl group \( \hat{W} \) acts on \( \Delta \) and on \( \hat{T}^* \).

Fix a subset \( \Delta_+ \subset \Delta \) of positive roots. There exists a unique element \( w_0 \in \hat{W} \) satisfying
\[
w_0 \Delta_+ = -\Delta_+.
\]
Define
\[
\hat{g}_+ := \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \hat{g}_- := \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}.
\]
Then we have the 'Gauss decomposition'
\[
\hat{g} = \hat{g}_- \oplus \hat{t} \oplus \hat{g}_+.
\]
On the Lie group level this implies that
\[
G_\times \cdot \hat{T} \cdot G_\times \subset \hat{G}
\]
is a dense open subset. Here \( \hat{G} \) is assumed to be simply-connected, containing a maximal complex torus \( \hat{T} \). Let \( g \mapsto g^* \) be an involution such that
\[
\hat{g} = \{ \gamma \in \hat{g} : \gamma^* = -\gamma \}
\]
is the Lie algebra of a compact form \( \hat{G} \) of \( \hat{G} \).

The Theorem of the highest weight is the following:

**Theorem 3.2.3.** For every \( \lambda \in \hat{T}^*_+ \) there is a finite-dimensional irreducible \( \hat{g} \)-module, denoted by \( \hat{G}_\lambda \), with highest weight \( \lambda \), and every finite-dimensional irreducible \( \hat{g} \)-module is isomorphic to \( \hat{G}_\lambda \) for a unique \( \lambda \in \hat{T}^*_+ \).

Thus any choice of positive roots yields a bijection
\[
\hat{T}^*_+ \rightarrow \hat{G}^*,
\]
where the right-hand side denotes the (discrete) set of all finite-dimensional irreducible \( \hat{g} \)-modules.

### 3.2.2 0-Cohomology: Borel-Weil theorem

**Lemma 3.2.4.** For every \( G^\mathbb{C} \)-module \( E \) there is a \( G \)-equivariant mapping
\[
E \rightarrow \mathcal{O}(G^\mathbb{C}, \mathbb{C}), \quad (\xi, \eta) \mapsto \xi^* \eta
\]
defined by
\[
(\xi^* \eta)_g := (\xi|g^* \eta)
\]
for all \( \xi, \eta \in E \). We have
\[
\rho_g(\xi^* \eta) = \xi^*(g^* \eta)
\]

**Proof.**
\[
(\rho_g(\xi^* \eta))_{g_1} = (\xi^* \eta)_{g_1 g} = (\xi|g_1 g^* \eta) = (\xi|g_1^* g^* \eta) = (\xi^*(g^* \eta))_{g_1} \]
\[\square\]
Assume that
\[ b^+ \xi = \chi(b_+) \xi \]
is a highest weight vector. Then \( \overline{\mathcal{B}}^- \xi \)
\[ (\xi^* \eta)_{\mathcal{B}^-} = (\xi | (b^- g)^* \eta) = (\xi | b^+ \xi^* g^* \eta) = (b^+ \xi^* | g^* \eta) \]
\[ = (\overline{\mathcal{B}}^- \xi | g^* \eta) = \chi(\overline{\mathcal{B}}^-) (\xi | g^* \eta) = \overline{\chi(\overline{\mathcal{B}}^-) (\xi | g^* \eta)}. \]
In particular,
\[ (\xi^* \eta)_{t^g} = \overline{\chi(t) (\xi^* \eta)} = t^{-\chi} (\xi^* \eta). \]
This shows that
\[ \xi^* \eta \in \mathcal{O}(G^C \chi \overline{B}^- \mathbb{C}). \]

Consider the simply-connected complex Lie group \( \hat{G} \) with Lie algebra \( \hat{\mathfrak{g}} \).

**Theorem 3.2.5. Borel-Weil Theorem:** Let \( \lambda \in \hat{T}^* \) and consider the induced line bundle \( \hat{G} \times_{\hat{G}^-}^\lambda \mathbb{C} \), with trivial action of \( \hat{G}^- \). Then
\[ H^0(\hat{G} \times_{\hat{G}^-}^\lambda \mathbb{C}) = \begin{cases} \hat{G}_\lambda & \lambda \in \hat{T}^*_+ \\ 0 & \lambda \notin \hat{T}^*_+ \end{cases} \]
Here \( \hat{G}_\lambda \) is 'the' irreducible \( \hat{G} \)-module with highest weight \( \lambda \).

**Proof.** The holomorphic sections \( H^0(\hat{G} \times_{\hat{G}^-}^\lambda \mathbb{C}) \) are identified with the subspace
\[ \{ f \in \mathcal{O}(\hat{G}, \mathbb{C}) : f_{gb} = b^{-\lambda} f_g \ \forall \ b \in \hat{G}^- \}. \]
Assume first that \( H^0(\hat{G} \times_{\hat{G}^-}^\lambda \mathbb{C}) \neq 0 \). Then there exists a (non-zero) highest weight vector \( f^0 \in H^0(\hat{G} \times_{\hat{G}^-}^\lambda \mathbb{C}) \) satisfying
\[ a \times f^0 = a^\chi f^0, \ f^0_{a^g} = a^{-\chi} f^0_g \]
for all \( a \in \hat{G}^+ \), where \( \chi \) is a character of \( \hat{G}^+ \). For \( c \in \hat{G}^+, \ t \in \hat{T}, \ d \in \hat{G}^- \) we have \( (ct)^\chi = t^\chi \) and \( (td)^\lambda = t^\lambda \). Hence (??) and (??) imply
\[ f^0_{ctd} = (ct)^{-\chi} f^0_d = t^{-\chi} f^0_e = (td)^{-\lambda} f^0_e = t^{-\lambda} f^0_c. \]
If \( f^0_c = 0 \) then \( f^0 \) vanishes on the dense open subset \( \hat{G}^+ T \hat{G}^- \subset \hat{G} \). Hence \( f^0 = 0 \) by continuity, a contradiction. Thus \( f^0_c \neq 0 \) and (??) shows \( \lambda = \chi \). Since \( \chi \) is dominant, \( \lambda \in \hat{T}^*_+ \) and the second assertion follows.

Conversely let \( \lambda \in \hat{T}^*_+ \) be a dominant weight such that (??) holds. Let \( \hat{G}_\lambda \) be an irreducible \( \hat{G} \)-module with highest weight \( \lambda \). Consider the involution \( g \mapsto g^* \) on \( \hat{G} \) such that the compact real form \( \hat{G} \) acts unitarily. Let \( v^0 \in \hat{G}_\lambda \) be a non-zero highest weight vector (unique up to a scalar multiple). For any \( v \in \hat{G}_\lambda \) define a holomorphic function \( \tilde{v} \) on \( \hat{G} \) by
\[ \tilde{v}_g := (v^0 | g^{-\lambda} v). \]
For all \( b = ct \in \hat{G}^- \) we have \( b^* \in \hat{G}^+ \) and hence
\[ b^{-\lambda^*} v^0 = b^* v^0 = t^{-\lambda} v^0. \]
It follows that
\[ \tilde{v}_{gb} = (v^0 | (gb)^{-\lambda} v) = (v^0 | b^{-\lambda} (g^{-\lambda} v)) = (b^* v^0 | g^{-\lambda} v) = (b^* v^0 | g^{-\lambda} v) = t^{-\lambda} (v^0 | g^{-\lambda} v) = t^{-\lambda} \tilde{v}_g. \]
Therefore, via the identification (??), we have \( \tilde{v} \in H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}) \). The computation
\[
(g \cdot \tilde{v})_g = \tilde{v}_{g^{-1}g'} = (v^0|(g^{-1}g')^{-\lambda}v) = (v^0|(g')^{-\lambda}(g^\lambda v)) = \tilde{g}v_{g'}
\]
for \( g, g' \in G \) shows
\[
g \cdot \tilde{v} = \tilde{g}v.
\]
It follows that the \( \mathbb{C} \)-linear mapping
\[
\hat{G}_\lambda \to H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}), \quad v \mapsto \tilde{v}
\]
is \( \hat{G} \)-equivariant. We have to show that it is an isomorphism. Suppose that \( \tilde{v} = 0 \) for some \( v \in \hat{G}_\lambda \).
Then
\[
(g^*v^0)(v) = (v^0)g^\lambda v = \tilde{v}_{g^{-1}} = 0
\]
for all \( g \in G \). By irreducibility, the orbit \( \hat{G}^\lambda v^0 \) is total in \( \hat{G}_\lambda \). It follows that \( v = 0 \). Thus (??) is also injective, and the range of (??) is a \( \hat{G} \)-submodule of \( H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}) \). For \( c \in \hat{G}_\to, t \in T, d \in \hat{G}_< \) we have \( c^{-\lambda}v^0 = v^0 = d^{-\lambda}v^0 \) and \( t^{-\lambda}v^0 = t^{-\lambda'}v^0 \). It follows that
\[
\tilde{v}^0_{ctd} = (v^0|(ctd)^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}c^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}v^0) = (v^0|d^{-\lambda}t^{-\lambda}v^0) = t^{-\lambda}(v^0|d^{-\lambda}v^0) = t^{-\lambda}(v^0|v^0).
\]
On the other hand, let \( f^0 \in H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}) \) be any highest weight vector. Then \( c^{-\lambda}f^0 = f^0 \) and hence
\[
f^0_{ctd} = (c^{-\lambda}f^0)_{td} = f^0_{td} = t^{-\lambda}f^0.
\]
Since \( \hat{G}_\to \hat{T}\hat{G}_< \) is dense in \( \hat{G} \), a continuity argument shows
\[
f^0 = \frac{f^0_t}{(v^0|v^0)} \tilde{v}^0.
\]
Thus all highest weight vectors in \( H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}) \) are proportional. Since distinct irreducible summands would contain distinct highest weight vectors, \( H^0(\hat{G} \times_{\hat{G}_-} \mathbb{C}) \) is irreducible. Therefore (??) defines a \( \hat{G} \)-equivariant isomorphism.

