ON SMOOTH PEAK-INTERPOLATION SETS FOR WEAKLY PSEUDOCONVEX DOMAINS

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Abstract

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A(\Omega)$ be the class of functions that are continuous on $\overline{\Omega}$ and holomorphic in Ω. A compact subset $K \subset \partial\Omega$ is called a peak-interpolation set for $A(\Omega)$ if given any function $f \in \mathcal{C}(K)$, $f \not\equiv 0$, there exists a function $F \in A(\Omega)$ such that $F|_K = f$ and $|F(\zeta)| < \sup_K |f|$ for every $\zeta \in \overline{\Omega} \setminus K$.

When Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, having a $\mathcal{C}^2$-boundary and $\boldsymbol{M}$ is a compact submanifold of $\partial\Omega$, it is known that $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$ if and only if $\boldsymbol{M}$ is complex-tangential, i.e. if $T_p(\boldsymbol{M}) \subset H_p(\partial\Omega)\ \forall p \in \boldsymbol{M}$. Here $H_p(\partial\Omega)$ denotes the maximal complex subspace of $T_p(\partial\Omega)$. Furthermore, it is known that for any bounded domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with $\mathcal{C}^2$-boundary, complex-tangency is a necessary condition for a submanifold $\boldsymbol{M} \subset \partial\Omega$ to be a peak-interpolation set for $A(\Omega)$. We extend known results to certain classes of *weakly* pseudoconvex domains. In particular, we show that for : (i) Smoothly bounded weakly pseudoconvex domains in $\mathbb{C}^2$, and (ii) Bounded convex domains in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundaries, if $\boldsymbol{M}$ is a sufficiently smooth submanifold of the boundary, being complex-tangential is still the sufficient condition for $\boldsymbol{M}$ to be a peak-interpolation set for $A(\Omega)$.

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Chapter 1

Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$, let $\overline{\Omega}$ denote its closure and $\partial\Omega$ its boundary. Let $A(\Omega)$ be the class of functions continuous on $\overline{\Omega}$ and holomorphic in Ω. If K is a compact set, let $\mathcal{C}(K)$ denote the class of continuous, complex-valued functions on K. A compact subset $K \subset \partial\Omega$ is called an **interpolation set for $\boldsymbol{A(\Omega)}$** if for any $f \in \mathcal{C}(K)$, there exists a function $F \in A(\Omega)$ such that $F|_K = f$. It is called a **peak-interpolation set for $\boldsymbol{A(\Omega)}$** if we can find an interpolating function $F \in A(\Omega)$ which satisfies $|F(\zeta)| < \sup_K |f|$ for every $\zeta \in \overline{\Omega} \setminus K$ (whenever $f \not\equiv 0$). It is called a **peak set for $\boldsymbol{A(\Omega)}$** if there exists a function $F \in A(\Omega)$ satisfying $F|_K \equiv 1$ and $|F(\zeta)| < 1$ for every $\zeta \in \overline{\Omega} \setminus K$. K is called a **zero set for $\boldsymbol{A(\Omega)}$** if there exists a function $F \in A(\Omega)$ such that $F^{-1}\{0\} = K$.

In this thesis, we study smooth peak-interpolation sets in the boundaries of smoothly bounded domains $\Omega \Subset \mathbb{C}^n$, $n \geq 2$. To make this statement precise, we consider the following questions :

(*) Let $\Omega \subset \mathbb{C}^n$, $n \geq 1$, be a bounded domain with smooth boundary. Can one characterize all peak and/or peak-interpolation sets for $A(\Omega)$?

(**) Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded domain with smooth boundary, and let $\boldsymbol{M}$ be a smooth submanifold of $\partial\Omega$. Under what conditions is $\boldsymbol{M}$ a peak-interpolation set for $A(\Omega)$?

Questions in the spirit of (*) above were first considered for domains in $\mathbb{C}$ – see, for instance, [8], [14], [28], [31]. Among the results concerning the algebra $A(\Omega)$, $\Omega \subset \mathbb{C}$, is the Rudin-Carleson characterization of interpolation sets for the disc algebra. Their results, that are contained in [8] and [28], are summarized by the following

Theorem *Let $\mathbb{U}$ be the open unit disc in $\mathbb{C}$. The interpolation, peak, peak-interpolation and zero sets for $A(\mathbb{U})$ coincide, and are precisely the subsets of $\partial\mathbb{U}$ having Lebesgue measure 0 (relative to $\partial\mathbb{U}$).*

This characterization motivates the question of whether there is a measure-theoretic characterization for the aforementioned sets for $A(\Omega)$ when $\Omega \subset \mathbb{C}^n$, $n \geq 2$. Yet, when $n > 1$ there are compact sets $K \subset \partial\Omega$ of Lebesgue measure 0 (relative to $\partial\Omega$) that are *not* zero sets. For instance, if $\Omega = \mathbb{B}^n$ – the open unit ball of $\mathbb{C}^n$, $n \geq 2$ – and $\zeta \in \partial\mathbb{B}^n$, the curve

$$K = \{\zeta e^{it} \in \mathbb{C}^n \mid t \in [-\pi, \pi]\ \} \subset \partial\mathbb{B}^n \tag{1.1}$$

cannot be a zero set, since it is the boundary of an analytic disc in $\mathbb{C}^n$.

One generalization of the Rudin-Carleson theorem is Bishop's result [3], which provides a characterization for a $K \subset \partial\Omega$, $\Omega \Subset \mathbb{C}^n$, to be a peak-interpolation set. Recall that $A(\Omega)$ is a proper closed subspace of $\mathcal{C}(\overline{\Omega})$. We can thus consider those bounded linear functionals of $\mathcal{C}(\overline{\Omega})$ – which are precisely the regular, complex Borel measures on $\overline{\Omega}$ – that annihilate $A(\Omega) \subset \mathcal{C}(\overline{\Omega})$. We will call a regular, complex Borel measure an **annihilating measure** if, viewed as a bounded linear functional of $\mathcal{C}(\overline{\Omega})$, it annihilates $A(\Omega)$. We can now state Bishop's theorem

Theorem (Bishop) *Let Ω be a bounded domain in $\mathbb{C}^n$. A compact set $L \subset \partial\Omega$ is a peak-interpolation set for $A(\Omega)$ if and only if for every annihilating measure $\mu \perp A(\Omega)$,*

$|\mu|(L) = 0$.

Recall that $|\mu|$ denotes the total variation of the measure μ.

Compact subsets $K \subset \partial\Omega$ that satisfy the measure-theoretic condition in the above theorem are called **null sets**. For domains $\Omega \Subset \mathbb{C}^n$, $n \geq 2$, the most that is known *in general* for $K \subset \partial\Omega$ is :

$$\begin{array}{ccccc}
 & & \begin{array}{c} K \text{ is a} \\ \text{peak set} \end{array} & & \\
 & & \Uparrow & \searrow & \\
\begin{array}{c} K \text{ is an} \\ \text{interpolation set} \end{array} & \Leftarrow & \begin{array}{c} K \text{ is a} \\ \text{peak-interpolation set} \end{array} & \Rightarrow & \begin{array}{c} K \text{ is a} \\ \text{zero set} \end{array} \\
 & & \Updownarrow \ \text{(Bishop's theorem)} & & \\
 & & \begin{array}{c} K \text{ is a} \\ \text{null set} \end{array} & &
\end{array}$$

and if Ω is simply connected, then additionally

$$\begin{array}{c} K \text{ is a} \\ \text{zero set} \end{array} \Rightarrow \begin{array}{c} K \text{ is a} \\ \text{peak set} \end{array}$$

Tumanov [32] constructed a peak/peak-interpolation set in the unit sphere of $\mathbb{C}^3$ having Hausdorff dimension 2.5. Henriksen [22] showed that for any smoothly bounded domain $\Omega \Subset \mathbb{C}^n$, $\partial\Omega$ contains a peak set of Hausdorff dimension $2n - 1$ (note that the Hausdorff dimension of $\partial\Omega$ is $2n-1$). The latter result suggests – in contrast to the counterexample (1.1), which has Hausdorff dimension 1 – that peak/peak-interpolation/zero sets can be quite large in a certain measure-theoretic sense. These examples suggest that it is highly non-trivial to provide a simple geometric characterization of *all* peak/peak-interpolation sets for bounded domains in $\mathbb{C}^n$, $n \geq 2$. It is reasonable, therefore, to restrict attention to *smooth* peak-interpolation sets. This leads to question (**). In this thesis, we provide some answers to the latter question.

The questions (*) and (**) have mostly been studied in the case of pseudoconvex domains. To define pseudoconvexity, we recall the following notations and definitions.

Let $S \subset \mathbb{C}^n,\ n \geq 2$, be a smooth real hypersurface, and let $p \in S$. In this thesis, $T_p(S)$ will denote the real tangent space to S at p. As $T_p(S)$ is a real subspace contained in $\mathbb{C}^n$, we can define

$$H_p(S) := T_p(S) \cap iT_p(S).$$

$H_p(S)$ is the maximal complex subspace of $T_p(S)$ and for any $p \in S$, $\dim_{\mathbb{C}}[H_p(S)] = n-1$.

Definition 1.1 Let $\Omega \subset \mathbb{C}^n,\ n \geq 2$, be a domain with a $\mathcal{C}^2$ boundary. Then, Ω can be expressed as $\Omega = \{z \in \mathbb{C}^n | \rho(z) < 0\}$, where ρ is a real-valued function of class $\mathcal{C}^2$ defined in a neighbourhood of Ω satisfying $\nabla\rho(\zeta) \neq 0\ \forall\zeta \in \partial\Omega$. Ω is **pseudoconvex** if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(\zeta) v_j \overline{v_k} \geq 0 \quad \forall v \in H_\zeta(\partial\Omega),\ \forall\zeta \in \partial\Omega.$$

If the form on the left-hand side above is strictly positive definite on $H_\zeta(\partial\Omega)$ for each $\zeta \in \partial\Omega$, we say that that Ω is **strictly pseudoconvex**.

Often, we will use the term **weakly pseudoconvex** for Ω to emphasize that Ω is not strictly pseudoconvex.

For *strictly* pseudoconvex domains, the null, peak, peak-interpolation and zero sets for $A(\Omega)$ coincide. This follows from the papers of Chaumat-Chollet [11] and Chollet [12].

In what follows, $\mathcal{C}^k$, $k = 1, 2, 3, ..., \infty$, will denote the class of functions that have continuous derivatives upto order k, while $\mathcal{C}_c^k$ will denote the class of functions in $\mathcal{C}^k$ having compact support. Given Ω, a closed subset $S \subset \partial\Omega$, $p \in S$ and a neighbourhood V of p, we will say that a function $G \in \mathcal{C}^\infty(V; \mathbb{C})$ **peaks on** S if $G|_{S\cap V} \equiv 1$ and $|G(z)| < 1 \; \forall z \in (\overline{\Omega} \setminus S) \cap V$.

The question (**) has been answered completely for strictly pseudoconvex domains. It turns out that to be a peak-interpolation set for $A(\Omega)$, it suffices for $\boldsymbol{M} \subset \partial\Omega$ to satisfy a certain directional condition. We say that a submanifold $\boldsymbol{M} \subset \partial\Omega$ is **complex-tangential** if $T_\zeta(\boldsymbol{M}) \subset H_\zeta(\partial\Omega) \; \forall \zeta \in \boldsymbol{M}$. Several authors have shown, under various regularity conditions on $\partial\Omega$ and $\boldsymbol{M}$, that if Ω is strictly pseudoconvex, and if $\boldsymbol{M} \subset \partial\Omega$ is complex-tangential, then every compact subset of $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$; see [7], [19], [21], [25], [29].

In his paper [25], Nagel shows that if $\Omega \Subset \mathbb{C}^n$ has a $\mathcal{C}^3$ boundary, and $\boldsymbol{M} \subset \partial\Omega$ is of class $\mathcal{C}^3$, then $\boldsymbol{M}$ is an interpolation set for $A(\Omega)$. That $\boldsymbol{M}$ is a peak-interpolation set (given that Ω is strictly pseudoconvex) follows from a theorem of Varopoulos [33]. Nagel shows, in fact, that given $f \in \mathcal{C}(\boldsymbol{M})$ the interpolating function can be obtained by integrating f with respect to a suitable complex measure on $\boldsymbol{M}$. By simplifying Nagel's integral expression, Rudin [29] showed that any complex-tangential manifold $\boldsymbol{M} \subset \partial\Omega$ of class $\mathcal{C}^1$ is a countable union of peak-interpolation sets for $A(\Omega)$. This implies, by Bishop's theorem, that $\boldsymbol{M}$ is a peak-interpolation set. Henkin & Tumanov's paper [21] uses quite a different strategy : they show that if $\boldsymbol{M} \subset \partial\Omega$ is complex-tangential, it has to be totally real (i.e. at no point $\zeta \in \boldsymbol{M}$ does $T_\zeta(\boldsymbol{M})$ contain a complex subspace), and they construct a smooth function G that is almost holomorphic with respect to $\boldsymbol{M}$ – i.e. G has the property that $\partial G/\partial\overline{z}_j$, $j = 1, ..., n$, vanish upto specified order on $\boldsymbol{M}$ –

and peaks on $\boldsymbol{M}$. This function G is used to set up a certain $\overline{\partial}$-equation which is used (together with Bishop's criterion) to show that $\boldsymbol{M}$ peak-interpolates.

Very little is known about (**) for weakly pseudoconvex domains. We know, in view of a result by Nagel & Rudin [26], that if $\boldsymbol{M}$ is a submanifold of $\partial\Omega$ of class $\mathcal{C}^2$ and is a peak-interpolation set for $A(\Omega)$, it is necessarily complex-tangential. We show, in this thesis, that for certain classes of weakly pseudoconvex domains and certain classes of submanifolds $\boldsymbol{M} \subset \partial\Omega$, being complex-tangential is sufficient for $\boldsymbol{M}$ to be a peak-interpolation set for $A(\Omega)$.