\[\square\]

### 3.2.3 Parabolic subgroups and flag manifolds

We now pass from a maximal torus to an arbitrary torus. Let \( \Pi \subset \Delta \) be a set of simple (positive) roots. Let \( \Phi \subset \Pi \) be any subset, including the empty set \( \Phi = \emptyset \). Then
\[
\Phi \mathfrak{t} := \{ H \in \mathfrak{t} : \alpha H = 0 \ \forall \ \alpha \in \Phi \}
\]
is a subtorus whose centralizer
\[
\mathfrak{c} := \{ X \in \mathfrak{g} : [X, \Phi \mathfrak{t}] = 0 \}
\]
is a reductive Lie algebra, with Levi decomposition
\[
\mathfrak{c} = \Phi \mathfrak{t} \oplus \mathfrak{g}^\Phi
\]
Its semi-simple commutator ideal \( \mathfrak{g}^\Phi \) has itself a Gauss decomposition
\[
\mathfrak{g}^\Phi = \mathfrak{g}_\to^\Phi \oplus \mathfrak{t}^\Phi \oplus \mathfrak{g}_\to^\Phi.
\]
Thus we have added one positive root space

\[ \hat{\mathfrak{g}}_\sigma = \mathfrak{g}_\sigma \oplus \mathfrak{g}_\sigma^\circ \oplus \mathfrak{g}_\sigma \]

and

\[ \hat{\mathfrak{g}}^- = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \hat{\mathfrak{g}}^0 = \sum_{\alpha \in \Delta_+ \setminus \Delta} \mathfrak{g}_\alpha. \]

On the other hand, define

\[ \hat{\mathfrak{g}}^- = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \hat{\mathfrak{g}}^0 = \sum_{\alpha \in \Delta_+ \setminus \Delta} \mathfrak{g}_\alpha. \]

Then the parabolic subalgebra is

\[ \hat{\mathfrak{g}}^- = \hat{\mathfrak{g}}^-_\sigma \oplus \hat{\mathfrak{g}}^-_\sigma^\circ \oplus \hat{\mathfrak{g}}^-_\sigma = \hat{\mathfrak{g}}^-_\sigma \oplus \hat{\mathfrak{g}}^-_\sigma^\circ \oplus \hat{\mathfrak{g}}^-_\sigma = \hat{\mathfrak{g}}^- \oplus \hat{\mathfrak{g}}^0 \]

since \( \hat{\mathfrak{g}}^- \equiv \hat{\mathfrak{g}}^-_\sigma \oplus \hat{\mathfrak{g}}^-_\sigma^\circ \subset \hat{\mathfrak{g}}^- \).

Thus in the non-empty case \( \Phi \neq \emptyset \) the reductive torus centralizer \( \hat{\mathfrak{t}}^\Phi \oplus \hat{\mathfrak{g}}^\Phi \) plays the role of the torus \( \hat{\mathfrak{t}} \) and \( \hat{\mathfrak{g}}^- \) is the unipotent radical. Compared to the line bundles in the case \( \Phi = \emptyset \), we now have vector bundles since the semi-simple part \( \hat{\mathfrak{g}}^\Phi \) has higher dimensional irreducible highest weight representations.

In the special case \( \Phi = \emptyset \) we have \( \emptyset \hat{t} = 0, \emptyset \hat{g} = 0 \), since \( \hat{t} \) is maximal. Therefore

\[ \emptyset \hat{g} = \hat{t} \oplus \sum_{\alpha \in \Delta_+} \hat{g}_\alpha = \hat{t} \oplus \hat{g}^- = \hat{g} \]

is a Borel subalgebra. In the opposite case \( \Phi = \Pi \) we have \( \Pi \hat{t} = \hat{t}, \Pi \hat{g} = 0 \) and hence \( \Pi \hat{g} = \hat{g}^- \) is the full Lie algebra.

### 3.2.4 q-Cohomology: Bott’s Theorem

For passing from 0-cohomology to q-cohomology, in case \( \lambda \) is not dominant, we use reflections by simple roots. Let \( \Phi = \{ \sigma \} \), where \( \sigma \in \Pi \) is a simple root. Then there is a splitting

\[ \hat{t} = \sigma \hat{t} \oplus \hat{t}^\sigma \]

where \( \hat{t}^\sigma := \mathfrak{c} \cdot H_\sigma \) and \( \sigma \hat{t} := \{ H \in \hat{t} : \sigma H = 0 \} \). The torus centralizer

\[ \mathfrak{c} := \{ X \in \hat{\mathfrak{g}} : [X, \sigma \hat{t}] = 0 \} = \sigma \hat{t} \oplus \hat{\mathfrak{g}}^\sigma \]

is a reductive Lie algebra, and the Gauss decomposition of its semi-simple commutator ideal \( \hat{\mathfrak{g}}^\sigma \) simplifies to

\[ \hat{\mathfrak{g}}^\sigma = \mathfrak{g}_\sigma \oplus \mathfrak{c} \cdot H_\sigma \oplus \mathfrak{g}_\sigma \equiv \mathfrak{sl}_2(\mathfrak{c}), \]

since \( \hat{\mathfrak{g}}^\sigma < \mathfrak{g}^- \) and \( \hat{\mathfrak{g}}^\circ = \mathfrak{g}^- \). On the other hand, define

\[ \sigma \hat{\mathfrak{g}} = \sum_{\alpha \in \Delta_+ \setminus \sigma} \mathfrak{g}_\alpha, \quad \sigma \hat{\mathfrak{g}}^\circ = \sum_{\alpha \in \Delta_+ \setminus \sigma} \mathfrak{g}_\alpha \]

The parabolic subalgebra is

\[ \sigma \hat{\mathfrak{g}}^- = \sigma \hat{t} \oplus \sigma \hat{\mathfrak{g}}^\circ \oplus \sigma \mathfrak{g}_\sigma = \hat{t} \oplus \hat{\mathfrak{g}}^\sigma \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha = \hat{\mathfrak{g}}^- \oplus \hat{\mathfrak{g}}^\circ \]

Thus we have added one positive root space \( \mathfrak{g}_\sigma \) to the Borel subalgebra \( \mathfrak{g}^- \). Since \( \dim \mathfrak{g}_\alpha = 1 \) we have

\[ \sigma \hat{\mathfrak{g}}^- / \hat{\mathfrak{g}}^- = \mathbb{P}^1. \]
Take $E_\sigma \in \mathfrak{g}_\sigma$, $F_\sigma \mathfrak{g}_{-\sigma}$ with $[E_\sigma, F_\sigma] = H_\sigma$. Since $\hat{G}$ is supposed to be simply-connected, $\sigma\hat{G}_-$ has a Levi decomposition

$$\sigma\hat{G}_- = \sigma\hat{T} \cdot \sigma\hat{G}_\sigma \cdot \sigma\hat{G}_<$$

where the semi-simple part $\sigma\hat{G}_\sigma \equiv \text{SL}_2(\mathbb{C})$ has the Lie algebra

$$\hat{\mathfrak{g}}_\sigma = \langle E_\sigma, F_\sigma, H_\sigma \rangle$$

and the complex torus $\sigma\hat{T} \subset \hat{T}$ has the Lie algebra

$$\sigma\hat{t} := \{ H \in \hat{t} : \sigma H = 0 \}.$$

Let $\lambda \in T^*$ satisfy $m := \lambda H_\sigma \geq 0$. Let

$$\sigma\hat{\mathfrak{g}}_m := \langle v_m, v_{m-2}, \ldots, v_{2-m}, v_{-m} \rangle$$

be the $m + 1$-dimensional ‘spin’ representation of $\sigma\hat{\mathfrak{g}} \equiv \mathfrak{sl}_2(\mathbb{C})$. Then

$$H_\sigma v_k = k v_k, \quad E_\sigma v_k \in C v_{k+2}, \quad F_\sigma v_k \in C v_{k-2}$$

for all $k$, putting $v_k = 0$ if $|k| > m$. Since $\hat{G}$ is assumed to be simply-connected, one can show that $\sigma\hat{G} \equiv \text{SL}_2(\mathbb{C})$ and hence the infinitesimal action on $\sigma\hat{\mathfrak{g}}_m$ can be integrated to an action $\pi$ of $\sigma\hat{G}$ denoted by $\sigma\hat{G}_m$. The highest weight vector $v_m$ satisfies $p(H_\sigma)v_m = p(m)v_m$ for all polynomials $p$ and hence

$$\exp(z H_\sigma)^\pi v_m = e^{2m}v_m$$

for all $z \in \mathbb{C}$. Now suppose that $t \in T^* \cap \sigma\hat{G}$. Then $t = \exp(z H_\sigma)$ for some $z \in \mathbb{C}$. This implies

$$t^\lambda \cdot v_m = \exp(z H_\sigma)^\lambda \cdot v_m = e^{z\lambda H_\sigma} \cdot v_m = e^{2m}v_m = \exp(z H_\sigma)^\pi v_m = t^\pi v_m.$$  

Since $\sigma\hat{G}$ centralizes $\sigma\hat{T}$, it follows that

$$t^\pi (s^\pi v_m) = (ts)^\pi v_m = (st)^\pi v_m = s^\pi t^\pi v_m = s^\pi (t^\lambda v_m) = t^\lambda \cdot (s^\pi v_m)$$

for all $s \in \sigma\hat{G}$. Since the set $\sigma\hat{G}^\pi v_m$ is total in $\sigma\hat{G}_m$, it follows that

$$t^\pi v = t^\lambda v$$

for all $v \in \sigma\hat{G}_m$. Thus the two representations agree on $\sigma\hat{G} \cap \sigma\hat{T}$ and therefore induce an irreducible representation of $\sigma\hat{G} \sigma\hat{T}$ which extends trivially to a representation of $\sigma\hat{G}_-$. We denote this module by $\sigma\hat{G}_m$.

**Lemma 3.2.6.** Let $\lambda \in T^*$ satisfy $m := \lambda H_\sigma \geq 0$. Then there is an exact sequence of $\hat{G}_-$-modules

$$0 \to M \to \sigma\hat{G}_m \to \hat{G}_\lambda \to 0$$

such that

$$\begin{cases} M = 0 & m = 0 \\ M = \hat{G}_{s,\lambda} & m = 1 \\ 0 \to \hat{G}_{s,\lambda} \to M \to \sigma\hat{G}_{m-\sigma} \to 0 & m \geq 2 \end{cases}$$

**Proof.** Define a $\hat{G}_-$-submodule

$$M := \langle v_{m-2}, \ldots, v_{2-m}, v_{-m} \rangle$$

Since

$$\sigma\hat{G}_\lambda^\pi / M = \langle v_m \rangle$$

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The calculation
Proof.

is an isomorphism.

We have

\[ \text{Lemma 3.2.7. Let } \pi : G \to GL(E) \text{ be a holomorphic representation and consider the restricted representation } \pi : H \to GL(E). \text{ Then the map} \]

\[ G \times E \to G/H \times E, [g,v] \mapsto (gH,g^\pi v) \]

is an isomorphism.

**Proof.** The calculation

\[ [g,v] = [gh,h^{-\pi}v] \mapsto (ghH,(gh)^{\pi}h^{-\pi}v) = (gH,g^\pi v) \]

shows that the map (??) is well-defined. It is clearly surjective. To show injectivity, let \((gH,g^\pi v) = (g_1H,g_1^\pi v_1)\). Then \(h := g^{-1}g_1 \in H\) and \(h^{-\pi} v = g_1^{-\pi} g^\pi v v_1\). Thus \([g,v] = [gh,h^{-\pi}v] = [g_1,v_1]\).

**Proposition 3.2.8.** Let \(V\) be a (holomorphic) \(\mathfrak{g}^-\)-module and let \(\lambda \in T^*\) satisfy \(\lambda H_\sigma = -1\). Then

\[ H^k(G \times (\mathfrak{g}^-/\mathfrak{g}^- \otimes V)) = 0 \quad \forall \ k \geq 0. \]

**Proof.** We have

\[ ^{\mathfrak{g}^-}G_- \times (\mathfrak{g}^-/\mathfrak{g}^- \otimes V) = (^{\mathfrak{g}^-}G_- \times \mathfrak{g}^-/\mathfrak{g}^-) \otimes (^{\mathfrak{g}^-}G_- \times V) \]

and the condition \(\lambda H_\sigma = -1\) implies that

\[ ^{\mathfrak{g}^-}G_- \times \mathfrak{g}^-/\mathfrak{g}^- = \mathcal{O}(-1) \]

as a line bundle over \(^{\mathfrak{g}^-}G_-/\mathfrak{g}^- \equiv P^1\). By Proposition ?? it follows that

\[ H^q(^{\mathfrak{g}^-}G_- \times \mathfrak{g}^-/\mathfrak{g}^-) = H^q(P^1,\mathcal{O}(-1)) = 0 \]

for \(q = 0,1\). On the other hand,

\[ ^{\mathfrak{g}^-}G_- \times V = P^1 \times V \]

is a trivial vector bundle by Lemma 3.2.7, since \(V\) carries a representation of \(^{\mathfrak{g}^-}G_- \supset \mathfrak{g}^-\). Therefore

\[ H^q(^{\mathfrak{g}^-}G_- \times (\mathfrak{g}^-/\mathfrak{g}^- \otimes V)) = H^q((^{\mathfrak{g}^-}G_- \times \mathfrak{g}^-/\mathfrak{g}^-) \otimes (^{\mathfrak{g}^-}G_- \times V)) = 0 \]
for all $q$. The fibration
\[ \sigma G_+/\hat{G}_- \to G/\hat{G}_- \to G/\sigma \hat{G}_- \]
induces the **Leray spectral sequence**
\[ H^{p+q}(G \times (\hat{G}_-^\lambda \otimes V)) = H^p(G \times H^{q}(\sigma \hat{G}_-^\lambda \times (\hat{G}_-^\lambda \otimes V)). \]
The assertion follows.

For each simple root $\sigma$, the reflection $s_\sigma \in W$ acts on $t^*$ by
\[ s_\sigma(\lambda) := \lambda - (\lambda H_\sigma)\sigma = \lambda - (\lambda|\sigma)\sigma. \]
These reflections generate the Weyl group $W_T(G)$. Define an affine action of $W$ on $t^*$ by
\[ w \cdot \lambda := w(\lambda + \rho) - \rho. \]

**Lemma 3.2.9.** Let $\sigma \in \Pi$ be a simple root and $\lambda \in T^*$ such that $(\lambda + \rho)H_\sigma \geq 0$. Then, as $G$-modules,
\[ H^k(G \times C) \cong H^{k+1}(G \times C_{s_\lambda}) \quad \forall \ k \in \mathbb{Z}. \]

**Proof.** Assume $m = (\lambda + \rho)H_\sigma \geq 2$. Then Lemma 3.2.6 yields exact $\hat{G}_-$-module sequences
\[ 0 \to M \to \sigma \hat{G}_{\lambda+\rho} \to \hat{G}_{\lambda+\rho} \to 0, \]
\[ 0 \to \hat{G}_{s_\lambda} \to M \to \sigma \hat{G}_{\lambda+\rho-\sigma} \to 0 \]
Tensoring with $\hat{G}_{-\rho}$ yields exact $\hat{G}_-$-module sequences
\[ 0 \to M \otimes \hat{G}_{-\rho} \to \sigma \hat{G}_{\lambda+\rho} \otimes \hat{G}_{-\rho} \to \hat{G}_{\lambda} \to 0, \]
\[ 0 \to \hat{G}_{s_\lambda} \to M \otimes \hat{G}_{-\rho} \to \sigma \hat{G}_{\lambda+\rho-\sigma} \otimes \hat{G}_{-\rho} \to 0. \]
The corresponding sequences of holomorphic $\hat{G}$-module sheaves are also exact. Since $\rho H_\sigma = 1$, Proposition 3.2.8 yields
\[ H^k(G \times (\sigma \hat{G}_{\mu} \otimes \hat{G}_{-\rho})) = 0 \]
for $\mu = \lambda + \rho$ and $\mu = \lambda + \rho - \sigma$. Therefore the corresponding exact cohomology sequence implies
\[ H^k(G \times \hat{G}_{\lambda}) = H^{k+1}(G \times (M \otimes \hat{G}_{-\rho}) \equiv H^{k+1}(G \times \hat{G}_{s_\lambda}) \]
for all $k \in \mathbb{Z}$.  

**Lemma 3.2.10.** Let $\lambda \in \hat{T}^*$ with $\lambda + \rho \in \hat{T}^*_+$. Then, as $\hat{G}$-modules, for all $w \in W$
\[ H^k(G \times C) = H^{k+|w|}(G \times \hat{G}_{\lambda} \otimes C) \quad \forall \ k \in \mathbb{Z} \]

**Proof.** The proof uses induction over $\ell \geq 1$. For $\ell = 1$, we have $w = s_\sigma$ for some simple root $\sigma$ and Lemma 3.2.9 applies. Now let $w = s_{\sigma_0} \cdots s_\ell$ be a product of minimal length $\ell + 1$, with $s_k = s_{\alpha_k}$ for simple roots $\alpha_k$. Suppose we have
\[ s_{k-1} \cdots s_1 \alpha_0 = \alpha_k \]
for some $1 \leq k \leq \ell$. Then $(s_{k-1} \cdots s_1)s_0(s_1 \cdots s_k) = s_k$ and hence

$$w = s_0 \cdots s_{k-1}s_{k+1} \cdots s_\ell$$

has length $\leq \ell$, a contradiction. Thus (3.1) cannot happen for any $k$. Since

$$s_\sigma(\Delta^+ - \sigma) = \Delta^+ - \sigma$$

for any simple root $\sigma$, it follows that $w' := s_1 \cdots s_\ell$ satisfies

$$w'^{-1}a_0 = s_\ell \cdot s_1a_0 \in \Delta^+.$$

Putting $\sigma = a_0$ we have

$$(w' \cdot \lambda + \rho)H_\sigma = w'(\lambda + \rho)H_\sigma = (\lambda + \rho)H_{w' - 1} \geq 0$$

since $\lambda + \rho \in \mathcal{T}^*$. Applying the induction hypothesis to $w'$, of length $\leq \ell$, and Lemma 3.2.9 to $w' \cdot \lambda$ we obtain $G$-module isomorphisms

$$\text{H}^k(G \times C) = \text{H}^{k+\ell}(G \times C) = \text{H}^{k+\ell+1}(G \times C) = \text{H}^{k+\ell+1}(G \times C)$$

\[ \square \]

**Corollary 3.2.11.** Let $\lambda + \rho \in \mathcal{T}_{\ell}^*$. Then

$$\text{H}^k(G \times C) = 0 \quad \forall k > 0.$$ 

**Proof.** An element $w \in W$ of maximal length satisfies $w(\Delta^+) = \Delta^-$. This implies that $\ell = \ell(w) = \dim \mathbb{C}G/\mathbb{C}G_\Delta$. Applying Lemma 3.2.10 we obtain

$$\text{H}^k(G \times C) = \text{H}^{k+\ell}(G \times C) = 0$$

for $k > 0$, since $k + \ell > \dim \mathbb{C}G/\mathbb{C}G_\Delta$. \[ \square \]

A linear form $\mu \in \mathcal{T}^*$ is called **regular**, if $\mu H_\alpha \neq 0$ for all $\alpha \in \Delta$. Then there exists a unique $w = w_\mu \in W$ such that $w(\mu) \in \mathcal{T}^*_{\ell}$. 