Consider a smoothly bounded, weakly pseudoconvex domain Ω of finite type. A definition of this notion for domains in $\mathbb{C}^2$ follows below (There are several notions of type for domains in $\mathbb{C}^n$, $n \geq 3$. We shall not define them at this juncture; the interested reader may refer to [6], [9], [13], [24].). A necessary condition for any (topologically) closed complex-tangential arc in $\partial\Omega$ to be a peak-interpolation set is that each point in $\partial\Omega$ be a peak point for $A(\Omega)$ – a point $\zeta \in \partial\Omega$ is called a **peak point** if there exists $F \in A(\Omega)$ that peaks at ζ. This we infer from the fact that any compact subset of a peak-interpolation set for $A(\Omega)$ is a peak-interpolation set (this is an elementary fact; see, for instance, [18, Theorem II/11.1] or [30, Theorem I/7.18], but it can also be inferred from Bishop's theorem). Showing that every boundary point is a peak point, for arbitrary pseudoconvex domains of finite type in $\mathbb{C}^n$, is an extremely difficult open problem in the theory of functions of several complex variables. In this thesis, we restrict ourselves to studying those domains Ω for which we know that every boundary point is a peak point for $A(\Omega)$. We remark here that the foregoing property is crucial to the proofs of the theorems we shall state presently. But first, we need a few more definitions.

Definition 1.2 Let $\Omega \subset \mathbb{C}^2$ be a bounded domain having a smooth boundary. Let $p \in \partial\Omega$. The **type** of p, denoted by $\tau(p)$, is the maximum order of contact that a 1-dimensional complex subvariety (of some open neighbourhood of p) can have with $\partial\Omega$ at p. The point p is said to be of finite type if $\tau(p) < \infty$. The domain Ω is said to be of **finite type** if there is an $N \in \mathbb{N}$ such that $\tau(p) \leq N$ for each $p \in \partial\Omega$.

Definition 1.3 A domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with a $\mathcal{C}^2$ boundary, given by $\Omega = \{x \in \mathbb{R}^N | \rho(x) < 0\}$, where ρ is a real-valued function of class $\mathcal{C}^2$ defined in a neighbourhood of Ω, is convex when the quadratic form

$$T_p(\partial\Omega) \ni v \mapsto \sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) v_j v_k, \quad p \in \partial\Omega,$$

is positive semi-definite at each $p \in \partial\Omega$. We say that Ω is **weakly convex** if there are boundary points at which the above form is *not* strictly positive definite.

The quadratic form presented above is known as the **second fundamental form for $\partial\Omega$**.

Every boundary point of a bounded, pseudoconvex finite-type domain in $\mathbb{C}^2$ having $\mathcal{C}^\infty$ boundary is a peak point for $A(\Omega)$. This fact comes from a paper by Fornaess & Sibony [16] (also see [1] and [15]). Every boundary point of a bounded convex domain having a smooth boundary that contains no complex-planar sets is a peak point for $A(\Omega)$. In the case of a smoothly bounded convex domain whose boundary contains no *line segments*; this fact is particularly easy to see – given a boundary point ζ, the function

$$G(\zeta, \cdot) \, : \, \overline{\Omega} \ni z \mapsto \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta)(\zeta_j - z_j)$$

(where ρ is a defining function for Ω) vanishes precisely on $H_\zeta(\partial\Omega)$ (viewed as an affine $\mathbb{C}$-hyperplane in $\mathbb{C}^n$) and for $z \in \overline{\Omega}$, $\mathrm{Re}[G(\zeta, z)] \geq 0$ and $G(\zeta, z) = 0 \Leftrightarrow z = \zeta$, whence

$$\overline{\Omega} \ni z \mapsto \exp^{-G(\zeta,z)}$$

peaks at ζ. The main theorems of this thesis concern the classes of bounded domains just discussed. They are :

Theorem 1.4 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ having smooth boundary, and let $\boldsymbol{M} \subset \partial\Omega$ be a smooth curve.*

(i) Let $\partial\Omega$ be of class $\mathcal{C}^\infty$ and Ω be of finite type. If $\boldsymbol{M}$ is complex-tangential, and if $\partial\Omega$ is of constant type along $\boldsymbol{M}$, then each compact subset of $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$.

(ii) Let Ω have real-analytic boundary, and let $\boldsymbol{M} \subset \partial\Omega$ be a real-analytic, complex-tangential curve. Then, each compact subset of $\boldsymbol{M}$ is a peak interpolation set for $A(\Omega)$.

Observe that in (ii) above, we do *not* assume that $\partial\Omega$ is of constant type along $\boldsymbol{M}$.

Theorem 1.5 *Let Ω be a bounded convex domain in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundary, and let $\boldsymbol{M}$ be a real-analytic submanifold of $\partial\Omega$. If $\boldsymbol{M}$ is complex tangential, then $\boldsymbol{M}$ (and thus, every compact subset of $\boldsymbol{M}$) is a peak-interpolation set for $A(\Omega)$.*

We remark that if a convex domain $\Omega \Subset \mathbb{C}^n$ is strongly convex, the conclusion of Theorem 1.5 is subsumed by the results stated earlier (since a strongly convex domain is strongly pseudoconvex). Theorem 1.5 therefore provides new information in the context of weakly convex domains.

The proof techniques for Theorems 1.4(ii) and 1.5 share some common features. Both proofs rely on the following three ingredients :

(a) Since $\boldsymbol{M}$ is real-analytic, for each $p \in \boldsymbol{M}$, $\boldsymbol{M}$ has the following local stratification near p :

$$\boldsymbol{M} \cap U_p = \cup_{j=0}^{\dim(\boldsymbol{M})} M_j, \qquad (U_p \text{ is some neighbourhood of } p) \tag{1.2}$$

where each M_j is a disjoint union of finitely many j-dimensional (not necessarily closed) real-analytic submanifolds of $\partial\Omega$ and each connected component of M_j is a closed submanifold of $U_p \setminus (\cup_{k=0}^{j-1} M_k)$, $j = 1, ..., \dim(\boldsymbol{M})$, so that each positive-dimensional stratum of $\boldsymbol{M} \cap U_p$ given by (1.2) satisfies a specific property dictated by the geometry of the problem. Specifically, if $\mathcal{M}$ is a positive-dimensional stratum, then

- In the setting of Theorem 1.4, $\partial\Omega$ is of constant type along $\mathcal{M}$.
- In the setting of Theorem 1.5, the second fundamental form for $\partial\Omega$ is strictly positive definite on $T_\zeta(\mathcal{M}) \subset T_\zeta(\partial\Omega)$ $\forall \zeta \in \mathcal{M}$.

(b) Each positive-dimensional stratum of $\boldsymbol{M} \cap U_p$ given by (1.2) is a countable union of peak-interpolation sets for $A(\Omega)$.

(c) Each point belonging to M_0 (as defined by (1.2)) is a peak point for the algebra $A(\Omega)$.

The organization of this thesis is as follows.

In Chapter 2, we present a technical fact which is instrumental to the proofs of both the main theorems. Let $p \in \boldsymbol{M}$ and let $\mathcal{M} \subset \partial\Omega$ be a positive-dimensional stratum of $\boldsymbol{M} \cap U_p$ given by the stratification (1.2). The proof of item (b) above relies on the ability – given $\zeta \in \mathcal{M}$ and an appropriately small open neighbourhood $V_\zeta \subset U_p$ of ζ and any bump function $\chi \in \mathcal{C}_c^\infty(V_\zeta;\mathbb{C})$ – to construct a sequence of functions $\{h_k\}_{k\in\mathbb{N}}$ such that : (i) $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$, and is uniformly bounded on $\overline{\Omega}$, (ii) $\lim_{k\to\infty} h_k(z) = 0 \ \forall z \in \overline{\Omega} \setminus (\mathcal{M} \cap V_\zeta)$, and (iii) $\lim_{k\to\infty} h_k(z) = \chi(z) \ \forall z \in \mathcal{M} \cap V_\zeta$. In Chapter 2, we demonstrate how the existence of such a sequence of $A(\Omega)$-functions results in item (b).

Chapter 3 is devoted to the proof of Theorem 1.4. The central idea of its proof is similar to that used in proving the main theorem in [21]. In Secn. 3.1 we show that for each $\zeta \in \boldsymbol{M}$, $\boldsymbol{M}$ as in Theorem 1.4(i), there exists a neighbourhood U of ζ and a function $G \in \mathcal{C}^\infty(U;\mathbb{C})$ that is almost holomorphic with respect to $\boldsymbol{M} \cap U$ and peaks on $\boldsymbol{M} \cap U$. In Secn. 3.2 we use the result of Chapter 2 to complete the proof of Theorem 1.4.

Chapters 4 and 5 relate to the proof of Theorem 1.5. In Chapter 4, we present certain geometric properties of smoothly bounded convex domains whose boundaries contain no line segments. Now let $\Omega \Subset \mathbb{C}^n$, $n \geq 2$, be a convex domain with real-analytic boundary. The local stratification of $\boldsymbol{M} \subset \partial\Omega$ – $\boldsymbol{M}$ real-analytic – referred to in item (a) above needs to be carried out with some care in the setting of Theorem 1.5. Specifically, we must find a local stratification of $\boldsymbol{M}$, as described in (1.2), so that if $\mathcal{M}$ is a positive-dimensional stratum, then the second fundamental form for $\partial\Omega$ is strictly positive definite on $T_\zeta(\mathcal{M}) \subset T_\zeta(\partial\Omega) \ \forall \zeta \in \mathcal{M}$. Theorem 4.3 shows that such a local stratification of $\boldsymbol{M}$ exists. It is in Chapter 4 that the hypothesis of real-analyticity gets used. Chapter 5 presents a series of estimates necessary for constructing a sequence $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ with the properties listed earlier. This construction, in view of Chapter

2, completes the proof of Theorem 1.5. The construction is carried out on each local stratum of $\boldsymbol{M}$ – as described above – at a time. The key result in this context is Theorem 5.5, which may be of independent interest.

Theorem 5.5 *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, weakly convex domain having a smooth boundary that contains no line segments, and let $\gamma : B_d(0; R) \to \partial\Omega$ be a smooth imbedding whose image is complex-tangential. Also assume that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}$ $\forall x$. There exists a $\varrho \equiv \varrho(\gamma) > 0$ such that if $f \in \mathcal{C}_c[B_d(0; \varrho); \mathbb{C}]$, then defining*

$$h_\delta(z) = \int_{B_d(0;\varrho)} \frac{\delta^d f(x)/G(x)\ dx}{\left\{\delta^2 + \sum_{j=1}^n \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]\right\}^d}\ , \quad z \in \overline{\Omega},$$

where G is defined as

$$G(x) = \int_{\mathbb{R}^d} \left\{1 + \frac{1}{2}\sum_{j,k=1}^n \Big(\partial^2_{j\bar{k}}\rho(\gamma(x))[d\gamma(x)v]_j\ \overline{[d\gamma(x)v]_k} + \partial^2_{jk}\rho(\gamma(x))[d\gamma(x)v]_j\ [d\gamma(x)v]_k\Big)\right\}^{-d} dv,$$

we have

(i) $\{h_\delta\}_{\delta>0} \subset A(\Omega)$ *and is uniformly bounded on* $\overline{\Omega}$,

(ii) $\lim_{\delta\to 0} h_\delta(z) = 0$ *if* $z \in \overline{\Omega} \setminus \gamma[B_d(0; \varrho)]$,

(iii) $\lim_{\delta\to 0} h_\delta[\gamma(s)] = f(s)$ $\forall s \in B_d(0; \varrho)$.

Notice that the γ in the result above plays the role of a local parametrization of a d-dimensional stratum ($d > 0$) $\mathcal{M}$. Theorem 5.5 is similar to a result by Rudin [29, Theorem 3.1], except that in our result Ω is *not* strongly convex. Yet, the condition $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}$ $\forall x$ (where $\mathfrak{N}_{\gamma(x)} \subseteq T_{\gamma(x)}(\partial\Omega)$ denotes the null space of the second fundamental form at $\gamma(x)$) allows us to obtain the same conclusions as in Rudin's theorem.

Chapter 2

A technical lemma

In this chapter, we present an abstract lemma that is instrumental to the proofs of both Theorem 1.4 and Theorem 1.5. In itself, it is a technical statement, but combined with explicit geometric and analytical propositions appearing in Secn. 3.2 and Theorem 5.5, it will be obvious how this theorem will yield the desired results.

Crucial to our proof is Bishop's theorem, which we stated in Chapter 1. Before stating the main theorem of this chapter, we make the following

Definition 2.1 Given an open set $V \subset \mathbb{R}^N$, a **bump function** f in V is a function belonging to $\mathcal{C}_c^\infty(V;[0,1])$ such that $\mathrm{int}[f^{-1}\{1\}] \neq \emptyset$.

The central theorem of this section is :

Lemma 2.2 *Let Ω be a bounded domain in $\mathbb{C}^n$ having smooth boundary and let $\mathcal{M}$ be a smooth submanifold of $\partial\Omega \cap \mathcal{U}$, where $\mathcal{U}$ is an open subset of $\mathbb{C}^n$. Assume that for each $\zeta \in \mathcal{M}$, there exists a small neighbourhood V_ζ of ζ with $V_\zeta \subset \mathcal{U}$ such that for any bump function $\chi \in \mathcal{C}_c^\infty(V_\zeta;[0,1])$ we can find a sequence of functions $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ (depending on χ) satisfying*

(i) $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ *is uniformly bounded on* $\overline{\Omega}$.

(ii) $\lim_{k\to\infty} h_k(z) = 0 \ \forall z \in \overline{\Omega} \setminus (\mathcal{M} \cap V_\zeta)$.

(iii) $\lim_{k\to\infty} h_k(z) = \chi(z)\ \forall z \in \mathcal{M} \cap V_\zeta$.

Then, $\mathcal{M}$ is a countable union of peak-interpolation sets for $A(\Omega)$.

Furthermore, if $\dim_{\mathbb{R}}(\mathcal{M}) = 1$, then, for the above conclusion to be true, it suffices to find sequences $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ associated with $\chi \in \mathcal{C}_c^\infty(V_\zeta; [0,1])$ that satisfy properties (i)–(iii) above only for those bump functions $\chi \in \mathcal{C}_c^\infty(V_\zeta; [0,1])$ for which $int[\chi^{-1}\{1\}] \cap \mathcal{M}$ is an arc.

Proof : Fix $\zeta \in \mathcal{M}$. Let $K \subset \mathcal{M} \cap V_\zeta$ be a compact subset of $\mathcal{M} \cap V_\zeta$. Let $\{\mathfrak{D}_\nu\}_{\nu\in\mathbb{N}}$ be a shrinking family of compact subsets of $\mathbb{C}^n$ such that

(a) $\mathfrak{D}_{\nu+1} \subset \mathrm{int}(\mathfrak{D}_\nu)$,

(b) $\cap_{\nu\in\mathbb{N}} \mathfrak{D}_\nu = K$,

(c) $\mathfrak{D}_\nu \subseteq V_\zeta$.