**Theorem 3.2.12.** (Bott) Let $\hat{G}_\lambda$ be irreducible with highest weight $\lambda$. If $\lambda + \rho$ is singular, then

$$\text{H}^k(\hat{G} \times \hat{G}_\lambda) = 0$$

for all $k \geq 0$. If $\lambda + \rho$ is regular, let $w \in W$ be the unique element such that $w(\lambda + \rho) \in \mathcal{T}^*_{\ell}$. Then

$$\text{H}^k(\hat{G} \times \hat{G}_\lambda) = \begin{cases} \hat{G}_{w,\lambda} & \text{if } k = |w| \\ 0 & \text{if } k \neq |w| \end{cases}.$$ 

Here $\hat{G}_{w,\lambda}$ is 'the' irreducible $\hat{G}$-module of highest weight $w \cdot \lambda$.

**Proof.** We first consider line bundles over $\hat{G}_\Delta$. $(\Phi = 0)$. Choose $w \in W$ with $w \cdot \lambda + \rho = w(\lambda + \rho) \in \mathcal{T}^*_{\ell}$. Assume first that $\lambda + \rho$ is singular. Then $w \cdot \lambda + \rho = w(\lambda + \rho)$ is also singular. Thus there exists a
simple root $\sigma$ with $(w \cdot \lambda + \rho)H_\sigma = 0$. Hence $(w \cdot \lambda)H_\sigma = -\rho H_\sigma = -1$. Applying Lemma 3.2.10 and Proposition 3.2.8 (for $V = C$) we obtain

$$H^k(\hat{G} \times C) = H^{k+\ell}(\hat{G} \times C) = 0.$$  

Now let $\lambda + \rho$ be regular. Then $w$ is unique. Since $w \cdot \lambda + \rho \in \hat{T}^*$, Lemma 3.2.10 implies

$$H^{k+\ell}(\hat{G} \times C) = H^{k+\ell}(\hat{G} \times C) = H^k(\hat{G} \times C)$$

for all $k \in \mathbb{Z}$. For $k < 0$ this vanishes trivially. For $k > 0$ this vanishes by Corollary 3.2.11. For $k = 0$ we obtain

$$H^0(\hat{G} \times C) = H^0(\hat{G} \times C) = \mathcal{G}_{\lambda \times \lambda}$$

by the Borel-Weil Theorem 3.2.5. Here we use that $w \cdot \lambda + \rho \in \hat{T}^*$ since $w \cdot \lambda + \rho$ is regular, so that $(w \cdot \lambda + \rho)H_\sigma \geq 1$ for all simple roots $\sigma$, and therefore $(w \cdot \lambda)H_\sigma = (w \cdot \lambda + \rho)H_\sigma - \rho H_\sigma = (w \cdot \lambda + \rho)H_\sigma - 1 \geq 0$.  

The final step in the proof is achieved by

**Proposition 3.2.13.** Let $\Phi \hat{G}_{-}$ be an irreducible holomorphic representation of $\Phi \hat{G}_{-}$ with highest weight $\lambda$. Then, as $G$-modules,

$$H^k(\hat{G} \times \hat{G}_{-}) = H^k(\hat{G} \times \hat{G}_{-}) \quad \forall k \geq 0.$$  

**Proof.** We first show that

$$H^0(\Phi \hat{G}_{-} \times C) = \Phi \hat{G}_{-}$$

is an irreducible $\Phi \hat{G}$-module of highest weight $\lambda$. The parabolic subgroup $\Phi \hat{G}_{-}$ has a Levi decomposition

$$\Phi \hat{G}_{-} = \Phi \hat{T} \Phi \hat{G}_{<}$$

where $\Phi \hat{G}$ is semi-simple and connected, $\Phi \hat{T} \subset \hat{T}$ is a complex torus and $\Phi \hat{G}_{<}$ is the unipotent radical of $\Phi \hat{G}_{-}$. We have

$$\Phi \hat{G}_{-} = \Phi \hat{T} \Phi \hat{G}_{-} = \Phi \hat{T} \Phi \hat{G}_{<} \Phi \hat{G}_{>} = \Phi \hat{T} \Phi \hat{G}_{<} \Phi \hat{G}_{>} = \Phi \hat{G}_{-}.$$  

Since the unipotent radical always acts trivially we have to check the actions of $\Phi \hat{G}$ and $\Phi \hat{T}$ on $H^0(\Phi \hat{G}_{-} \times C)$. The semi-simple Lie group $\Phi \hat{G}$ has the Borel subgroup

$$\Phi \hat{G}_{-} = \Phi \hat{T} \Phi \hat{G}_{<}.$$  

It follows that

$$\Phi \hat{G}_{-} / \hat{G}_{-} = \Phi \hat{G}_{>} = \Phi \hat{G} / \Phi \hat{G}_{-}.$$  

Hence the inclusion map $\iota : \Phi \hat{G} \rightarrow \Phi \hat{G}_{-}$ induces a biholomorphic map $\iota : \Phi \hat{G} / \Phi \hat{G}_{-} \rightarrow \Phi \hat{G}_{-} / \hat{G}_{-}$ satisfying

$$\iota^*(\Phi \hat{G}_{-} \times C) = \Phi \hat{G}_{-} \times C,$$

where $\lambda' := \lambda|_{\Phi \hat{T}}$. This implies

$$H^0(\Phi \hat{G}_{-} \times C) = H^0(\Phi \hat{G} \times C)$$
Recall that for a positive definite hermitian metric

$$H^0(\mathcal{O}_G) = \mathcal{O}_G$$

is an irreducible $G$-module of highest weight $\lambda$. Let $f^0 \in \mathcal{O}_G$ be a highest weight vector. Since $H^0(\mathcal{O}_G)$ is irreducible under $G$, it is 'a fortiori' irreducible under $G\_\_$. In order to find its highest weight, recall that

$$H^0(\mathcal{O}_G) = \{ f \in \mathcal{O}_G : f(p) = b^{-\Phi} \} \forall p \in G\_\_,$$  

For $t \in \mathcal{O}_G$ and $p = sc{u}$, with $s \in \mathcal{O}_G$, $c \in \mathcal{O}_G$, $u \in \mathcal{O}_G<\_\_$ we have $t^{-1} = sc{u}^{-1}$ since $\mathcal{O}_G$ is the centralizer of $\mathcal{O}_G$ in $G$. Let $\lambda'' := \lambda_{s1}$. Then

$$(\cdot f^0)_p = f^0(t^{-1}_p) = f^0(t^{-1}_sc{u}) = f^0(t^{-1}_sc{u}) = f^0(sc{u}) = t^{\lambda''} f^0(sc{u}) = t^{\lambda''} f^0(p).$$

Hence $f^0$ is a highest weight vector for the weight $\lambda = (\lambda', \lambda'')$ under the action of $T = G/T \_\_$. Therefore (??) holds. Under the inclusion map $\iota : \mathcal{O}_G/G\_\_ \rightarrow G/G\_\_$ the pull-back is the homogeneous line bundle

$$\iota^*(\mathcal{O}_G) = \mathcal{O}_G/G\_\_. $$

Applying the Leray spectral sequence

$$H^{p+q}((\mathcal{O}_G) \times \mathcal{O}_G) = H^p(\mathcal{O}_G \times H^q(\mathcal{O}_G))$$

to the special case $q = 0$ and using (??) yields the assertion

$$H^k(\mathcal{O}_G) = H^k(\mathcal{O}_G \times \mathcal{O}_G) = H^k(\mathcal{O}_G \times \mathcal{O}_G).$$

Let $G/T$ be a compact flag manifold, where $T$ is the centralizer of a torus. Consider the complexified Lie algebra

$$\mathfrak{g}_C,$$

with Cartan subalgebra $\mathfrak{t}_C$ and Weyl group $W := N(T)/T$. For every $w \in W$ we obtain a Borel subalgebra $\mathfrak{g}_w \subset \mathfrak{g}_C$ such that

$$\mathfrak{g}_w \cap \mathfrak{g}_w = \mathfrak{t}_C.$$  

Torus, centralizer $C_G(T) G/C_G(T)$ flag domain

### 3.3 Compact Kähler Manifolds and Kodaira Embedding Theorem

#### 3.3.1 Chern Classes, Divisors and Positivity

Recall that for a positive definite hermitian metric

$$\sum_{i,j} h_{i,j} dz^i \overline{dz}^j$$
with \((h_{ij}) > 0\) positive definite, the associated \((1,1)\)-form

\[-i\omega := \sum_{i,j} h_{ij} dz^i \wedge d\overline{z}^j\]

is called a Kähler form if \(d\omega = 0\).

**Lemma 3.3.1.** On a ball \(U \subset \mathbb{C}^n\) a \((1,1)\)-form \(\omega\) is closed if and only if \(-i\omega = \partial \overline{\partial} K\) for some smooth real function \(K\).

**Proof.**

**Corollary 3.3.2.** The \((1,1)\)-form \(\omega\) associated with a 0-metric \(h\) is a Kähler form if and only if for a covering \((V_a)\) there exist smooth functions \(K_a : V_a \to \mathbb{R}\) such that

\[-i\omega|_{V_a} = \partial \overline{\partial} K_a\]

for all \(a\).

**Proof.** \(d\omega = 0\) if and only if \(\omega|_{V_a}\) is closed for all \(a\). □

A smooth function \(K : M \to \mathbb{R}\) is called 0-plurisubharmonic if the hermitian Levi form

\[\sum_{i,j} \frac{\partial^2 K}{\partial z^i \partial \overline{z}^j} dz^i d\overline{z}^j\]

is 0-positive. In this case

\[-i\omega = \partial \overline{\partial} K = \sum_{i,j} \frac{\partial^2 K}{\partial z^i \partial \overline{z}^j} dz^i \wedge d\overline{z}^j\]

is a Kähler form.

**Proposition 3.3.3.** \(\mathbb{P}^n\) is a Kähler manifold.

**Proof.** For \(0 \leq a \leq n\) let \(z' = (z^0, \ldots, z^a, \ldots, z^n)\) with \(z^j = \frac{\zeta^j}{\zeta^a}\). Define \(K_a : V_a \to \mathbb{R}\) by

\[K_a[\zeta] = \log(1 + (z'|z')) = \log(1 + \sum_{j \neq a} |z^j|^2) = \log(1 + \sum_{j \neq a} |\frac{\zeta^j}{\zeta^a}|^2) = \log|\zeta|^2 - \log|\zeta^a|^2.\]

Then on \(V_a \cap V_b\) we have

\[K_a[\zeta] - K_b[\zeta] = \log|\frac{\zeta^b}{\zeta^a}|^2 = \log|\sigma^b_a(z)|^2 = \log|\sigma^b_a(z)|^2 + \log|\sigma^a_b(z)|^2\]

evaluated on \(U_a \cap U_b\). Hence \(\partial \overline{\partial} K_a = \partial \overline{\partial} K_b\) on \(U_a \cap U_b\) and we obtain a global \((1,1)\)-form \(\omega\) with

\[-i\omega = \partial \overline{\partial} K\]

on \(U_a\), satisfying \(-i d\omega = (\partial + \overline{\partial}) \partial \overline{\partial} K_a = \partial \overline{\partial} \partial \overline{\partial} K_a = -\partial \overline{\partial} \partial \overline{\partial} K_a = 0\). For positivity, we compute

\[\partial \overline{\partial} K_a = \partial \log(1 + (z'|z')) = \frac{(z'|dz')}{1 + (z'|z')},\]

and hence

\[\partial \overline{\partial} K_a = \frac{\partial (z'|dz')}{1 + (z'|z')} + \left( \frac{1}{1 + (z'|z')} \right) \wedge (z'|dz') = \frac{1}{1 + (z'|z')} \sum_{j \neq a} dz^j \wedge d\overline{z}^j - \frac{(dz'|z') \wedge (z'|dz')}{(1 + (z'|z'))^2}\]

\[= \frac{1}{(1 + (z'|z'))^2} \sum_{i,j \neq a} (\delta_{ij} (1 + (z'|z')) - \overline{\sigma}^i z^j) dz^i \wedge d\overline{z}^j.\]

By Cauchy-Schwarz, the \(n \times n\) matrix \((\delta_{ij} (1 + (z'|z')) - \overline{\sigma}^i z^j)\) (for indices \(0 \leq i, j \leq n\) distinct from \(a\)) is positive definite. Hence \(\omega\) is a Kähler form. □
Definition 3.3.4. The Chern class of a cocycle line bundle $\mathcal{V} \times^\beta \mathbb{C}$ is the integral 2-cocycle

$$c(\mathcal{V} \times^\beta \mathbb{C}) \sim \frac{1}{2\pi i} (\log \beta_a^b + \log \beta_b^c + \log \beta_c^a) \in H^2(M, \mathbb{Z}).$$

Lemma 3.3.5. In terms of a metric $h^a$ satisfying $h^a|\beta|^2 = h^b$ the Chern class is cohomologous to the family of closed $(1, 1)$-forms

$$c(F) \sim \frac{1}{2\pi i} \overline{\partial} \partial \log h^a.$$

Proof. Identifying the Cech and Dolbeault description, the closed $(1, 1)$-forms $\partial \overline{\partial} \log h^a$ correspond to the Cech 2-cocycle $\log \beta^b_a + \log \beta^c_b + \log \beta^a_c$. \hfill \Box

Definition 3.3.6. A line bundle $L$ on $M$ is said to be 0 positive on an open subset $V \subset M$, if

$$ic(L) \sim \sum_{i,j} h_{ij} dz^i \wedge d\overline{z}^j,$$

where $(h_{ij})$ is 0 positive on $V$.

In general, let $D \subset M$ be a divisor (irreducible subvariety of codimension 1) in a compact complex manifold $M$. For a coordinate cover $(V_a)$ there exist holomorphic functions $f_a : V_a \to \mathbb{C}$ such that $D \cap V_a = \{m \in V_a : f_a(m) = 0\}$. We may choose $f_a$ such that

$$\beta_a^b(m) := \frac{f_a(m)}{f_b(m)}$$

is holomorphic and nowhere zero on $V_a \cap V_b$. Then the cocycle $(\beta_a^b) \in H^1(M, \mathbb{C}^\times)$ defines a line bundle $[D] = \mathcal{V} \times^{\beta^a} \mathbb{C}$ which corresponds to the divisor $D$.

3.3.2 Blow-up process

Let $L = \mathbb{C}^n$. For projective space $\mathbb{P}(L) = \mathbb{P}^{n-1}$ consider the open subsets

$$V_i := \{[\zeta] \in \mathbb{P}(L) : \zeta^i \neq 0\} \subset \mathbb{P}(L)$$

for $1 \leq i \leq n$, with coordinate charts

$$\tau^i : \mathbb{C}^{n\setminus i} \to V_i, \quad \zeta^{n\setminus i} \mapsto [\zeta^{n\setminus i}; 1].$$

The set

$$N := \{(z, [\zeta]) \in L \times \mathbb{P}(L) : z \in [\zeta]\} = \{(z, [\zeta]) \in L \times \mathbb{P}(L) : z_i \zeta_j = z_j \zeta_i \forall 1 \leq i, j \leq n\} = \{(z, [\zeta]) \in L \times \mathbb{P}(L) : \text{rank} \begin{pmatrix} z_1, \ldots, z_n \\ \zeta_1, \ldots, \zeta_n \end{pmatrix} \leq 1\}.$$

is an $n$-dimensional submanifold of $L \times \mathbb{P}(L)$. The canonical projection

$$\pi : N \to \mathbb{P}(L), \quad (z, [\zeta]) \mapsto [\zeta]$$

is a submersion. Consider the open covering

$$N_i := \{(z, [\zeta]) \in N : \zeta^i \neq 0\} = \pi^{-1}(V_i)$$

of $N$.
Lemma 3.3.7. We have coordinate charts
\[ \tau^i : \mathbb{C}^n \to N_i, \quad \tau^i(t', t^i) := (t^i(t', 1^i), [t', 1^i]), \]
where \( t' \in \mathbb{C}^{n\setminus i} \).

**Proof.** Let \( t'' \in \mathbb{C}^{n\setminus i,j} \). The equality
\[ \tau^i(t'', t^i, t^j) = (t^i(t'', 1^i, 1^j), [t'', 1^i, 1^j]) = \tau^i(s'', s^i, s^j) = \tau^i(s'', s^i, 1^j) \]
for \( t^i \neq 0 \neq t^j \) shows
\[ \tau^i_j(t'', t^i, t^j) = \left( \frac{1}{\partial t''}, \frac{1}{\partial t^j}, t^j \right). \]

Note that
\[ (s'', s^i, 1^j) = \left( \frac{1}{\partial t''}, \frac{1}{\partial t^j}, 1^j \right) = \frac{1}{\partial t''}(t'', 1^i, t^j). \]

\[ \square \]

**Proposition 3.3.8.** Let \( M \) be a complex \( n \)-manifold and \( p \in M \). Choose a chart \( \hat{\sigma} : \hat{U} \to \hat{V} \subset M \) such that \( 0 \in \hat{U} \subset \hat{L} \) and \( p = \hat{\sigma}_0 \in \hat{V} \). Put
\[ \hat{N} := \{(z, [\zeta]) \in N_1 : z \in \hat{U}_i \}. \]

Then the disjoint union
\[ \hat{M} := (M \setminus p) \cup \mathfrak{P}(L) \]
becomes a manifold such that \( M \setminus p \subset \hat{M} \) is an open subset and the bijective map
\[ F : \hat{N} \to (\hat{V} \setminus p) \cup \mathfrak{P}(L) \subset \hat{M}, \]
defined by
\[ F(z, [\zeta]) := \begin{cases} \hat{\sigma}_z \in \hat{V} \setminus p \subset M \setminus p & z \neq 0 \\ \{[\zeta] \in \mathfrak{P}(L) & z = 0 \end{cases}, \]
is biholomorphic.