Let $\chi_\nu \in \mathcal{C}_c^\infty(V_\zeta;\ [0,1])$ be a bump function with

$$\chi_\nu|_{\mathfrak{D}_{\nu+1}} \equiv 1, \qquad \text{supp } \chi_\nu \subseteq \mathfrak{D}_\nu.$$

Finally, define $\{h_{k,\nu}\}_{k\in\mathbb{N}}$ to be the sequence of functions corresponding to χ_ν given by the hypothesis of this lemma.

Choose any $\mu \perp A(\Omega)$. By the bounded convergence theorem

$$0 = \lim_{k\to\infty} \int_{\overline{\Omega}} h_{k,\nu} d\mu = \int_{\mathcal{M}\cap V_\zeta} \chi_\nu d\mu.$$

Another passage to the limit yields $\mu(K) = 0$, and this is true for any $\mu \perp A(\Omega)$ and any $K \Subset \mathcal{M} \cap V_\zeta$. Let $\mathcal{V}_\zeta$ be any neighbourhood of $\zeta \in \mathcal{M}$ such that $\mathcal{V}_\zeta \Subset V_\zeta$. We have just shown that for any $\mu \perp A(\Omega)$, $|\mu|(\mathcal{M} \cap \overline{\mathcal{V}_\zeta}) = 0$. By Bishop's theorem and the fact that, by hypothesis, $\mathcal{M}\cap\overline{\mathcal{V}_\zeta}$ is a compact subset of $\partial\Omega$, it is a peak-interpolation set for $A(\Omega)$. Letting ζ vary over a countable dense subset of $\mathcal{M}$, we have the desired result.

Now consider the case when $\dim_{\mathbb{R}}(\mathcal{M}) = 1$. Fix some neighbourhood $\mathcal{U}_\zeta$ of $\zeta \in \mathcal{M}$ such that $\mathcal{U}_\zeta \Subset V_\zeta$ and $\mathcal{M} \cap \mathcal{U}_\zeta$ is an arc. Under the hypothesis that we can find $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ that satisfy properties (i)–(iii) for bump functions $\chi \in \mathcal{C}^\infty_c(V_\zeta; [0,1])$ for which $\mathrm{int}[\chi^{-1}\{1\}] \cap \mathcal{M}$ is an arc, and by reiterating the above argument, we conclude that $\mu(\mathcal{C}) = 0$ for any closed sub-arc $\mathcal{C} \subset \mathcal{M} \cap V_\zeta$, and any $\mu \perp A(\Omega)$. As μ is a regular measure, $\mu(\mathcal{A}) = 0$ for any *open* sub-arc $\mathcal{A} \subset \mathcal{M} \cap V_\zeta$; in particular $\mu(\mathcal{M} \cap \mathcal{U}_\zeta) = 0$. Consider a compact $K \subset \mathcal{M} \cap \mathcal{U}_\zeta$. Then

$$K = (\mathcal{M} \cap \mathcal{U}_\zeta) \ \setminus \ \coprod_{k\in\mathbb{N}} \mathcal{A}_k,$$

where each $\mathcal{A}_k$ is an open sub-arc of $\mathcal{M} \cap \mathcal{U}_\zeta$. By the additivity of μ, we conclude that $\mu(K) = 0$, and this is true for any $\mu \perp A(\Omega)$. Let $\mathcal{V}_\zeta$ be any neighbourhood of $\zeta \in \mathcal{M}$ such that $\mathcal{V}_\zeta \Subset \mathcal{U}_\zeta$. We have just shown that for any $\mu \perp A(\Omega)$, $|\mu|(\mathcal{M} \cap \overline{\mathcal{V}_\zeta}) = 0$. As before, this implies that $\mathcal{M}$ is a countable union of peak-interpolation sets.

$\blacksquare$

Chapter 3

Smooth peak-interpolation sets for domains in $\mathbb{C}^2$

In this chapter, we give a proof of Theorem 1.4. The domains considered in this chapter will, unless indicated otherwise, be bounded, smoothly bounded pseudoconvex domains in $\mathbb{C}^2$ of finite type (here, the notion of finite type is the one given in Definition 1.2).

Firstly, some notation : for a $\mathcal{C}^2$ function ϕ defined in some open set in $\mathbb{C}^n$, we will adopt the following notation

$$\partial_j \phi = \frac{\partial \phi}{\partial z_j}\ , \qquad \partial_{\bar{j}} \phi = \frac{\partial \phi}{\partial \overline{z}_j}\ ,$$

$$\partial^2_{jk} \phi = \frac{\partial^2 \phi}{\partial z_j \partial z_k}\ , \quad \partial^2_{j\bar{k}} \phi = \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}\ , \quad \partial^2_{\bar{j}k} \phi = \frac{\partial^2 \phi}{\partial \overline{z}_j \partial \overline{z}_k}\ .$$

If F is a smooth function defined in a neighbourhood of $0 \in \mathbb{R}^N$, we define

$\mathrm{In}(F) :=$ the leading homogeneous polynomial in the Taylor expansion of F around 0,

$\mathrm{ord}(F) :=$ the degree of $\mathrm{In}(F)$.

In what follows, $B(p;r)$ will denote the open Euclidean ball in $\mathbb{C}^2$ centered at $p \in \mathbb{C}^2$ and having radius r, while $D(a;r)$ will denote the open disc in $\mathbb{C}$ centered at $a \in \mathbb{C}$ and having radius r. Several parameters occur in our analysis and the independence of the quantitative estimates, occuring in the results below, from these parameters is of concern. We will express such estimates via the notation $X \lesssim Y$ – meaning that there is a constant $C > 0$, independent of all parameters, so that $X \leq CY$.

Definition 3.1 Let U be an open subset of $\mathbb{C}^n$ and let $S \subset U$ be a totally real submanifold of U of class $\mathcal{C}^\infty$. A function $f \in \mathcal{C}^\infty(U; \mathbb{C})$ is said to be **almost holomorphic** with respect to S if $\overline{\partial} f := \sum_{j=1}^n (\partial_{\overline{j}} f) d\overline{z}_j$ vanishes to infinite order on S. If $\phi \in \mathcal{C}^\infty(S; \mathbb{C})$, a $\mathcal{C}^\infty$ function F defined in some tubular neighbourhood of S is called an **almost holomorphic extension of ϕ** if F is almost holomorphic with respect to S and $F|_S = \phi$.

If every $\phi \in \mathcal{C}^\infty(S; \mathbb{C})$ admits an almost holomorphic extension, it is necessary for S to be totally real, since S must not admit any tangential Cauchy-Riemann equations induced by the presence of complex tangent vectors at some point in S. Conversely, every smooth function on a smooth, totally real submanifold S admits an almost holomorphic extension. See, for instance, Harvey & Wells [20]. We will need a much more modest version of this result in our proof of Theorem 1.4, namely

Proposition *Let U be an open set in $\mathbb{R}^n \subset \mathbb{C}^n$ and let ϕ be a complex-valued $\mathcal{C}^\infty$ function on U. There exists a $\mathcal{C}^\infty$ function $F : U + i\mathbb{R}^n \to \mathbb{C}$ that extends ϕ and such that $\overline{\partial} F$ vanishes to infinite order on $\{z \in U + i\mathbb{R}^n \mid Im(z) = 0\}$.*

A very brief sketch of the proof of the above result for is as follows : Let $\chi : \mathbb{R}^n \to [0, 1]$ be a $\mathcal{C}^\infty$ function with compact support satisfying $\chi \equiv 1$ near $0 \in \mathbb{R}^n$. We define

$$F(x + iy) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \phi}{\partial x^\alpha}(x)(iy)^\alpha \chi(c_{|\alpha|} y), \quad x \in U, \tag{3.1}$$

where $\{c_j\}_{j \in \mathbb{N}}$ is an increasing sequence of positive numbers that tends to $+\infty$ sufficiently rapidly to ensure that $F \in \mathcal{C}^\infty(U + i\mathbb{R}^n)$. By construction, F is the desired extension of ϕ.

Remark 3.2 Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Suppose $p \in \partial\Omega$, $\tau(p) = N$ (recall that $\tau(p)$ denotes the type of the point p) and that there exist local holomorphic coordinates $(U; \zeta_1, \zeta_2)$, near p, with respect to which $p = 0$ and with respect to which $(U \cap \partial\Omega)$ is defined by

$$\rho(\zeta) = A(\zeta_1) + O(v_2^2, |\zeta_1||v_2|) - u_2, \tag{3.2}$$

where $\zeta_k := u_k + iv_k$, and $A(\zeta_1) = O(|\zeta_1|^2)$, then

(1) N is the leading order in ζ_1 of A.

(2) Owing to the pseudoconvexity of Ω, N is an even number.

The above are the consequences of a computation on smoothly bounded pseudoconvex domains in $\mathbb{C}^2$ of finite type at $p \in \partial\Omega$, which is given in [17, Lecture 28]. Examining this calculation, we can infer that

(3) Suppose $\Phi = (\phi_1, \phi_2) : (U, p) \to (\mathbb{C}^2, 0)$ is a smooth change of coordinate – and write $\zeta_j = \phi_j(z_1, z_2)$, $j = 1, 2$ – such that $\overline{\partial}\phi_j$, $j = 1, 2$, vanishes to infinite order at p, and such that $(U \cap \partial\Omega)$ (with respect to these new coordinates) has a defining function of the form (3.2). Then, conclusions (1) and (2) above continue to hold.

3.1 Constructing an almost holomorphic function that peaks locally on M

Let $p \in \partial\Omega$. In this section, we will study $\partial\Omega$ near p with respect to a convenient system of local coordinates that are almost holomorphic with respect to $\boldsymbol{M}$ (near p) – where Ω

and $\boldsymbol{M}$ are as in Theorem 1.4(i). We first prove the following lemma which asserts the existence of local coordinates having the desired properties.

Lemma 3.3 *Let Ω be a bounded domain in $\mathbb{C}^2$ having smooth boundary and let $\boldsymbol{M} \subset \partial\Omega$ be a complex-tangential curve. Let $p \in \boldsymbol{M}$. There is a neighbourhood $\omega \ni p$ and a $\mathcal{C}^\infty$-diffeomorphism $\Phi : (\omega, p) \to (\mathbb{C}^2, 0)$ which is almost holomorphic with respect to $(\boldsymbol{M} \cap \omega)$ and so that, writing $(\zeta_1, \zeta_2) := \Phi(z_1, z_2)$, we have*

(1) $\Phi(\boldsymbol{M} \cap \omega) \subset \{(\zeta_1, \zeta_2) : Im(\zeta_1) = \zeta_2 = 0\}$.

(2) $\Phi(\partial\Omega \cap \omega)$ is defined by a defining function of the form

$$\rho(\zeta) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) - u_2,$$

where $\zeta_k := u_k + iv_k,\ k = 1, 2$; $A(\zeta_1) = O(|\zeta_1|^2)$, $R(\zeta_1, v_2) = O(|v_2|^2)$, and

$$A(u_1) = B(u_1) = 0,$$
$$\nabla A(u_1) = \nabla B(u_1) = 0,$$

for every u_1 near 0.

Proof : Without loss of generality, we may let p be the origin, and assume that, near p, $\partial\Omega$ is defined by

$$r(z_1, z_2) = h(z_1, \mathrm{Im}(z_2)) - \mathrm{Re}(z_2),$$

where $h(0) = 0$ and $\nabla h(0) = 0$.

Let ω be a neighbourhood of $p = 0$ and let $\mathcal{M} \subset \partial\Omega$ be the smooth 2-manifold of ω formed by the integral curves to the vector-field $-\mathbb{J}(\nabla r)$ passing through $(\boldsymbol{M} \cap \omega)$. $\mathcal{M}$ is totally real. Let $\gamma = (\gamma_1, \gamma_2) : (B(0; \varepsilon), (u_1, u_2) = 0) \to ((\mathcal{M} \cap \omega), p = 0)$ parametrize

$\mathcal{M}$ near $p = 0$ in such a way that Image$(\gamma|_{\{u_2=\mathrm{const}\}})$ are all integral curves to the unit vector field of $T(\mathcal{M}) \cap H(\partial\Omega)|_{\mathcal{M}}$, such that Image$(\gamma|_{\{u_2=0\}}) = (\boldsymbol{M} \cap \omega)$ and such that $\frac{\partial\gamma(0,0)}{\partial u_2} = -\mathbb{J}(\nabla r)(0,0)$. Shrinking ω if necessary, we construct a diffeomorphism $\varPhi : (\omega, p = 0) \to (\mathbb{C}^2, 0)$ of class $\mathcal{C}^\infty$, which is almost holomorphic with respect to $(\mathcal{M} \cap \omega)$, by defining

$$\varPhi^{-1}(\zeta_1, \zeta_2) = (\Gamma_1(\zeta_1, -i\zeta_2),\ \Gamma_2(\zeta_1, -i\zeta_2)) := \eta(\zeta_1, \zeta_2),$$

where $\zeta_k := u_k + iv_k,\ k = 1, 2$; and Γ_k is an almost holomorphic extension of $\gamma_k,\ k = 1, 2$. Notice that by construction

$$\begin{aligned} \varPhi(\mathcal{M} \cap \omega) &\subset \{(\zeta_1, \zeta_2) : v_1 = u_2 = 0\}, \\ \varPhi(\boldsymbol{M} \cap \omega) &\subset \{(\zeta_1, \zeta_2) : v_1 = \zeta_2 = 0\}. \end{aligned} \tag{3.3}$$

Now, $\varPhi(\partial\Omega \cap \omega)$ is defined by

$$\rho(\zeta_1, \zeta_2) = r \circ \varPhi^{-1}(\zeta_1, \zeta_2).$$

We expand ρ around the origin in a Taylor series. We make use of the fact that Γ_k are almost holomorphic with respect to $\{(\zeta_1, \zeta_2)|\ v_1 = v_2 = 0\}$ to get

$$\rho(\zeta) = \ 2\ \mathrm{Re}\left[\sum_{j=1}^{2} \frac{\partial r}{\partial z_j}(\eta(0,0)) \left\{ \frac{\partial \Gamma_j}{\partial \zeta_1}(0,0)\zeta_1 + (-i)\frac{\partial \Gamma_j}{\partial \zeta_2}(0,0)\zeta_2 \right\}\right] + O(|\zeta|^2).$$