**Proof.** Put \( \hat{U}_i := \tau_i(\hat{N}_i) \) and define charts \( \rho^i : \hat{U}_i \to \hat{M} \) by
\[ \rho^i(t) = F(t^i(t', 1^i), [t', 1^i]) = \begin{cases} \hat{\sigma}(t^i(t', 1^i)) & t^i \neq 0 \\ [t', 1^i] & t^i = 0 \end{cases}. \]

Then \( W_i := F(\hat{N}_i) = \rho^i(\hat{U}_i) \subset \hat{M} \) are open subsets and, in view of (??) and (??), there is a commuting diagram
\[
\begin{diagram}
\hat{N}_i \arrow{e}{F} \arrow{se}{\tau^i} \arrow{s}{\rho^i} & W_i \nend{diagram}
\]

We also have the charts \( \sigma^a : U_a \to V_a \) covering \( W_0 := M \setminus p \). Thus
\[ \hat{M} = W_0 \cup W_1 \cup \ldots \cup W_n. \]

We show that the collection \( \sigma^a, \rho^i \) are local charts for \( \hat{M} \) (the chart \( \hat{\sigma} \) is not needed any more.) Since \( \rho^i = (F \cup I) \circ \tau^i \) the transition maps
\[ \rho^i : = \rho \circ \rho^i = ((F \cup I) \circ \tau^i)^{-1} \circ ((F \cup I) \circ \tau^i) = \tau_i \circ \tau^i = \tau_i^i. \]

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are biholomorphic. Now let \( m \in V_a \cap \hat{M}_T = V_a \cap \hat{M}_S \). On \( V'_i := \tau_i(N_i \setminus \hat{T}) \), the diagram (??) simplifies to

\[
\begin{array}{ccc}
\hat{N}_T & \xrightarrow{F} & \hat{M}_S \\
\cup & & \downarrow \rho' \\
N_i \setminus \hat{T} & \xrightarrow{\tau_i} & V'_i 
\end{array}
\]

Thus the identity \( \sigma^a(m) = \rho'(w) = F(\tau^i(w)) \) implies

\[
\sigma_a \circ \rho^i(w) = z = (\sigma_a \circ F^{-1} \circ \tau^i)(w), \quad \rho_i \circ \sigma^a(m) = w = (\rho_i \circ \tau_i \circ F)(m).
\]

Since \( f \) is biholomorphic, the assertion follows.

The manifold

\[
\hat{M} = (M \setminus p) \cup P(L) = \hat{U}/\sim
\]

is called the blow-up of \( M \) at the point \( p \).

**Lemma 3.3.9.** The collection of holomorphic functions

\[
\hat{\beta}_0^i : W_i \cap W_0 \to \mathbb{C}^*, \quad \hat{\beta}_0^j(F(z, [\zeta])) := z^i,
\]

\[
\hat{\beta}_i^j : W_i \cap W_j \to \mathbb{C}^*, \quad \hat{\beta}_i^j(F(z, [\zeta])) := \frac{\zeta^j}{\zeta^i}
\]

form a cocycle on \( \hat{M} = (M \setminus p) \cup P(L) \).

**Proof.** Note that \( \hat{\beta}_0^i(F(z, [\zeta])) = z^i \) is non-zero since \( V_a \cap P(L) = \emptyset \). On \( W_0 \cap W_i \cap W_j \) we have

\[
\hat{\beta}_0^i(m) \hat{\beta}_i^j(m) = \beta^i(F^{-1}(m)) \beta_i^j(F^{-1}(m)) = \beta^i(F^{-1}(m)) = \hat{\beta}_0^j(m)
\]

and

\[
\hat{\beta}_0^i(m) \hat{\beta}_i^j(m) = \beta^i(F^{-1}(m)) \frac{1}{\beta^j(F^{-1}(m))} = \hat{\beta}_i^j(m).
\]

**Lemma 3.3.10.** The line bundle \( \hat{U} \times \mathbb{C} \) over \( \hat{M} = \hat{U}/\sim \) associated with the cocycle (??) corresponds to the divisor \( P(L) \subset \hat{M} \). In formulas

\[
[P(L)] = \hat{U} \times \mathbb{C}
\]

**Proof.** We have \( W_0 \cap P(L) = \emptyset \). If \( i > 0 \), then every point in \( W_i \) has the form

\[
m = F(t^i(t', 1^i), [t', 1^i]) = \begin{cases} \hat{\sigma}(t^i(t', 1^i)) & t^i \neq 0 \\ [t', 1^i] & t^i = 0 \end{cases}.
\]

Thus the intersection \( W_i \cap P(L) \) on the coordinate chart \( W_i \) correspond to \( t^i = 0 \). Therefore, on \( W_i \cap W_j \), the cocycle associated with \( P(L) \) is given by \( \frac{\hat{\beta}_i^j}{\hat{\beta}_0^j} = \hat{\beta}_i^j(m) \).

Our next goal is to determine the Chern class of this line bundle in terms of a metric. Choose a smooth function \( h : M \to \mathbb{R}^+ \) satisfying

\[
h(m) = 1
\]

for \( m \in M \setminus \hat{V} \), and

\[
h(\hat{\sigma}z) = (z|z)
\]

for all \( z \in \hat{U} \) with \( (z|z) < \epsilon \).
Lemma 3.3.11. The smooth functions

\[ \tilde{h}^0 : W_0 \to \mathbb{R}^+, \quad \tilde{h}^0(m) := h(m), \]

\[ \tilde{h}^i : W_i \to \mathbb{R}^+, \quad \tilde{h}^i(\rho'(t', t'')) := \frac{h(\hat{\sigma}(t'(t', 1''))}{|\rho'|^2} \]

define a 0-metric on the line bundle \([P(L)] = \tilde{U} \times \tilde{C} \).

Proof. If \(0 < |t| < \epsilon\) then (??) implies

\[ \frac{h(\hat{\sigma}(t'(t', 1'))}{|\rho'|^2} = \frac{\|t'(t', 1')\|^2}{|\rho'|^2} = \|t', 1')\|^2. \]

Therefore (??) defines a smooth function on \(W_i\). By Proposition ?? we need to verify the property

\[ \hat{h}^i_m = |\beta^i_j(m)|^2 \tilde{h}^j_m \]

for \(0 \leq i, j \leq n\) and \(m \in W_i \cap W_j\). Assume first \(i, j > 0\). Let \(m = F(z, [\zeta]) = \rho'(t', t', t) = \rho'(s', s', s') \in W_i \cap W_j\). Then \(z = t'(t', t', 1') = s'(s', s', 1')\) and \([\zeta] = [t''', t', t'] = [s'', s', 1']\), therefore \(t' = s' s', s' = t't'\) and \(\zeta = \zeta'(t', t', t') = \zeta'(s', s', 1')\). This implies \(t' \zeta' = s' s' \zeta = s' \zeta'\) and hence

\[ |\tilde{h}^i_j(m)|^2 \tilde{h}^j_i(m) = \left( \frac{\zeta^j_i}{\zeta^i_j} \right)^2 \frac{h(\hat{\sigma}(m))}{|s'|^2} = \frac{h(\hat{\sigma}(m))}{|\rho'|^2} = \tilde{h}^i_j(m). \]

On the other hand, if \(m \in W_0 \cap W_i\), for \(i > 0\), then \(m = \hat{\sigma}(s', t')\) for \(z = t'(t', 1') \in \tilde{U} \setminus 0\). Hence \(z' = t'\) and

\[ |\tilde{h}^i_0(m)|^2 \tilde{h}^0_i(m) = |z'|^2 \frac{h(\hat{\sigma}(m))}{|\rho'|^2} = h(\hat{\sigma}(m)) = \tilde{h}^0_i(m). \]

\[ \square \]

Corollary 3.3.12. The Chern class is given by the family of (1,1)-forms

\[ c([P(L)]) = c(\tilde{\mathcal{U}} \times \tilde{C}) \sim \left( \frac{1}{2\pi i} \tilde{\partial} \tilde{\partial} \log \tilde{h}^\ell \right)_\ell = 0. \]

Lemma 3.3.13. Let \(\pi : \tilde{M} \to M\) be the canonical projection, mapping \(P(L)\) to \(p\). Then the (1,1)-form

\[ \tilde{\partial} \partial (h \circ \pi + \log \tilde{h}^\ell) = \pi^* (\tilde{\partial} \partial h + \tilde{\partial} \partial \log \tilde{h}^\ell) \]

on \(W_\ell\) is 0-positive on a neighborhood of \(P(L) \subset \tilde{M}\).

Proof. For fixed \(\ell > 0\) and \((|t| t) < \epsilon\) the condition (??) implies

\[ \tilde{h}^\ell(\rho^\ell(t', t'')) = \|(t', 1')\|^2 = 1 + (t'|t'). \]

Putting \((t'|dt') := \sum_{j \neq \ell} t^j dt^j, (dt'|t') := \sum_{i \neq \ell} t^i dt^i\), we have

\[ \tilde{\partial} \log (1 + (t'|t')) = \frac{\tilde{\partial}(t'|t')}{1 + (t'|t')} = \frac{(t'|dt')}{1 + (t'|t')} \]

and hence

\[ \partial \tilde{\partial} \log (1 + (t'|t')) = \frac{\partial((t'|dt')}{1 + (t'|t')} - \frac{(dt'|t')}{1 + (t'|t') \wedge (t'|dt')} \]

\[ = \frac{1}{1 + (t'|t')} \sum_{i,j \neq \ell} \delta^j_i dt^i \wedge dt^j - \frac{(dt'|t') \wedge (t'|dt')}{(1 + (t'|t'))^2} \]

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\[
\frac{1}{(1 + (t'|t'))^2} \sum_{i,j \neq \ell} dt^i \wedge d\bar{t}^j \left( \delta^j_i (1 + (t'|t')) - \bar{t}^i t^j \right).
\]

The matrix \( A^i_\ell := \delta^j_i (1 + (t'|t')) - \bar{t}^i t^j \) corresponds to the hermitian form
\[
(\xi, \eta) \mapsto (\xi'|\eta')(1 + (t'|t')) - (\xi'|t') - (\xi'|\eta') = (\xi'|\eta') + \left( (\xi'|\eta')(t'|t') - (\xi'|t')(t'|\eta') \right).
\]

By Cauchy-Schwarz, this is semi-positive but vanishes on the hyperplane \( t^\ell = 0 \). We need the extra \( h \)-term for positivity: Near \( P(L) \) we have \( \|t\|^2 < \epsilon \) and hence \( P^*(\partial \partial h) = P^*(\partial \partial (z|z)) \), with \( z = t^\ell (t', t^\ell) \) and \( (z|z) = |t^\ell|^2 (1 + (t'|t')) \). Therefore
\[
\partial (z|z) = \partial \left( |t^\ell|^2 (1 + (t'|t')) \right) = (\bar{\partial} |t^\ell|^2)(1 + (t'|t')) + |t^\ell|^2 \partial (t'|t') = (t^\ell d\bar{t}^\ell)(1 + t'|t') + |t^\ell|^2 (t'|dt')
\]
and hence
\[
\partial (z|z) = \partial (t^\ell d\bar{t}^\ell)(1 + t'|t') - (t^\ell d\bar{t}^\ell) \wedge \partial (1 + t'|t') + (\partial |t^\ell|^2) \wedge (t'|dt') + |t^\ell|^2 \partial (t'|dt')
\]
\[
= dt^\ell \wedge d\bar{t}^\ell (1 + t'|t') - (t^\ell d\bar{t}^\ell) \wedge (dt^\ell|t') + (t^\ell d\bar{t}^\ell) \wedge (t'|dt') + |t^\ell|^2 \sum_{j \neq \ell} dt^j \wedge d\bar{t}^j.
\]

By (??), the divisor \( P(L) \) corresponds to \( z = 0 \). On \( W_\ell \) this is equivalent to \( t^\ell = 0 \). If \( t^\ell = 0 \) then
\[
\partial (z|z) = dt^\ell \wedge d\bar{t}^\ell (1 + t'|t').
\]

By continuity, it follows that the sum
\[
\partial (h \circ \pi + \log h) = \partial (z|z) + \log (1 + (t'|t'))
\]
is positive definite on a neighborhood of \( P(L) \) in \( W_\ell \).

**Proposition 3.3.14.** The Chern class of the divisor \( P(L) \subset \bar{M} \) is negative:
\[
c[P(L)] < 0
\]
near \( P(L) \).

**Proof.** By Lemma ?? \( P^*(\partial \partial h) + \partial \partial (\log h) \) is strictly positive near \( P(L) \). Now
\[
\partial (h \circ \pi) = \partial (\pi^* h) = d\bar{\partial} (\pi^* h) \sim 0
\]
is null-cohomologous in \( \bar{M} \). Hence \( c[-P(L)] \) is cohomologous to a positive line bundle. \( \square \)

### 3.3.3 Proof of the Kodaira embedding theorem

**Proposition 3.3.15.** The canonical line bundles \( K(\bar{M}) \) and \( K(M) \) are related by
\[
K(\bar{M}) = \pi^* K(M) + (n - 1)[P(L)],
\]
where \( \pi : \bar{M} \to M \) is the canonical projection.

**Proof.** Relative to the coordinate charts \( \sigma^a : U_a \to V_a \subset M \setminus p \) and \( \hat{\sigma} : \hat{U} \to \hat{V} \ni p \), the canonical line bundle of \( M \) is given by the cocycle
\[
\begin{cases}
J^a_b = \det \frac{\partial \sigma_b}{\partial \sigma_a} & \text{on } V_a \cap V_b \\
J_a = \det \frac{\partial \sigma_a}{\partial \hat{\sigma}} & \text{on } V_a \cap \hat{V}.
\end{cases}
\]

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Passing to $\tilde{M}$, with coordinate charts $\sigma^a : U_a \to V_a \subset W_0$ and $\rho^i : V_i \to W_i$, the cocycle (??) is supplemented by

$$
\begin{cases}
J^j_i := \det \frac{\partial \rho^a_i}{\partial \rho^a_j} & \text{on } W_i \cap W_j \\
J^i_a := \det \frac{\partial \rho^i}{\partial \rho^a} & \text{on } M_i \cap V_a.
\end{cases}
$$

Now let $s = \rho^j_i(t) = \rho^i \circ \rho^j(t)$. Then $\dot{\sigma}(\tau^i(t)) = \dot{\sigma}(\tau^j(s))$ and hence

$$(t^i t^j, t^i t^j) = (s^i s^j, s^i s^j).$$

Therefore

$$s^j = t^i t^j, s^i = \frac{t^i}{s^j} = \frac{1}{t^j}, s'' = \frac{t^i}{s^j} t'' = \frac{1}{t^j} t''$$

The partial derivatives $\frac{\partial}{\partial t^r} s^i = 0, \frac{\partial}{\partial t^r} s^j = 0, \frac{\partial}{\partial t^r} s^k = 0, \frac{\partial}{\partial t^r} s^i = \frac{-1}{(t^j)^2}, \frac{\partial}{\partial t^r} s^j = t^i, \frac{\partial}{\partial t^r} s^k = \frac{-k}{(t^j)^2}, \frac{\partial}{\partial t^r} s^i = 0, \frac{\partial}{\partial t^r} s^j = 0, \frac{\partial}{\partial t^r} s^i = \delta^i_k t^j$. Therefore

$$\frac{\partial \rho^i}{\partial t^r} = \begin{pmatrix}
0 & 1/t^j & -t''/t^j^2 \\
0 & 1/t^j & t''/t^j \\
1 & 0 & 0
\end{pmatrix}$$

with determinant

$$J^i_j = \det \frac{\partial \rho^i}{\partial t^r} = \frac{1}{(t^j)^n-2} \det \begin{pmatrix}
0 & t^i \\
-1/(t^j)^2 & t^i
\end{pmatrix} = \frac{1}{(t^j)^{n-1}}.$$

Finally, let $\sigma^a(w) = \rho^i(t) = \dot{\sigma}(m)$, where $z = (t^i t^j, t^i)$. Since $z^i = t^i, z^j = t^i t^j$ we obtain $\frac{\partial z^i}{\partial t^r} = 1, \frac{\partial z^j}{\partial t^r} = t^j, \frac{\partial z^i}{\partial t^r} = 0, \frac{\partial z^i}{\partial t^r} = t^j \delta^i_k$. Therefore

$$\dot{\sigma} = \begin{pmatrix}
1 & t^j \\
0 & t^i \end{pmatrix}$$

has the determinant

$$\det \dot{\sigma} = (t^j)^{n-1}.$$

It follows that

$$\det \frac{\partial w}{\partial t} = \det \frac{\partial w}{\partial z} \det \frac{\partial z}{\partial t} = (t^j)^{n-1} \det \frac{\partial w}{\partial z} = (t^i)^{n-1} \det \frac{\partial \sigma}{\partial z}.$$

$\square$

**Proposition 3.3.16.** Let $E$ be a strictly positive line bundle on $M$. For distinct $p, q \in M$ consider $\tilde{M} = (M_{(p)}^p \cup M_{(q)}^q)$, with canonical projection $\pi : \tilde{M} \to M$. Then for $k$ sufficiently large, the bundle $k\pi^*E - K_M - [P_p] - [P_q]$ on $\tilde{M}$ is strictly positive.

**Proof.** By Proposition ?? we have

$$F := k\pi^*E - K_M - [P_p] - [P_q] = \pi^*(kE - K_M) - n [P_p] - n [P_q].$$

It follows that

$$c(F) = \pi^*c(kE - K_M) - n c[P_p] - n c[P_q].$$

Since $E > 0$, there exists $k$ so large that $kE - K_M > 0$. Then $\pi^*(kE - K_M) \geq 0$ on $\tilde{M}$ and $\pi^*(kE - K_M) > 0$ on $\tilde{M} \setminus (P_p \cup P_q)$, where $\pi$ is biholomorphic. By Proposition 3.3.13, we have $c[P_p] < 0$ near $P_p$, and similarly, $c[P_q] < 0$ near $P_q$. Therefore (??) is strictly positive on $\tilde{M}$. $\square$

**Lemma 3.3.17.** Let $\pi : \tilde{M} = M_P^p \to M$ where $P \subset M_P^p$ is a divisor isomorphic to $P(L)$. For a line bundle $L$ over $M$ let $O_P \otimes L$ denote the sheaf of holomorphic sections $M \to L$ which vanish at $p$. Let $O_P \otimes \pi^*L$ denote the sheaf of holomorphic sections $\tilde{M} \to \pi^*L$ which vanish on $P$. Then $H^1(M, O_P \otimes \pi^*L) = 0$ implies $H^1(M, O_p \otimes L) = 0$. 