Using the fact that

$$\frac{\partial \Gamma_j}{\partial \zeta_k}(0,0) = \frac{\partial \gamma_j}{\partial u_k}(0,0), \quad j, k = 1, 2$$

$$
\begin{aligned}
\rho(\zeta) = {} & 2\ \mathrm{Re}\left[\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial\gamma_j}{\partial u_1}(0,0)\zeta_1 + (-i)\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial\gamma_j}{\partial u_2}(0,0)\zeta_2\right] + O(|\zeta|^2)\\
= {} & 2\ \mathrm{Re}\left[(-i)\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(0,0))\frac{\partial\gamma_j}{\partial u_2}(0,0)\zeta_2\right] + O(|\zeta|^2)\\
= {} & -u_2 + O(|\zeta|^2).
\end{aligned}
$$

The second equality follows from the complex-tangency of Image$[\gamma(\boldsymbol{.},0)]$ which implies

$$
\sum_{j=1}^{2}\frac{\partial r}{\partial z_j}(\gamma(u_1,0))\frac{\partial\gamma_j}{\partial u_1}(u_1,0) = 0 \quad \forall u_1 \in (-\varepsilon,\varepsilon),
$$

and the last equality follows from the normalization condition on $\frac{\partial\gamma(0,0)}{\partial u_2}$. We see that the only term in the above expansion of first-order in either ζ_1 or ζ_2 is $-u_2$. Hence, the hypersurface $\Phi(\partial\Omega\cap\omega)$ is tangent to the hyperplane $\{(\zeta_1,\zeta_2)\in\mathbb{C}^2\,|u_2=0\}$ (in the remainder of this section, we will refer to this hyperplane as $\boldsymbol{H}$). Thus, we can find, near $0\in\mathbb{C}^2$, a defining function – and for convenience of notation, we will continue to call it ρ – which has the form

$$
\rho(\zeta) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1,v_2) - u_2, \tag{3.4}
$$

where $A(\zeta_1) = O(|\zeta_1|^2)$ and $R(\zeta_1,v_2) = O(|v_2|^2)$. Then, since $\Phi(\boldsymbol{M}\cap\omega)\subset\Phi(\partial\Omega\cap\omega)$, setting $v_1=\zeta_2=0$ in (3.4), we get

$$
A(u_1) = 0 \quad \forall (u_1,0)\in\Phi(\boldsymbol{M}\cap\omega). \tag{3.5}
$$

And since $\Phi(\mathcal{M}\cap\omega)\subset\Phi(\partial\Omega\cap\omega)$, setting $v_1=u_2=0$ in (3.4), we see that $B(u_1)v_2 + O(|v_2|^2) = 0\ \forall (u_1,v_2)$ belonging to a small neighbourhood 0. Thus

$$
B(u_1) = 0 \quad \forall (u_1,0)\in\Phi(\boldsymbol{M}\cap\omega). \tag{3.6}
$$

By construction, $(\nabla\rho)(u_1, v_2)$ is a normal vector to $\Phi(\mathcal{M}\cap\omega)$ $\forall(u_1, v_2) \in \Phi(\mathcal{M}\cap\omega)$. This implies that $T_{(u_1,v_2)}[\Phi(\partial\Omega\cap\omega)] = \boldsymbol{H}$ $\forall(u_1, v_2) \in \Phi(\mathcal{M}\cap\omega)$. Computing $(\nabla\rho)(u_1, v_2)$, we see that $\nabla A(u_1) + \nabla B(u_1)v_2 = 0$ $\forall(u_1, v_2)$ belonging to a neighbourhood of 0. Thus

$$\nabla A(u_1) = \nabla B(u_1) = 0 \quad \forall(u_1, 0) \in \Phi(\boldsymbol{M} \cap \omega). \tag{3.7}$$

By (3.5), (3.6) and (3.7), we have the desired result.

■

The central idea in the proof of Theorem 1.4 is to show that for each $p \in \boldsymbol{M}$, $\boldsymbol{M}$ as in Theorem 1.4(i), there is a small neighbourhood $U \ni p$ and a function $G \in \mathcal{C}^\infty(U;\mathbb{C})$ that is almost holomorphic with respect to $\boldsymbol{M} \cap U$ and peaks on $\boldsymbol{M} \cap U$. We now state this idea precisely.

Lemma 3.4 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type, and let $\partial\Omega$ be of class $\mathcal{C}^\infty$. Let $\boldsymbol{M} \subset \partial\Omega$ be a complex-tangential curve of class $\mathcal{C}^\infty$, and let $\partial\Omega$ be of constant type $2M$ along $\boldsymbol{M}$. Let $p \in \boldsymbol{M}$. There exists a neighbourhood $V \equiv V(p)$ of p and a uniform constant $C > 0$, and for any open set $U \Subset V$ such that $\boldsymbol{M} \cap U$ is an arc, there is a neighbourhood $V_1 \equiv V(p, U)$ of p satisfying $\boldsymbol{M} \cap V_1 = \boldsymbol{M} \cap V$ and a function $G \in \mathcal{C}^\infty(V_1)$ – G depending on p and U – which satisfies*

(1) $G^{-1}\{1\} = \boldsymbol{M} \cap \overline{U}$.

(2) $\overline{\partial}G$ *vanishes to infinite order on* $V \cap \boldsymbol{M}$.

(3) $|G(z)| \leq 1 - C\,dist[z, \boldsymbol{M} \cap V]^{2M}$ *for each* $z \in \overline{\Omega} \cap V_1$.

Proof : Let $\omega \ni p$ and $\Phi : (\omega, p) \to (\mathbb{C}^2, 0)$ be the change of coordinate described in Lemma 3.3. Let $\Phi(\partial\Omega \cap \omega)$ be defined by

$$\rho(\zeta_1, \zeta_2) = A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) - u_2. \tag{3.8}$$

Consider a point $(x_0, 0) \in \Phi(\boldsymbol{M} \cap \omega)$ and let

$$\varrho_{x_0}(\zeta_1^*, \zeta_2) = \mathcal{A}(\zeta_1^*) + \mathcal{B}(\zeta_1^*)v_2 + \mathcal{R}(\zeta_1^*, v_2) - u_2 \tag{3.9}$$

represent the expansion of ρ in (3.8) around $(x_0, 0)$, where $\zeta_1^* := \zeta_1 - x_0$.

Claim 1. Shrinking ω if necessary, there is a $c > 0$ such that

$$A(u_1 + iv_1) \geq cv_1^{2M}, \ \forall \zeta_1 \text{ such that } \zeta \in \Phi(\omega). \tag{3.10}$$

As $A(x_0) = B(x_0) = 0$ and $\nabla A(x_0) = \nabla B(x_0) = 0$ for each $(x_0, 0) \in \Phi(\partial\Omega \cap \omega)$, the right-hand side of (3.9) represents a defining function of the form (3.2). By Remark 3.2(3), the function $\mathcal{A}$ in (3.9) must vanish to order $2M$ at 0, whereby the function A in (3.8) must vanish *precisely* to order $2M$ at each $(u_1, 0) \in \Phi(\partial\Omega \cap \omega)$. Now write

$$A(u_1 + iv_1) = a_J(u_1)v_1^J + O(|v_1|^{J+1}), \tag{3.11}$$

where J is the least positive integer k such that $a_k \not\equiv 0$ near $u_1 = 0$. By our above remarks, it is clear that $J \leq 2M$. But, if $J < 2M$, then if $\widetilde{u_1}$ is such that $a_J(\widetilde{u_1}) \neq 0$, then A vanishes to order $< 2M$ at $u_1 + iv_1 = \widetilde{u_1}$, which contradicts our remarks above. Thus, $J = 2M$ in (3.11) and

$$A(u_1 + iv_1) = a_{2M}(u_1)v_1^{2M} + O(|v_1|^{2M+1}).$$

Furthermore, if $a_{2M}(0) = 0$, then A would vanish to order $> 2M$ at the origin, which is impossible. Now recall that Φ is almost-holomorphic with respect to $(\mathcal{M} \cap \omega)$. If, in fact

$(u_1 + iv_1, u_2 + iv_2)$ were *holomorphic* coordinates, then the pseudoconvexity of Ω would have implied that

$$\alpha : (u_1, v_1) \mapsto a_{2M}(u_1) v_1^{2M} \text{ is subharmonic,}$$

$$\Delta\alpha(u_1, v_1) > 0 \text{ off } \{v_1 = 0\}, \text{ and close to } 0.$$

This would have implied that $a_{2M}(u_1) > 0$ for u_1 close to 0 (the second statement above follows from an obvious calculation). In our present situation, the coordinates $(u_1 + iv_1, u_2 + iv_2)$ differ from holomorphic ones by terms vanishing to arbitrarily high order along $(\boldsymbol{M} \cap \omega)$. From the foregoing facts, we can conclude that shrinking ω if necessary

$$a_{2M}(u_1) > 0, \quad \forall (u_1, 0) \in \Phi(\partial\Omega \cap \omega).$$

From this final fact, we deduce (3.10). Hence the claim.

Claim 2. We can find a $\omega_1 \Subset \omega$ and a uniform constant $T > 0$ such that

$$B(\zeta_1)^2 \leq TA(\zeta_1), \ \forall \zeta \in \Phi(\overline{\Omega} \cap \omega_1). \tag{3.12}$$

To see this, we use a procedure originating in [4, Secn.4]. Write $q = (x_0, 0) \in \Phi(\boldsymbol{M} \cap \omega)$. The positivity of the Levi-form for $\partial\Omega$ on the complex tangent vectors implies that *were* $(u_1 + iv_1, u_2 + iv_2)$ *holomorphic coordinates* there would be a $\delta > 0$ such that the function $\mathfrak{L} : D(x_0; \delta) \times (-\delta, \delta) \to \mathbb{R}$ defined by

$$\mathfrak{L} = |\partial_{\bar{2}}\rho|^2 \ \partial^2_{1\bar{1}}\rho + |\partial_{\bar{1}}\rho|^2 \ \partial^2_{2\bar{2}}\rho - 2\mathrm{Re}[\partial_1\rho \ \partial_{\bar{2}}\rho \ \partial^2_{\bar{1}2}\rho]$$

would be non-negative (notice that $\mathfrak{L}$ is independant of u_2). In our present situation, however, $\mathfrak{L}(u_1, v_2) \geq 0 \ \forall (u_1, v_2) \in \Phi(\mathcal{M} \cap \omega)$.

Write

$$\mathfrak{L}(\zeta_1, v_2) = \mathfrak{L}^{(0)}(\zeta_1) + v_2 \mathfrak{L}^{(1)}(\zeta_1) + v_2^2 \mathfrak{L}^{(2)}(\zeta_1) + O(|v_2|^3).$$

It has been shown in [4] that if $\mathrm{ord}(B) < \mathrm{ord}(A)$, then

$$\mathrm{In}(\mathfrak{L}^{(0)}) = \frac{1}{4}\mathrm{In}(\partial^2_{1\bar{1}}A), \qquad \mathrm{ord}(\mathfrak{L}^{(0)}) = \mathrm{ord}(A) - 2, \tag{3.13}$$
$$\mathrm{In}(\mathfrak{L}^{(1)}) = \frac{1}{4}\mathrm{In}(\partial^2_{1\bar{1}}B), \qquad \mathrm{ord}(\mathfrak{L}^{(1)}) = \mathrm{ord}(B) - 2.$$

If already $2\mathrm{ord}(B) \geq \mathrm{ord}(A)$, then (3.12) would follow trivially. Thus, *assume* that $2\mathrm{ord}(B) < \mathrm{ord}(A)$. Write $r = \mathrm{ord}(B)$. We have

$$\frac{1}{\lambda^{2r-2}}\mathfrak{L}(\lambda(u_1 + i0), \lambda^r v_2) \geq 0, \quad \forall (u_1, v_2) \in (x_0 - \delta, x_0 + \delta) \times (-\delta, \delta), \ \forall \lambda \in \mathbb{R}_+.$$

But from (3.13) and our assumption

$$\lim_{\lambda \to 0_+} \frac{1}{\lambda^{2r-2}}\mathfrak{L}(\lambda u_1, \lambda^r v_2) = \frac{v_2}{4}\mathrm{In}(\partial^2_{1\bar{1}}B)(\zeta_1). \tag{3.14}$$

Write

$$B(u_1 + iv_1) = b_J(u_1)v_1^J + O(|v_1|^{J+1}),$$

where J is the least positive integer k such that $b_k \not\equiv 0$ near $u_1 = 0$. By Lemma 3.3(2), $J \geq 2$, whence $\mathrm{In}(B)$ is non-harmonic near 0. So, as v_2 occurs linearly in the right-hand side of (3.14), it is impossible that

$$\frac{v_2}{4}\mathrm{In}(\partial^2_{1\bar{1}}B)(u_1) \geq 0, \quad \forall (u_1, v_2) \in (x_0 - \delta, x_0 + \delta) \times (-\delta, \delta).$$

This results in a contradiction. So $2\mathrm{ord}(B) \geq \mathrm{ord}(A)$, which, in conjunction with the positivity of A, viz. (3.10), yields (3.12).