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Proof. Let \( V = (V_a) \) be an open covering of \( M \) such that \( \mathcal{L}|_{V_a} \) is trivial. Then \( \tilde{V}_a := \pi^{-1}(V_a) \) form an open covering \( \tilde{V} \) of \( M \). Let \( \Phi_{ab} \in Z^1(V, \mathcal{O}_p \otimes \mathcal{L}) \) be a 1-cocycle. Thus \( \Phi_{ab} : V_a \cap V_b \to \mathcal{L} \) are holomorphic sections vanishing on \( V_{ab} \cap \{p\} \). Therefore \( \Phi_{ab} \circ \pi : \tilde{V}_a \cap \tilde{V}_b \to \pi^* \mathcal{L} \) are holomorphic sections vanishing on \( \tilde{V}_{ab} \cap P \). For any sheaf \( S \), the canonical map
\[
H^1(\tilde{V}, S) \to H^1(\tilde{M}, S)
\]
is injective. Hence the assumption implies \( H^1(\tilde{V}, \mathcal{O}_p \otimes \pi^* \mathcal{L}) = 0 \). It follows that there exist holomorphic sections \( \psi_a : \tilde{V}_a \to \pi^* \mathcal{L} \) vanishing on \( \tilde{V}_a \cap P \) such that
\[
\Phi_{ab} \circ \pi = \psi_a - \psi_b.
\]
Suppose first that \( p \notin V_a \). Then \( \tilde{V}_a \subset \tilde{M} \setminus P \) and \( \pi : \tilde{V}_a \to V_a \) is biholomorphic. Therefore
\[
\Phi_a := \psi_a \circ \pi^{-1} : V_a \to \mathcal{L}
\]
is a holomorphic section vanishing on \( V_a \cap \{p\} = \emptyset \). Suppose now that \( p \in V_a \). Then the restriction \( \pi : \tilde{V}_a \setminus P \to V_a \setminus p \) is biholomorphic. Thus \( \psi_a \circ \pi^{-1} : V_a \setminus p \to \mathcal{L} \) is a holomorphic section. Since \( \mathcal{L}|_{V_a} \) is trivial, we may apply Hartogs’ extension theorem (for \( n > 2 \)) to obtain a holomorphic section \( \Phi_a : V_a \to \mathcal{L} \) satisfying
\[
\Phi_a|_{V_a \setminus p} = \psi_a \circ \pi^{-1}.
\]
Therefore \( \Phi_a \circ \pi = \psi_a \) on \( \tilde{V}_a \setminus P \). By continuity (or analytic continuation) it follows that
\[
\Phi_a \circ \pi = \psi_a
\]
on \( \tilde{V}_a \). This implies \( \Phi_a(p) = \psi_a(P) = 0 \). Thus we obtain a family \( (\Phi_a) \in H^0(V, \mathcal{O}_p \otimes \mathcal{L}) \) such that \( \Phi_{ab} = \Phi_a - \Phi_b \). Therefore \( (\Phi_{ab}) = 0 \in H^1(V, \mathcal{O}_p \otimes \mathcal{L}) \). Since \( V \) is arbitrary and, in general,
\[
H^q(M, S) = \lim_{\tilde{V}} H^q(V, S),
\]
the assertion follows. \( \square \)

**Theorem 3.3.18.** (Kodaira Vanishing Theorem) Let \( L > 0 \). Then
\[
H^q(X, \mathcal{O}_p \otimes L) = 0 \quad \forall \ p + q > n.
\]

**Proof.** Since \( L > 0 \) the square \( (d^A)^2 \) of its Chern connexion is the exterior multiplication \( \epsilon_\omega \) by a positive \((1,1)\)-form \( \omega \), which is therefore a Kähler metric. Consider the operators
\[
\square^A := \partial^A \ast \partial^A + \ast \partial^A \ast \partial^A, \quad \square^A := \overline{\partial}^A \ast \overline{\partial}^A + \ast \overline{\partial}^A \ast \overline{\partial}^A
\]
(Hodge-Laplacian). Thus
\[
\begin{align*}
\square^A : & \Lambda^p q \longrightarrow \Lambda^{p+1, q} \\
& |_\omega \quad |_{\epsilon_\omega} \\
\square^A & : \Lambda^{p+1, q} \longrightarrow \Lambda^{p-1, q-1} \\
& |_\omega \\
\square^A & : \Lambda^{p, q} \longrightarrow \Lambda^{p+1, q+1} \\
& |_{\epsilon_\omega + \epsilon_\omega} \\
\square^A & : \Lambda^{p-1, q-1} \longrightarrow \Lambda^{p, q} \\
& |_\omega
\end{align*}
\]
For Kähler manifolds we have the **Bochner-Kodaira-Nakano Identity**

\[ \square^A - \square^A = [\epsilon_\omega, \tau_\omega]. \]

By Dolbeault and Hodge theory we have

\[ H^p(M, \Omega^q \otimes L) = H^p_{\overline{\partial}}(M, L) \]

Now let \( u \in H^p_{\overline{\partial}}(M, L) = H^p(M, \Omega^q \otimes L) \). Then

\[ \int_M dm(\square^A u|u)_m = \int_M dm(\overline{\partial}^A \ast \overline{\partial}^A u + \ast \overline{\partial}^A \partial^A u|u)_m = \int_M dm\left( (\ast \overline{\partial}^A u|\ast \overline{\partial}^A u) + (\overline{\partial}^A u|\partial^A u) \right) \geq 0 \]

and hence

\[ \int_M (\square^A u|u) = \int_M (\square^A u + [\epsilon_\omega, \tau_\omega]|u) = \int_M (\square^A u|u) + \int_M ([\epsilon_\omega, \tau_\omega]|u) \geq \int_M ([\epsilon_\omega, \tau_\omega]|u). \]

If \( [\epsilon_\omega, \tau_\omega] \) is positive definite on each fibre, then \( \square^A u = 0 \) implies \( u = 0 \), i.e., \( H^p(M, \Omega^q \otimes L) = 0 \). Since \((\epsilon_\omega, \tau_\omega, (\text{deg} - n)L)\) is a so-called sl2-triple, we have

\[ ([\epsilon_\omega, \tau_\omega]|u|u) = (p + q - n)\|u\|^2 \]

which is positive for \( p + q > n \). \( \square \)

**Corollary 3.3.19.** If \( F - K_M > 0 \) then

\[ H^q(M, \mathcal{O} \otimes F) = 0 \quad \forall \ q > 0. \]

The **Kodaira map** is defined as follows: Let \( M \) be a Kähler manifold such that for all \( m \in M \) there exists \( \Phi \in \mathcal{O}(V_\omega \times \overline{\mathbb{C}}) \) with \( \Phi_m \neq 0 \). Then \( \mathcal{O}(V_\omega \times \overline{\mathbb{C}}) \) has finite dimension. We define a holomorphic map

\[ K : M \to \mathbb{P}(\mathcal{O}(V_\omega \times \overline{\mathbb{C}})^*) \]

by the hyperplane

\[ K^* := \{ \Phi \in \mathcal{O}(V_\omega \times \overline{\mathbb{C}}) : \Phi_m = 0 \} = \text{Ker} K^*_m \]

as the kernel of the evaluation map.

The **Kodaira embedding theorem** (first half) is the following:

**Theorem 3.3.20.** Let \( E > 0 \) be a positive line bundle on a compact Kähler manifold \( M \). Then, for \( k \) large enough, the Kodaira map (??) for \( F = kE \) is injective.

**Proof.** For \( p \neq q \) in \( M \) consider the subsheaf \( \mathcal{O}_{p,q} \otimes F \subset \mathcal{O} \otimes F \) of germs vanishing at \( p \) and \( q \). Then the so-called 'skyscraper sheaf' \( S = \mathcal{O} \otimes F/\mathcal{O}_{p,q} \otimes F \) has stalks \( S_p = \mathbb{C} \equiv S_q \), whereas \( S_m = 0 \) for \( m \in M \setminus \{ p, q \} \). The exact sheaf sequence

\[ 0 \to \mathcal{O}_{p,q} \otimes F \to \mathcal{O} \otimes F \to S \to 0 \]

induces an exact cohomology sequence

\[ 0 \to H^0(M, \mathcal{O}_{p,q} \otimes F) \to H^0(M, \mathcal{O} \otimes F) \xrightarrow{\kappa_{p,q}} H^0(M, S) = \mathbb{C} \to H^1(M, \mathcal{O}_{p,q} \otimes F) \to H^1(M, \mathcal{O} \otimes F) \to H^1(M, S) \to \ldots, \]

where \( \kappa_{p,q} (\Phi) = (\Phi_p, \Phi_q) \), for \( p \in V_a, \ q \in V_b \), is the 'double' evaluation map. In order to show that the Kodaira map (??) is injective, it thus suffices to show that \( H^1(M, \mathcal{O}_{p,q} \otimes F) = 0 \), since then \( \kappa_{p,q} \) is
surjective for every pair \( p \neq q \). Let \( \pi : \tilde{M} \to M \) be the canonical projection, with \( P := \pi^{-1}(p) = P_p \) and \( Q := \pi^{-1}(q) = P_q \). By Lemma ?? it suffices to show that \( H^1(\tilde{M}, \mathcal{O}_{P\cup Q} \otimes F) = 0 \). Now \( \tilde{F} := \pi^*(kE) - [P] - [Q] \) satisfies \( \tilde{F} - K_{\tilde{M}} > 0 \) for \( k \) large enough, and hence, by corollary ??, we have \( H^1(\tilde{M}, \mathcal{O} \otimes \tilde{F}) = 0 \). Since

\[
\mathcal{O} \otimes \tilde{F} = \mathcal{O} \otimes (\pi^*(kE) - [P] - [Q]) = \mathcal{O}_{P\cup Q} \otimes \pi^*(kE)
\]

we finally obtain \( H^1(\tilde{M}, \mathcal{O}_{P\cup Q} \otimes \pi^*(kE)) = 0 \) and hence \( H^1(M, \mathcal{O}_{p,q} \otimes (kE)) = 0 \). It follows that the sheaf \( \mathcal{O} \otimes (k\pi^*E - [P] - [Q]) \) on \( \tilde{M} \) satisfies

\[
H^1(M, \mathcal{O} \otimes (k\pi^*E - [P] - [Q])) = 0.
\]