Finally, define $H : \Phi(\overline{\Omega} \cap \omega_1) \to \mathbb{C}$ by

$$H(\zeta) = \zeta_2 - \alpha\zeta_2^2 \ ,$$

for $\alpha > 0$ chosen appropriately large. We choose α as follows : Observe that

$$\begin{aligned}
&\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2 \\
&\qquad = \left(\frac{T}{\sqrt{2}}v_2 + \frac{B(\zeta_1)}{\sqrt{2}T}\right)^2 + \frac{1}{2T}[TA(\zeta_1) - B(\zeta_1)^2] - \frac{T^2}{2}v_2^2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2.
\end{aligned}$$

The first two terms of the right-hand side of the above equation are positive, in view of (3.12). So, we shrink ω_1 appropriately, and choose $\alpha > 0$ so large that

$$\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2 \geq 0, \quad \forall \zeta \in \Phi(\overline{\Omega} \cap \omega_1). \tag{3.15}$$

Now consider :

Case (i). $u_2 \geq 0$. Let $\varepsilon_1 > 0$ be so small that $(u_2 - \alpha u_2^2) \geq u_2/2$ for $\zeta \in \Phi(\overline{\Omega} \cap B(p;\varepsilon_1))$ and such that $B(p;\varepsilon_1) \subset \omega_1$. Then, for all such ζ, we have :

$$\begin{aligned}
\mathrm{Re}[H(\zeta)] &= (u_2 - \alpha u_2^2) + \alpha v_2^2 \qquad\qquad (3.16)\\
&\geq \frac{1}{2}u_2 + \alpha v_2^2 \\
&= \frac{1}{4}u_2 + \frac{\alpha}{2}v_2^2 + \frac{1}{4}(u_2 + 2\alpha v_2^2) \\
&\geq \frac{1}{4}u_2 + \frac{\alpha}{2}v_2^2 + \frac{1}{4}\left[\{A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2)\} + 2\alpha v_2^2\right] \\
&= \frac{1}{4}u_2 + \frac{1}{8}A(\zeta_1) + \frac{\alpha}{2}v_2^2 + \frac{1}{4}\left[\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + 2\alpha v_2^2\right] \\
&\gtrsim u_2^2 + v_2^2 + A(\zeta_1) \quad [\text{using } (3.15)].
\end{aligned}$$

Case (ii). $u_2 < 0$. Let $\varepsilon_2 > 0$ be so small that $(u_2 - \alpha u_2^2) \geq 2u_2$ for $\zeta \in \Phi(\overline{\Omega} \cap B(p;\varepsilon_2))$ and such that $B(p;\varepsilon_2) \subset \omega_1$. Then, for all such ζ, we have (we argue exactly as before)

$$\begin{aligned}
\mathrm{Re}[H(\zeta)] &\geq -u_2 + \frac{\alpha}{2}v_2^2 + 3\left(u_2 + \frac{\alpha}{6}v_2^2\right) \qquad\qquad (3.17)\\
&\geq -u_2 + \frac{3}{2}A(\zeta_1) + \frac{\alpha}{2}v_2^2 + 3\left[\frac{1}{2}A(\zeta_1) + B(\zeta_1)v_2 + R(\zeta_1, v_2) + \frac{\alpha}{6}v_2^2\right] \\
&\gtrsim u_2^2 + v_2^2 + A(\zeta_1) \quad [\text{using } (3.15)].
\end{aligned}$$

Now let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. From (3.10), (3.16) and (3.17) we see that there is a uniform constant $\kappa > 0$ such that

$$\begin{aligned} \mathrm{Re}[H(\zeta)] \geq\ & \kappa(u_2^2 + v_2^2 + v_1^{2M}) \\ \geq\ & \kappa \ \mathrm{dist}[\zeta, \varPhi(\boldsymbol{M} \cap B(p; \varepsilon_0))]^{2M} \quad \forall \zeta \in \varPhi(\overline{\Omega} \cap B(p; \varepsilon_0)). \end{aligned} \tag{3.18}$$

Write $\varPhi(\boldsymbol{M} \cap U) = (a, b),\ a < 0 < b$. Define the function ϕ as follows

$$\phi(u_1) = \begin{cases} \exp\{1/(u_1 - a)\}, & \text{if } u_1 < a \\ 0, & \text{if } a \leq u_1 \leq b \\ \exp\{-1/(u_1 - b)\}, & \text{if } u_1 > b, \end{cases}$$

Let $r > 0$ such that $B(0; r) \supset \varPhi[B(p; \varepsilon_0)]$, and let $R(\sigma)$ be the rectangle

$$R(\sigma) = \{(u_1 + iv_1) \in \mathbb{C} \mid |u_1| < r,\ |v_1| < \sigma\}.$$

Let $F \equiv F(u_1 + iv_1)$ be an almost holomorphic extension of ϕ as given by (3.1). We now appeal to the following fact

Fact : *For any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists a $\sigma \equiv \sigma(\varepsilon, N) > 0$ such that*

$$\mathit{Re}[F(u_1 + iv_1)] \geq -\varepsilon v_1^{2N}, \quad u_1 + iv_1 \in R(\sigma).$$

This follows from a computation as given in Noell [27, Lemma 4.7]. There is, thus, a $\sigma > 0$ sufficiently small such that

$$\mathrm{Re}[F(u_1 + iv_1)] \geq -\frac{\kappa}{2} v_1^{2M}, \quad u_1 + iv_1 \in R(\sigma). \tag{3.19}$$

We set

$$V_1(p,U) = B(p;\varepsilon_0) \cap \Phi^{-1}[\mathrm{Image}(\Phi) \cap (R(\sigma) \times \mathbb{C})].$$

From (3.18) and (3.19), we infer that the function $G(z) = (1-F) \circ \Phi(z) - H \circ \Phi(z)$ satisfies (1)-(3).

■

3.2 The proof of Theorem 1.4

The proof of Theorem 1.4(i) : Let $\boldsymbol{M}$ be as in Theorem 1.4(i), and fix $p \in \boldsymbol{M}$. Let $V(p)$ be the neighbourhood of p as given by Lemma 3.4. We will use Lemma 2.2 to provide a proof. Take V_p, in the notation of that theorem, to be $V(p)$. In the notation of Lemma 2.2, let $\chi \in \mathcal{C}_c^\infty(V_p;[0,1])$ be a bump function such that $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{M}$ is an arc. Write $U = \mathrm{int}[\chi^{-1}\{1\}]$. Now let $V_1 = V_1(p,U)$ and $G \in \mathcal{C}^\infty(V_1)$ be as given by Lemma 3.4.

Define

$$G_k(z) = \begin{cases} [G(z)]^k \ \chi(z), & \text{if } z \in \overline{\Omega} \cap V_1 \\ 0, & \text{if } z \in \overline{\Omega} \setminus V_1. \end{cases}$$

Also define

$$f_k(z) = \overline{\partial} G_k(z) = k[G(z)]^{k-1} \ \overline{\partial} G(z) \ \chi(z) + [G(z)]^k \ \overline{\partial}\chi(z). \tag{3.20}$$

For a $(0,1)$ form $\phi(z) = \phi_1(z_1)d\overline{z}_1 + \phi_2(z_2)\overline{z}_2$ defined on $\overline{\Omega}$, define

$$\|\phi\|_{\overline{\Omega}} := \max\{\sup_{\overline{\Omega}}|\phi_1(z)|, \ \sup_{\overline{\Omega}}|\phi_2(z)|\}.$$

By construction

$$\|G^k\,\overline{\partial}\chi\|_{\overline{\Omega}} \longrightarrow 0 \text{ as } k \to \infty. \tag{3.21}$$

Notice that $\overline{\partial}G$ vanishes to infinite order wherever $G(z) = 1$. Thus, for $j = 1, 2$

$$|k[G(z)]^{k-1}\,\partial_{\bar{j}}G(z)\,\chi_(z)| \lesssim k[1-C\mathrm{dist}[z, \boldsymbol{M}\cap V_p]^{2M}]^{k-1}|\partial_{\bar{j}}G(z)| \longrightarrow 0 \text{ uniformly as } k \to \infty. \tag{3.22}$$

From (3.21) and (3.22)

$$\|f_k\|_{\overline{\Omega}} \longrightarrow 0 \text{ as } k \to \infty. \tag{3.23}$$

Now consider the following $\overline{\partial}-$equations on Ω

$$\overline{\partial}u_k = f_k.$$

We need Lipschitz estimates on the solution of the $\overline{\partial}$-equation on pseudoconvex domains in $\mathbb{C}^2$ of finite type. Such estimates may be found, for instance, in the results of Chang, Nagel & Stein [10], which imply that

$$\|u_k\|_{\overline{\Omega}} \;\le \|u_k\|_{\Lambda^{1/N}(\overline{\Omega})} \;\le C^*\|f_k\|_{\overline{\Omega}}\;, \tag{3.24}$$

where N is a positive integer such that $\tau(\zeta) \le N$ for each $\zeta \in \partial\Omega$, $\Lambda^{1/N}(\overline{\Omega})$ is the class of Lipschitz functions and $C^* > 0$ is a constant depending only on Ω. From (3.23) and (3.24) we see that $\|u_k\|_{\overline{\Omega}} \to 0$, whence, defining

$$h_k(z) = G_k(z) - u_k(z), \quad \forall z \in \overline{\Omega}$$

we have a sequence of $A(\Omega)$ functions with

$$\lim_{k\to\infty} h_k(z) = \lim_{k\to\infty} G_k(z) = \begin{cases} \chi(z), & \text{if } z \in \boldsymbol{M}\cap V_p \\ 0, & \text{if } z \in \overline{\Omega}\setminus(\boldsymbol{M}\cap V_p). \end{cases}$$

Notice that, by construction, the sequence $\{h_k\}_{k\in\mathbb{N}}$ is uniformly bounded. $\{h_k\}_{k\in\mathbb{N}} \subset A(\Omega)$ satisfies properties (i)–(iii) in the hypothesis of Lemma 2.2 for the bump function $\chi \in \mathcal{C}_c^\infty(V_p;[0,1])$ such that $\mathrm{int}[\chi^{-1}\{1\}] \cap \boldsymbol{M}$ is an arc. Thus, we conclude (from the second part of Lemma 2.2) that any compact subset of $\boldsymbol{M}$ is a peak-interpolation for $A(\Omega)$.

■

The proof of Theorem 1.4(ii) : In the present situation, Ω is a bounded domain having a real-analytic boundary and $\boldsymbol{M}$ is a real-analytic complex-tangential curve. Let B be an open ball in $\mathbb{C}^2$ and let $\gamma : (-2\varepsilon, 2\varepsilon) \to \boldsymbol{M}$ be an injective real-analytic parametrization of $\boldsymbol{M}$ locally such that $\mathrm{Image}(\gamma|_{[-\varepsilon,\varepsilon]}) = (\boldsymbol{M} \cap \overline{B})$. Let $p \in (\boldsymbol{M} \cap \overline{B})$ be such that

$$\tau(p) = \min_{q\in \boldsymbol{M}\cap\overline{B}} \tau(q).$$

Write $\tau(p) = 2M$.

Recall that $H_p \otimes \mathbb{C}(\partial\Omega) = H_p^{1,0}(\partial\Omega) \oplus H_p^{0,1}(\partial\Omega)$, where $H \otimes \mathbb{C}(\partial\Omega)$ is the complexification of $H(\partial\Omega)$, and that $H_p^{1,0}(\partial\Omega)$ and $H_p^{0,1}(\partial\Omega)$ are the eigenspaces of the complex-structure map $\mathbb{J}$ corresponding to $+i$ and $-i$ respectively. Without loss of generality, we may assume that there is an open set $U \supset \overline{B}$ and a real-analytic section $\mathbf{L}$ of $H^{1,0}(\partial\Omega)|_U$ such that $\mathbf{L}(q)$ spans $H_q^{1,0}(\partial\Omega)$ and $\mathbf{L}(q) \in \{v \in H_q^{1,0}(\partial\Omega) : \|v\| = 1\}$ for each $q \in (\partial\Omega \cap U)$. Now consider the real-analytic function $\mathfrak{L} : S^1 \times I \to \mathbb{R}$ defined by

$$\mathfrak{L}(\zeta, t) = \sum_{\substack{j+k=2M \\ 1\leq j<2M}} \mathbf{L}^{j-1}\overline{\mathbf{L}}^{k-1}\langle[\mathbf{L}, \overline{\mathbf{L}}]\ ,\ \partial\rho\rangle(\gamma(t))\zeta^j\overline{\zeta}^k,$$

where I is an open interval around $[-\varepsilon, \varepsilon]$, S^1 is the unit circle in $\mathbb{C}$ and ρ is a defining function of $\partial\Omega$. Let t_0 be such that $\gamma(t_0) = p$. By the theorem of Bloom [5, Theorem

3.3], $\tau(p) = 2M$ implies that there exists a $\zeta_0 \in S^1$ such that $\mathfrak{L}(\zeta_0, t_0) \neq 0$. Then, by the real-analyticity of $\mathfrak{L}$, we conclude that

$$\{t \in [-\varepsilon, \varepsilon] : \mathfrak{L}(\zeta_0, t) = 0\} \text{ is a finite set } \mathfrak{S} \subset [-\varepsilon, \varepsilon].$$

Write $\mathfrak{S} = \{t_1, ..., t_N\}$. By [5, Theorem 3.3] again, in each connected component of $(\boldsymbol{M} \cap \overline{B}) \setminus \{\gamma_1(t_1), ..., \gamma(t_N)\}$, $\partial\Omega$ is of constant type $2M$. By Theorem 1.4(i), therefore

$$(\boldsymbol{M} \cap \overline{B}) \setminus \{\gamma(t_1), ..., \gamma(t_N)\} \text{ is a countable union of peak-interpolation sets.} \tag{3.25}$$

Recall that Ω is a bounded domain with real-analytic boundary. By Bedford & Fornaess [1], therefore, every point of $\partial\Omega$ is a peak point for $A(\Omega)$. So, each $\gamma(t_j)$, $j = 1, ..., N$; is a peak point for $A(\Omega)$. This last fact, in conjunction with (3.25), implies that $\boldsymbol{M}$ is a countable union of peak-interpolation sets for $A(\Omega)$, and that each compact subset of $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$.

$\blacksquare$

Chapter 4

Some results about convex domains in $\mathbb{C}^n$

We will divide the proof of Theorem 1.5 into two chapters. In order to show that a complex-tangential submanifold $\boldsymbol{M} \subset \partial\Omega$ – with $\boldsymbol{M}$ and $\partial\Omega$ real-analytic in the present context and Ω convex – we stratify $\boldsymbol{M}$ locally as described in Chapter 1, and show that each local stratum is a countable union of peak-interpolation sets. Proving that a local stratification having the desired properties exists requires several steps. The convexity of Ω and the real-analyticity of $\boldsymbol{M}$ and $\partial\Omega$ are essential to this process. This process and the results leading up to it may be of independent interest; hence we present them separately in this chapter.

In this chapter and the next, the notation $\langle\ ,\ \rangle$ will denote the usual real inner product on $\mathbb{R}^d$. Furthermore, given vectors $v, w \in \mathbb{R}^d$ and a real $d \times d$ matrix $M = [m_{jk}]$, the notation $\langle v \mid M \mid w\rangle$ will be defined as

$$\langle v \mid M \mid w\rangle := \sum_{j,k=1}^{d} m_{jk} v_j w_k.$$

$B_d(a; r)$ will denote the open ball in $\mathbb{R}^d$ centered at $a \in \mathbb{R}^d$ and having radius r, while $\mathbb{B}_d(a; r)$ will denote the closure of $B_d(a; r)$. Recall that ρ is a defining function for Ω. Let $\mathfrak{H}\rho$ denote the real Hessian of ρ. For a convex domain Ω, the second fundamental form for $\partial\Omega$ (see Definition 1.3) is positive semi-definite at each $p \in \partial\Omega$. Recall that

$\mathfrak{N}_p \subseteq T_p(\partial\Omega)$ is the null space of the second fundamental form at p, i.e. $\mathfrak{N}_p = \{v \in T_p(\partial\Omega) : \langle v \mid (\mathfrak{H}\rho)(p) \mid v\rangle = 0\}$. If U an open subset of $\mathbb{R}^d$ and $\phi : U \to \mathbb{R}^N$ is a mapping of class $\mathcal{C}^1$, then $d\phi$ will denote the derivative of ϕ.

The proof of Theorem 1.5 relies on three ingredients. We need to show that :

(1) If $\Omega \subset \mathbb{R}^N,\ N \geq 2$, is a convex domain having a smooth boundary that contains no line segments, there cannot be a smooth curve $\sigma : I \to \partial\Omega$ of class $\mathcal{C}^1$ with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ on an entire interval. Consequently – as we will show in Section 4.1 – if $\boldsymbol{M} \subset \partial\Omega$ is a smooth submanifold, then $\mathfrak{N}_\zeta \cap T_\zeta(\boldsymbol{M}) = \{0\}$ for each ζ belonging to an open, dense subset of $\boldsymbol{M}$.

(2) If $\Omega \subset \mathbb{R}^N,\ N \geq 2$, is a bounded convex domain with real-analytic boundary, $\boldsymbol{M} \subset \partial\Omega$ is a real-analytic submanifold and $p \in \boldsymbol{M}$, there is a neighbourhood $V \ni p$ and a stratification of $\boldsymbol{M} \cap V$ into finitely many real-analytic submanifolds (not necessarily closed) of $\partial\Omega\cap V$ such that if $\mathcal{M}$ is a stratum of positive dimension, $T_\zeta(\mathcal{M}) \cap \mathfrak{N}_\zeta = \{0\}\ \forall\zeta \in \mathcal{M}$.

(3) Let $\Omega \subset \mathbb{C}^n,\ n \geq 2$, be a bounded, weakly convex domain having a smooth boundary that contains no line segments, and let $\gamma : B_d(0;R) \to \partial\Omega$ be a smooth imbedding whose image is complex-tangential. Also assume that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x$. There exists a $\varrho > 0$ such that if $f \in \mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$, then defining

$$h_\delta(z) = \int_{B_d(0;\varrho)} \frac{\delta^d f(x)/G(x)\ dx}{\left\{\delta^2 + \sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]\right\}^d}\ ,\ z \in \overline{\Omega},$$

where G is defined as

$$G(x) = \int_{\mathbb{R}^d} \left\{ 1 + \frac{1}{2} \sum_{j,k=1}^{n} \Big(\partial^2_{j\overline{k}} \rho(\gamma(x))[d\gamma(x)v]_j \ \overline{[d\gamma(x)v]}_k + \right.$$
$$\left. \partial^2_{jk} \rho(\gamma(x))[d\gamma(x)v]_j \ [d\gamma(x)v]_k \Big) \ \right\}^{-d} dv,$$

we have :

(i) $\{h_\delta\}_{\delta>0} \subset A(\Omega)$ and is uniformly bounded on $\overline{\Omega}$,

(ii) $\lim_{\delta\to 0} h_\delta(z) = 0$ if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$,

(iii) $\lim_{\delta\to 0} h_\delta[\gamma(s)] = f(s)$ $\forall s \in B_d(0;\varrho)$.

Observe that the claim made in item (3) does *not* require real-analyticity of $\partial\Omega$. We remark that the object $\gamma[\mathbb{B}_d(0;\varrho)]$ – γ, ϱ as above – is the prototype for a d-dimensional stratum of the local stratification in item (2). If we take f in item (3) to be the pull-back, under γ, of a bump-function supported near p belonging to a d-dimensional local stratum, then the sequence $\{h_{1/k}\}_{k\in\mathbb{N}} \subset A(\Omega)$ would satisfy (i)–(iii) of Lemma 2.2. We use this lemma to complete our proof.

We prove item (1) in Lemma 4.1 below, (2) in Secn. 4.2, and we prove item (3) (and Theorem 1.5) in the next chapter.

4.1 Geometric lemmas

Lemma 4.1 *Let Ω be a convex domain in $\mathbb{R}^N$, $N \geq 2$, having a $\mathcal{C}^2$ boundary and containing no line segments in its boundary. Then, there is no smooth curve $\sigma : I \to \partial\Omega$ of class $\mathcal{C}^1$ with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ $\forall t \in I$ (where I is some interval of the real line).*

Proof : Assume the result is false. Let $\sigma : I \to \partial\Omega$ be a curve with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ $\forall t \in I$ (I is some interval). Let ρ be a defining function for Ω with $\|\nabla\rho\| = 1$ on $\partial\Omega$. For $t \in I$, set

$$n(t) = \nabla\rho(\sigma(t)) \in \mathbb{R}^N,$$
$$H(t) = (\mathfrak{H}\rho)(\sigma(t)) \in \mathbb{R}^{N\times N}.$$

We compute that

$$n'(t) = H(t)\sigma'(t). \tag{4.1}$$

Now, notice that as $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ by assumption, and as $H(t)$ is a symmetric matrix and is positive semi-definite on $T_{\sigma(t)}(\partial\Omega)$ $\forall t \in I$, we have

$$\langle H(t)\sigma'(t), v\rangle = \langle \sigma'(t),\ H(t)v\rangle = 0 \quad \forall v \in T_{\sigma(t)}(\partial\Omega),\ \forall t \in I.$$

The last equality follows from the fact that, since $H(t)$ is positive semi-definite on $T_{\sigma(t)}(\partial\Omega)$, for any $v \in T_{\sigma(t)}(\partial\Omega)$ we have

$$0 \le \langle \sigma'(t) + \lambda v\ ,\ H(t)(\sigma'(t) + \lambda v)\rangle \ = \ \lambda^2\langle v,\ H(t)v\rangle + 2\lambda\langle \sigma'(t),\ H(t)v\rangle \quad \forall \lambda \in \mathbb{R},$$

which forces $\langle \sigma'(t),\ H(t)v\rangle$ to vanish. Thus, by (4.1), $n'(t)$ is orthogonal to $T_{\sigma(t)}(\partial\Omega)$, $\forall t \in I$.

Next, observe that

$$\begin{aligned} &\langle n(t), n(t)\rangle = 1 \\ \Rightarrow\ &2\langle n'(t), n(t)\rangle = 0, \quad \text{[by differentiating the above equation]} \end{aligned}$$

whence $n'(t)$ is orthogonal to the outward unit normal at $\sigma(t)$ for each $t \in I$. We infer, thus, that $n'(t) = 0$ $\forall t \in I$. Thus n is constant on I.

Write $c = n(t)$, and define a function

$$G(s,t) = \langle \sigma(s) - \sigma(t), c \rangle, \;\; s,t \in I.$$

Clearly

$$\frac{\partial G}{\partial s} = \frac{\partial G}{\partial t} = 0, \quad [\text{since } \sigma'(\centerdot) \perp n(\centerdot)]$$

whence $G \equiv$ const. Since $G(s,s) = 0$, $G \equiv 0$. Thus,

$$\sigma(s) - \sigma(t) \in T_{\sigma(t)}(\partial\Omega). \tag{4.2}$$

By the convexity of Ω, the line segment joining $\sigma(s)$ and $\sigma(t)$ must lie in $\overline{\Omega}$. In view of (4.2), this means that the line segment joining $\sigma(s)$ to $\sigma(t)$ lies in $\partial\Omega$. This is a contradiction, whence the initial assumption if false.

■

Lemma 4.2 *Let Ω be a convex domain in $\mathbb{R}^N$, $N \geq 2$, having a $\mathcal{C}^2$ boundary and containing no line segments in its boundary, and let $\boldsymbol{M}$ be a submanifold of $\partial\Omega$ of class $\mathcal{C}^2$. Then, the set $\{p \in \boldsymbol{M} \mid T_p(\boldsymbol{M}) \cap \mathfrak{N}_p = \{0\}\ \}$ is open and dense in $\boldsymbol{M}$.*

Proof : Let $\dim_{\mathbb{R}}(\boldsymbol{M}) = d > 0$. Define

$$\mathfrak{S} = \{p \in \boldsymbol{M} \mid T_p(\boldsymbol{M}) \cap \mathfrak{N}_p \supsetneq \{0\}\ \}.$$

Let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be a non-singular parametrization of $\boldsymbol{M}$ near $p \in \boldsymbol{M}$ of class $\mathcal{C}^2$. We will show that $\mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]$ cannot contain an open subset of $\boldsymbol{M}$. Define

$$H(s) = d\gamma(s)^T (\mathfrak{H}\rho)(\gamma(s))\; d\gamma(s),$$

$$\mathcal{N}_s = \{v \in \mathbb{R}^d : \langle v \mid H(s) \mid v \rangle = 0\},$$

Consider the function $G : B_d(0;\varepsilon) \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ defined by

$$G : (s\ ;\ v_1, ..., v_d) \mapsto \langle d\gamma(s)v \mid (\mathfrak{H}\rho)(\gamma(s)) \mid d\gamma(s)v\rangle.$$

Observe that the matrix of derivatives with respect to v

$$d_v G(s,v) = 2v^T H(s) \in \mathbb{R}^{1\times d}.$$

Assume that $\mathrm{int}[\mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]] \neq \emptyset$. Without loss of generality, we may assume that there exists an $\varepsilon_* \in (0,\varepsilon]$ such that $\gamma[B_d(0;\varepsilon_*)] \subset \mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]$. We then have the following situations :

Case (i). There is a $v^0 \in S^{d-1}$ and $s_0 \in B_d(0;\varepsilon_*)$ such that $d_v G(s_0, v^0) \neq 0$. This means that $\mathrm{rank}_{\mathbb{R}} G(s_0, v^0)$ is maximal. Therefore, by the implicit function theorem, there is a $\delta \in (0,\varepsilon_*]$ and a non-vanishing vector-field $F = (F_1, ..., F_d) : B_d(s_0;\delta) \to \mathbb{R}^d \setminus \{0\}$ of class $\mathcal{C}^2$ such that

$$\begin{aligned} G(s, F(s)) &= 0 \quad \forall s \in B_d(s_0;\delta), \\ F(s_0) &= v^0. \end{aligned}$$

Case (ii). For each $s \in B_d(0;\varepsilon_*)$ and $v \in \mathbb{R}^d \setminus \{0\}$, $d_v G(s,v) = 0$. In this situation, $\mathcal{N}_s = \mathbb{R}^d$ $\forall s \in B_d(0;\varepsilon_*)$. In this case let $F = (F_1, ..., F_d) : B_d(0;\varepsilon_*) \to \mathbb{R}^d \setminus \{0\}$ be any non-vanishing vector field of class $\mathcal{C}^2$.

In either of the above cases, we have

$$F_1(s)\frac{\partial\gamma}{\partial s_1}(s) + \cdots + F_d(s)\frac{\partial\gamma}{\partial s_d}(s) \in \mathfrak{N}_{\gamma(s)} \quad \forall s \in B(s_0;\delta). \tag{4.3}$$

Let $\sigma : (-a,a) \to B_d(s_0;\delta)$ be the integral curve to F through s_0, i.e.

$$\begin{aligned} \sigma(0) &= \ s_0, \\ \sigma'(t) &= \ F(\sigma(t)) \quad \forall t \in (-a,a). \end{aligned}$$

Then

$$(\gamma \circ \sigma)'(t) = F_1(\sigma(t))\frac{\partial \gamma}{\partial s_1}(\sigma(t)) + \cdots + F_d(\sigma(t))\frac{\partial \gamma}{\partial s_d}(\sigma(t)) \quad \forall t \in (-a,a). \tag{4.4}$$

From (4.3) and (4.4),

$$(\gamma \circ \sigma)'(t) \in \mathfrak{N}_{\gamma\circ\sigma(t)} \quad \forall t \in (-a,a)$$

which is impossible, by Lemma 4.1. Thus, $\mathfrak{S}$ does not contain any open subsets of $\boldsymbol{M}$.

In particular $\boldsymbol{M} \setminus \mathfrak{S} \neq \emptyset$. Consider a point $p \in \boldsymbol{M} \setminus \mathfrak{S}$, and let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be as before. Consider $G : B_d(0;\varepsilon) \times S^{d-1} \to \mathbb{R}$ defined exactly as above. $G^{-1}[\mathbb{R} \setminus \{0\}]$ is an open set and $G^{-1}[\mathbb{R} \setminus \{0\}] \supset \{0\} \times S^{d-1}$, since $p \in \boldsymbol{M} \setminus \mathfrak{S}$. From this, we infer that there is an $\boldsymbol{M}$-open neighbourhood of p contained in $\boldsymbol{M} \setminus \mathfrak{S}$. This last fact completes the proof.

■

4.2 A stratification theorem

In this section, we shall prove the fact stated in item (2) above. A key fact that we will use to do so is the structure theorem for real-analytic subvarieties of $\mathbb{R}^N$, $N \geq 2$. This theorem is due to Łojasiewicz [23], which we now state.

Theorem (Łojasiewicz) : *Let F be a non-constant real-analytic function defined in a neighbourhood of $0 \in \mathbb{R}^N$, and assume that $V(F) = F^{-1}\{0\} \ni 0$. Then, there is a small* *neighbourhood* $U \ni 0$ *such that* $V(F) \cap U$ *has the decomposition*

$$V(F) \cap U = \cup_{j=0}^{N-1} S_j,$$

where each S_j *is a finite, disjoint union of (not necessarily closed)* j*-dimensional real-analytic submanifolds contained in* U*, such that each connected component of* S_j *is a closed real-analytic submanifold of* $U \setminus \left(\cup_{k=0}^{j-1} S_k\right)$*,* $j = 1, \ldots, (N-1)$*.*

We note that although the above theorem describes the local structure of the zero-set of a single real-analytic function, it, in fact, describes the local structure of a variety near the origin. This is because, given finitely many real-analytic functions $f_1, \ldots, f_M$ that vanish at the origin, their set of common zeros is exactly the zero-set of the real-analytic function $F := |f_1|^2 + \cdots + |f_M|^2$.

We remark that the idea of using Łojasiewicz's theorem to stratify the set of weakly pseudoconvex points for domains with real-analytic boundaries has appeared in several works in the past. We adapt this idea to serve an analogous purpose for the submanifold $\boldsymbol{M} \subset \partial\Omega$. The interested reader may refer to [2, Secn. 2] for a modern exposition of Łojasiewicz's theorem.

The following theorem is a precise statement of item (2). We clarify here that if $U \subset \mathbb{R}^N$ is an open subset, the phrase "$S \subset U$ is a not necessarily closed submanifold contained in U" – encountered below – implies that there is an open subset $V \subset U$ of which S is a submanifold.

Theorem 4.3 *Let* Ω *be a bounded convex domain in* $\mathbb{R}^N$*,* $N \geq 2$*, having real-analytic boundary, and let* $\boldsymbol{M} \subset \partial\Omega$ *be a* d*-dimensional real-analytic submanifold. Let* $p \in \boldsymbol{M}$*. There is a neighbourhood* $V \ni p$ *such that*

$$\boldsymbol{M} \cap V = \cup_{j=0}^{d} M_j, \tag{4.5}$$

where

(i) Each M_j is a disjoint union of finitely many (not necesarily closed) j-dimensional real-analytic submanifolds contained in V.

(ii) Each connected component of M_j is a closed, real-analytic submanifold of $V \setminus \left(\cup_{k=0}^{j-1} M_k\right)$, $j = 1, ..., d$.

(iii) For any $j \neq 0$, if $M_{j,\alpha}$ is a connected component of M_j, $\mathfrak{N}_\zeta \cap T_\zeta(M_{j,\alpha}) = \{0\}$ $\forall \zeta \in M_{j,\alpha}$.

Proof : Fix $p \in \boldsymbol{M}$. Let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be a real-analytic parametrization of $\boldsymbol{M}$ near p such that $\text{rank}_{\mathbb{R}}[d\gamma(x)]$ is maximal $\forall x$. Consider the real-analytic function $\mathfrak{F} : B_d(0;\varepsilon) \to \mathbb{R}$ defined by

$$\mathfrak{F}(x) = \det\left[d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x))\ d\gamma(x)\right].$$

The matrix in the above expression is simply the pull-back of the Hessian $\mathfrak{H}\rho$ by γ. By Lemma 4.2, $\mathfrak{F} \not\equiv 0$. Without loss of generality, we may assume that if $\mathfrak{F}^{-1}\{0\} \neq \emptyset$, then $\mathfrak{F}^{-1}\{0\} \ni 0$. By Łojasiewicz's theorem [23], there is a neighbourhood $U \ni 0$, $U \subseteq B_d(0;\varepsilon)$, such that

$$\mathfrak{F}^{-1}\{0\} \cap U = \cup_{j=0}^{d-1} \mathcal{S}_j, \tag{4.6}$$

where each $\mathcal{S}_j$ is a disjoint union of finitely many j-dimensional real-analytic submanifolds, and each connected component of $\mathcal{S}_j$ is a closed submanifold of $U \setminus \left(\cup_{k=0}^{j-1} \mathcal{S}_k\right)$, $j = 1, ..., d-1$. Write $\mathcal{S}_d = U \setminus \left(\cup_{j=0}^{d-1} \mathcal{S}_j\right)$.

We plan to demonstrate the present result by induction. We make the following inductive hypothesis :

For $m < d-1$ we have, shrinking U if necessary, a stratification of U

$$U = \cup_{j=0}^{d} S_j, \tag{4.7}$$

where,

(a) Each S_j is a disjoint union of finitely many (not necesarily closed) j-dimensional real-analytic submanifolds contained in U.

(b) Each connected component of S_j is a closed, real-analytic submanifold of $U \setminus \left(\cup_{k=0}^{j-1} S_k\right)$, $j = 1, ..., d$.

$(c)_m$ For each $k = 0, ..., m$ and each connected component $S_{d-k,\alpha}$, of S_{d-k} , $T_{\gamma(x)}[\gamma(S_{d-k,\alpha})] \cap \mathfrak{N}_{\gamma(x)} = \{0\}$ $\forall x \in S_{d-k,\alpha}$.

$(d)_m$ $\cup_{j=0}^{(d-m-1)} S_j$ is a real-analytic subvariety of U.

Consider the real-analytic subvariety $\widetilde{V}$ of U given by

$$\widetilde{V} = \left(\cup_{j=0}^{(d-m-1)} S_j\right) \cap \{x \in U : \mathrm{rank}_{\mathbb{R}} \left[d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x))\ d\gamma(x)\right] \leq (d-m-2)\},$$

where the S_j's come from the stratification in (4.7). We consider $S_{d-m-1,\alpha}$: a connected component of S_{d-m-1}. $\mathcal{M}_{d-m-1}^{\alpha} = \gamma(S_{d-m-1,\alpha})$ is a real-analytic submanifold contained in $\partial\Omega$. By Lemma 4.2, there is an $\mathcal{M}_{d-m-1}^{\alpha}$-open set $\mathcal{V}$ such that $T_\zeta(\mathcal{M}_{d-m-1}^{\alpha}) \cap \mathfrak{N}_\zeta = \{0\}$, $\forall \zeta \in \mathcal{V}$. Write $\mathcal{U} = \left(\gamma|_{S_{d-m-1,\alpha}}\right)^{-1}(\mathcal{V})$. $\mathcal{U}$ is open in $S_{d-m-1,\alpha}$, and for any $x \in \mathcal{U}$ and any $v \in [T_x(S_{d-m-1,\alpha}) \setminus \{0\}]$, $d\gamma(x)v \notin \mathfrak{N}_{\gamma(x)}$. In other words, for each $x \in \mathcal{U}$,

$$\ker \left\{d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x))\ d\gamma(x)|_{T_x(S_{d-m-1,\alpha})}\right\} = \{0\},$$

where we identify the matrices $d\gamma(x)^T(\mathfrak{H}\rho)(\gamma(x))\ d\gamma(x)$ with linear transformations. So, for each $S_{d-m-1,\alpha}$, the real-analytic subvariety $(\widetilde{V} \cap S_{d-m-1,\alpha}) \subsetneq S_{d-m-1,\alpha}$. From this, we infer that $\dim_{\mathbb{R}}(\widetilde{V}) < (d-m-1)$. By Łojasiewicz's theorem, shrinking U if necessary, we have

$$\widetilde{V} \cap U = \cup_{j=0}^{(d-m-2)} \widetilde{\mathcal{S}}_j$$

where each connected component of $\widetilde{\mathcal{S}}_j$ is a closed submanifold of $U \setminus \left(\cup_{k=0}^{j-1} \widetilde{\mathcal{S}}_k \right)$, $j = 1, ..., d-m-2$. Now write

$$\widetilde{S}_j = \begin{cases} S_j \cap U, & \text{if } j \geq (d-m) \\ (S_{d-m-1,\alpha} \setminus \widetilde{V}) \cap U, & \text{if } j = (d-m-1) \\ (S_j \cup \widetilde{\mathcal{S}}_j) \cap U, & \text{if } j \leq (d-m-2), \end{cases}$$

shrinking U further if necessary so that

$$U = \cup_{j=0}^{d} \widetilde{S}_j$$

is a stratification of U that satisfies (a) and (b) above with $\widetilde{S}_j$ replacing S_j. By construction, each connected component $\widetilde{S}_{d-m-1,\alpha}$, of $\widetilde{S}_{d-m-1}$, satisfies $(c)_{m+1}$ and $\left(\cup_{j=0}^{(d-m-2)} \widetilde{S}_j \right) = \widetilde{V} \cap U$ satisfies $(d)_{m+1}$.

Notice that the stratification in (4.6) establishes the case $m = 0$ for the inductive hypothesis above. By induction, therefore, we can find, shrinking U if necessary, a stratification

$$U = \cup_{j=0}^{d} S_j \tag{4.8}$$

where each connected component $S_{j,\alpha}$, of S_j, $j = 1, ..., d$, is a closed, real-analytic submanifold of $U \setminus \left(\cup_{k=0}^{j-1} S_k \right)$, and for each $j \geq 1$ and each α, $T_\zeta[\gamma(S_{j,\alpha})] \cap \mathfrak{N}_\zeta = \{0\}$, $\forall \zeta \in \gamma(S_{j,\alpha})$. We now find a suitably small neighbourhood, say V, of p so that writing

$$M_j = \gamma(S_j) \cap V, \qquad \boldsymbol{M} \cap V = \cup_{j=0}^{d} M_j$$

[where the S_j's come from (4.8)] gives us the result.

$\blacksquare$

Chapter 5

The proof of Theorem 1.5

In this chapter, we give a proof of the claim made in item (3) of the last chapter. That claim is valid in a broader class of domains that contains certain *smoothly* bounded convex domains (and not just domains with real-analytic boundary). Except in the final proof of this chapter, we will thus work with bounded convex domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$, having smooth boundaries containing no line segments. Let $\gamma : (B_d(0;R), 0) \to (\partial\Omega, q)$ be a smooth imbedding whose image is complex-tangential, and for which $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x$. For the remainder of this chapter, γ and $R > 0$ will have the specific meanings just introduced. In the context of Theorem 1.5, given a point $p \in \boldsymbol{M}$, $\gamma[B_d(0;R)]$ serves as the prototype for an open subset of a stratum of positive dimension in the local stratification (4.5) of $\boldsymbol{M}$ near p. All notations in this chapter will be the same as defined in Chapter 4.

For Ω as above, ρ a smooth defining function for $\partial\Omega$, $\zeta \in \partial\Omega$ and $z \in \mathbb{C}^n$, we write $G(\zeta, z) = \sum_{j=1}^n \partial_j\rho(\zeta)(\zeta_j - z_j)$. For a fixed $\zeta \in \partial\Omega$, the equation $G(\zeta, z) = 0$ defines $H_\zeta(\partial\Omega)$, and the real part of $G(z, \zeta)$ is the perpendicular distance of z from $T_\zeta(\partial\Omega)$. Thus, by the convexity of Ω, if $z \in \overline{\Omega}$, $\mathrm{Re}[G(\zeta, z)] \geq 0$ and $G(\zeta, z) = 0 \ \Leftrightarrow\ z = \zeta$.

We now prove a technical lemma, which we will need later in this chapter.

Lemma 5.1 *Let Ω, γ and R be as described above. For each $r \in (0, R/2)$, there exists an open set $\mathcal{U}(r) \supset \gamma[B_d(0;2r)]$ such that for each $z \in \overline{\Omega} \cap \mathcal{U}(r)$, there exists a $y_r^z \in B_d(0;2r)$*

satisfying

$$Re\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y_r^z))[\gamma_j(y_r^z)-z_j]\right\}\leq Re\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-z_j]\right\}\quad \textit{for every } x$$

$$\textit{belonging to a small neighbourhood, } U_z\subseteq B_d(0;R),\ \textit{of } y_r^z. \quad (5.1)$$

Proof : In what follows, we will write $\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\zeta)(\zeta_j-z_j)\right\} = F(\zeta,z)$. For $\zeta\in\gamma[B_d(0;R)]$, we define

$$N_\zeta(\gamma;\varepsilon) := \{z\in N_\zeta(\gamma[B_d(0;R)]) : |z-\zeta|<\varepsilon\},$$

where $N_\zeta(\gamma[B_d(0;R)])$ denotes the normal space of $\gamma[B_d(0;R)]$ in $\mathbb{C}^n$ at ζ. Let $\sigma>0$ be so small that if z lies in a tube around $\gamma[B_d(0;R)]$,

$$\mathrm{dist}[z,\gamma[B_d(0;R)]]\leq\sigma \ \Rightarrow\ \text{there is a unique } x\in B_d(0;R) \text{ such that } z\in\overline{N_{\gamma(x)}(\gamma;\sigma)}.$$

Also we will assume (shrinking $R>0$ if necessary) that for each $r\in(0,R/2)$ and each $x\in B_d(0;2r)$, $N_{\gamma(x)}(\gamma;2\sigma)\cap\gamma[\partial\mathbb{B}_d(0;2r)]=\emptyset$. We now fix $r\in(0,R/2)$ for the remainder of this proof. For each $t\in(0,2r)$, define the function $\mathfrak{F}_t:\overline{\Omega}\cap\left(\cup_{|x|\leq t}\overline{N_{\gamma(x)}(\gamma;\sigma)}\right)\times\gamma[\partial\mathbb{B}_d(0;2r)]\to[0,\infty)$ by

$$\mathfrak{F}_t:(z,\xi)\mapsto F(\xi,z).$$

For a fixed $t\in(0,2r)$, $\mathfrak{F}_t(z,\xi)>0$, by convexity and by the foregoing choices for R and σ. Thus, there exists a $m_t>0$ such that $\mathfrak{F}_t(z,\xi)\geq m_t\ \forall(z,\xi)\in\overline{\Omega}\cap\left(\cup_{|x|\leq t}\overline{N_{\gamma(x)}(\gamma;\sigma)}\right)\times\gamma[\partial\mathbb{B}_d(0;2r)]$. Write $s_t=\min\{m_t/2,\sigma\}$. Then

$$z\in\overline{\Omega}\cap\left(\cup_{|x|\leq t}\overline{N_{\gamma(x)}(\gamma;s_t)}\right)$$

$$\Rightarrow\ F(\zeta^z,z)=\mathrm{dist}[z,T_{\zeta^z}(\partial\Omega)]\leq s_t<F(\xi,z),\quad\forall\xi\in\gamma[\partial\mathbb{B}_d(0;2r)],\quad (5.2)$$

where $\zeta^z \in \gamma[\mathbb{B}_d(0;t)]$ such that $z \in N_{\zeta^z}(\gamma[B_d(0;R)])$. We define

$$\mathcal{U}(r) = \mathrm{int}\left[\cup_{t\in(0,2r)}\left\{\cup_{|x|\leq t} N_{\gamma(x)}(\gamma; s_t)\right\}\right],$$

and for each $z \in \overline{\Omega} \cap \mathcal{U}(r)$, we define y_r^z by

$$F(\gamma(y_r^z), z) = \inf_{x\in\mathbb{B}_d(0;2r)} F(\gamma(x), z).$$

If we could show that $y_r^z \notin \partial\mathbb{B}_d(0;2r)$, then we would be done. For each $z \in \mathcal{U}(r)$, let $x^z \in B_d(0;2r)$ be such that $z \in N_{\gamma(x^z)}(\gamma[B_d(0;R)])$. If $z \in \cup_{|x|\leq t} N_{\gamma(x)}(\gamma; s_t)$ for some $t \in (0,2r)$, then $|x^z| \leq t$. In view of (5.2)

$$F(\gamma(y_r^z), z) \;\leq\; F(\gamma(x^z), z) \;<\; F(\gamma(s), z) \quad \forall s \in \partial\mathbb{B}_d(0;2r).$$

Hence, $y_r^z \notin \partial\mathbb{B}_d(0;2r)$, and we have our result.

■

So far, we have not made use of the fact that $\gamma[B_d(0;R)]$ is complex-tangential. We shall do so in the next three lemmas.

Lemma 5.2 *Let Ω, γ, $y_r^z \in B_d(0;2r)$ and $\mathcal{U}(r)$ be as in Lemma 5.1. Then, for $z \in \overline{\Omega} \cap \mathcal{U}(r)$*

$$Re\left\{\sum_{j,k=1}^{n} \partial_{jk}^2\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z) - z_j] + \partial_{j\bar{k}}^2\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z) - z_j]\right\} = 0,$$

$$\mu = 1, ..., d. \quad (5.3)$$

Proof : If $z \in \overline{\Omega} \cap \mathcal{U}(r)$, then y_r^z is a local minimum of the function

$$B_d(0;2r) \ni x \mapsto F(\gamma(x), z).$$

Therefore, taking the partial derivative of the above with respect to x_μ and evaluating at $x = y_r^z$, we get

$$\mathrm{Re}\left\{\sum_{j,k=1}^{n} \partial^2_{jk}\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z)-z_j] + \sum_{j,k=1}^{n} \partial^2_{j\overline{k}}\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z)-z_j]\right.$$
$$\left. + \sum_{j=1}^{n} \partial_j\rho(\gamma(y_r^z))\frac{\partial\gamma_j}{\partial x_\mu}(y_r^z)\right\} = 0,\ \ \mu = 1, ..., d. \quad (5.4)$$

Since $\gamma[B_d(0;R)]$ is complex-tangential, we have

$$\sum_{j=1}^{n} \partial_j\rho(\gamma(y_r^z))\ \frac{\partial\gamma_j}{\partial x_\mu}(y_r^z) = 0,\ \ \mu = 1, ..., d. \quad (5.5)$$

The result follows from (5.4) and (5.5).

■

In the next lemma, we exploit the fact that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x \in B_d(0;R)$.

Lemma 5.3 *Let Ω and γ be as described above. There exist uniform constants $\delta \equiv \delta(\gamma) > 0$ and $C \equiv C(\gamma) > 0$ such that*

$$Re\left\{\sum_{j=1}^{n} \partial_j\rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)]\right\} \geq C\ |x-y|^2\ \forall x, y \in B_d(0,\delta). \quad (5.6)$$

Proof : Let $\eta \in \partial\Omega$ and $z \in \overline{\Omega}$. We Taylor expand the function

$$\eta \mapsto \sum_{j=1}^{n} \partial_j\rho(\eta)(\eta_j - z_j)$$

about $\eta = \xi$ to get

$$\begin{aligned}\sum_{j=1}^{n} \partial_j \rho(\eta)(\eta_j - z_j) &= \sum_{j=1}^{n} \partial_j \rho(\xi)(\xi_j - z_j) + \sum_{j=1}^{n} \partial_j \rho(\xi)(\eta_j - \xi_j)\\ &\quad + \sum_{j,k=1}^{n} \partial^2_{jk} \rho(\xi)(\xi_j - z_j)(\eta_k - \xi_k) + \sum_{j,k=1}^{n} \partial^2_{j\bar{k}} \rho(\xi)(\xi_j - z_j)(\overline{\eta_k} - \overline{\xi_k})\\ &\quad + \sum_{j,k=1}^{n} \partial^2_{jk} \rho(\xi)(\eta_j - \xi_j)(\eta_k - \xi_k) + \sum_{j,k=1}^{n} \partial^2_{j\bar{k}} \rho(\xi)(\eta_j - \xi_j)(\overline{\eta_k} - \overline{\xi_k})\\ &\quad + O(|\xi - z||\eta - \xi|^2,\ |\eta - \xi|^3).\end{aligned}$$

Substituting $z = \gamma(y),\ \eta = \gamma(x)$ and $\xi = \gamma(y)$ in the above expression, we get

$$\begin{aligned}&\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x) - \gamma_j(y)] \qquad (5.7)\\ &= \sum_{j=1}^{n} \partial_j \rho(\gamma(y))[\gamma_j(x) - \gamma_j(y)] + \sum_{j,k=1}^{n} \partial^2_{jk} \rho(\gamma(y))[\gamma_j(x) - \gamma_j(y)][\gamma_k(x) - \gamma_k(y)]\\ &\quad + \sum_{j,k=1}^{n} \partial^2_{j\bar{k}} \rho(\gamma(y))[\gamma_j(x) - \gamma_j(y)][\overline{\gamma_k(x)} - \overline{\gamma_k(y)}] + O(|\gamma(x) - \gamma(y)|^3),\end{aligned}$$

for $x, y \in B_d(0; R)$. Taylor expanding γ around $y \in B_d(0; R)$, and substituting in (5.7), we have

$$\begin{aligned}&\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x) - \gamma_j(y)] \qquad (5.8)\\ &= \sum_{j=1}^{n} \partial_j \rho(\gamma(y)) \left\{ \sum_{\mu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)(x_\mu - y_\mu) + \frac{1}{2} \sum_{\mu,\nu=1}^{d} \frac{\partial^2 \gamma_j}{\partial x_\mu \partial x_\nu}(y)(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &\quad + \sum_{j,k=1}^{n} \partial^2_{jk} \rho(\gamma(y)) \left\{ \sum_{\mu,\nu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \frac{\partial \gamma_k}{\partial x_\nu}(y)(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &\quad + \sum_{j,k=1}^{n} \partial^2_{j\bar{k}} \rho(\gamma(y)) \left\{ \sum_{\mu,\nu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \overline{\frac{\partial \gamma_k}{\partial x_\nu}(y)}(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &\quad + O(|x - y|^3) \quad \forall x, y \in B_d(0; R).\end{aligned}$$

We now use the fact that $\gamma[B_d(0;R)]$ is complex-tangential. For $y \in B_d(0;R)$, we have

$$\sum_{j=1}^{n} \partial_j \rho(\gamma(y))\ \frac{\partial \gamma_j}{\partial x_\mu}(y) = 0, \quad \mu = 1, ..., d. \tag{5.9}$$

Differentiating the above expression with respect to x_ν and evaluating at $x = y$ yields

$$\sum_{j,k=1}^{n} \partial^2_{jk}\rho(\gamma(y))\frac{\partial \gamma_j}{\partial x_\mu}(y)\ \frac{\partial \gamma_k}{\partial x_\nu}(y) + \sum_{j,k=1}^{n} \partial^2_{j\bar{k}}\rho(\gamma(y))\frac{\partial \gamma_j}{\partial x_\mu}(y)\ \overline{\frac{\partial \gamma_k}{\partial x_\nu}(y)}$$
$$+ \sum_{j=1}^{n} \partial_j\rho(\gamma(y))\frac{\partial^2 \gamma_j}{\partial x_\mu \partial x_\nu}(y) = 0, \quad \mu, \nu = 1, ..., d. \tag{5.10}$$

From (5.8), (5.9) and (5.10), we get

$$\mathrm{Re}\left\{\sum_{j=1}^{n} \partial_j\rho(\gamma(x))[\gamma_j(x) - \gamma_j(y)]\right\} = \frac{1}{2}\langle\ d\gamma(y)(x-y) \mid (\mathfrak{H}\rho)(\gamma(y)) \mid d\gamma(y)(x-y)\ \rangle$$
$$+ O(|x-y|^3).$$

This statement, in conjunction with the strict positivity of $(\mathfrak{H}\rho)(\zeta)$ on $T_\zeta(\gamma[B_d(0;R)]) \subset T_\zeta(\partial\Omega)\ \forall \zeta \in \gamma[B_d(0;R)]$, allows us to infer that there are uniform constants $\delta \equiv \delta(\gamma) > 0$ and $C \equiv C(\gamma) > 0$ such that

$$\mathrm{Re}\left\{\sum_{j=1}^{n} \partial_j\rho(\gamma(x))[\gamma_j(x) - \gamma_j(y)]\right\} \geq C\ |x-y|^2 \quad \forall x, y \in B_d(0;\delta).$$

■

Lemma 5.4 *Let Ω, γ and R be as described above. Then*

$$\lim_{\delta\to 0}\ \frac{1}{\delta^2}\sum_{j=1}^{n} \partial_j\rho(\gamma(x+\delta v))[\gamma_j(x+\delta v) - \gamma_j(x)]$$
$$= \frac{1}{2}\sum_{j,k=1}^{n}\left(\partial^2_{j\bar{k}}\rho(\gamma(x))[d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k + \partial^2_{jk}\rho(\gamma(x))[d\gamma(x)v]_j\ [d\gamma(x)v]_k\right) \tag{5.11}$$

for any $x \in B_d(0;R)$ and $v \in \mathbb{R}^d$.

Proof : Follows from (5.8), (5.9) and (5.10) in the proof of Lemma 5.3.

■

Theorem 5.5 *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, weakly convex domain having a smooth boundary that contains no line segments, and let $\gamma : B_d(0;R) \to \partial\Omega$ be a smooth imbedding whose image is complex-tangential. Also assume that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x$. There exists a $\varrho \equiv \varrho(\gamma) > 0$ such that if $f \in \mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$, then defining*

$$h_\delta(z) = \int_{B_d(0;\varrho)} \frac{\delta^d f(x)/G(x)\ dx}{\left\{\delta^2 + \sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]\right\}^d}\ ,\ z \in \overline{\Omega}, \tag{5.12}$$

where G is defined as

$$G(x) = \int_{\mathbb{R}^d} \left\{1 + \frac{1}{2}\sum_{j,k=1}^n \left(\partial^2_{j\overline{k}}\rho(\gamma(x))[d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k + \right.\right.$$
$$\left.\left.\partial^2_{jk}\rho(\gamma(x))[d\gamma(x)v]_j\ [d\gamma(x)v]_k\right)\right\}^{-d} dv, \tag{5.13}$$

we have

(i) $\{h_\delta\}_{\delta>0} \subset A(\Omega)$ *and is uniformly bounded on* $\overline{\Omega}$,

(ii) $\lim_{\delta\to 0} h_\delta(z) = 0$ *if* $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$,

(iii) $\lim_{\delta\to 0} h_\delta[\gamma(s)] = f(s)\ \forall s \in B_d(0;\varrho)$.

Proof : Consider any $r \in (0, R/3)$ and any $z \in \overline{\Omega} \cap \mathcal{U}(r)$, where $\mathcal{U}(r)$ is as described in Lemma 5.1. We first estimate the quantity

$$\mathrm{Re}\left\{\sum_{j=1}^n [\partial_j\rho(\gamma(y_r^z + x)) - \partial_j\rho(\gamma(y_r^z))]\,[\gamma_j(y_r^z) - z_j]\right\},$$

(where y_r^z is as introduced in Lemma 5.1) given that $(y_r^z + x) \in B_d(0;R)$. Taylor expanding about y_r^z and using the complex-tangency hypothesis for $\gamma[B_d(0;R)]$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y_r^z+x)) - \partial_j\rho(\gamma(y_r^z))\right]\left[\gamma_j(y_r^z) - z_j\right]\right\}\\
&= \mathrm{Re}\left\{\sum_{\mu=1}^{d}\left[\sum_{j,k=1}^{n}\partial^2_{jk}\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z)-z_j]\right.\right.\\
&\qquad\qquad\left.\left. + \ \partial^2_{j\overline{k}}\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z)-z_j]\right] x_\mu\right\}\\
&\quad + O(|x|^2), |\gamma(y_r^z) - z|).
\end{aligned}$$

In view of Lemma 5.2, we can find uniform constants $c > 0$ and $\varepsilon_* > 0$ such that

$$\left|\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y_r^z+x)) - \partial_j\rho(\gamma(y_r^z))\right]\left[\gamma_j(y_r^z) - z_j\right]\right\}\right| \ \leq \ c|x|^2 \ |\gamma(y_r^z) - z| \tag{5.14}$$

$$\forall r \in (0, R/3),$$
$$\forall |x| \leq \varepsilon_*,$$
$$\forall z \in \overline{\Omega} \cap \mathcal{U}(r).$$

Let $r_* \in (0, R/3)$ be so small that for every $r \in (0, r_*]$,

$$|\gamma(x) - z| \leq \frac{C(\gamma)}{2c} \quad \forall x \in B_d(0;r),$$
$$\forall z \in \overline{\Omega} \cap \mathcal{U}(r),$$

where c is as in (5.14) and $C(\gamma)$ is the constant appearing in Lemma 5.3. We now define a constant

$$\varrho \equiv \varrho(\gamma) := \min\{1/2, r_*, \delta(\gamma)/2, \varepsilon_*/3\},$$

where $\delta(\gamma)$ is the constant appearing in Lemma 5.3. In what follows, we will use the notation $y^z \in B_d(0;2\varrho)$ to mean $y^z = y_\varrho^z$. From the preceding estimate and (5.14), we have the estimate

$$\left| \operatorname{Re}\left\{ \sum_{j=1}^{n} [\partial_j \rho(\gamma(y^z + x)) - \partial_j \rho(\gamma(y^z))]\,[\gamma_j(y^z) - z_j] \right\} \right| \le \frac{C(\gamma)}{2}\,|x|^2 \tag{5.15}$$

$$\forall |x| \le \varepsilon_*,$$

$$\forall z \in \overline{\Omega} \cap \mathcal{U}(\varrho).$$

Now, consider any $f \in \mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$. For each $\delta > 0$, define h_δ according to (5.12) above. We remark that by our assumption on γ, the form

$$\mathbb{R}^d \ni v \mapsto \operatorname{Re} \sum_{j,k=1}^{n} \left(\partial^2_{j\bar{k}} \rho(\gamma(x))[d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k + \partial^2_{jk} \rho(\gamma(x))[d\gamma(x)v]_j\ [d\gamma(x)v]_k \right)$$

is strictly positive definite for each $x \in B_d(0;\varrho)$. Using this fact, it can be shown – see [29, Lemma 2.4] –that the integrals (5.13) converge, and that $G(x) \neq 0$. From the discussion at the beginning of this chapter, we conclude that the real part of $\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]$, which occurs in the denominator of the integral in (5.12) is non-negative when $z \in \overline{\Omega}$. Thus, $h_\delta \in A(\Omega)$ for each $\delta > 0$.

Claim (i) $\{h_\delta\}_{\delta>0}$ is uniformly bounded on $\overline{\Omega}$.

We first consider the case when $z \in \overline{\Omega} \cap \mathcal{U}(\varrho)$. We indulge in a slight abuse of notation : we will define the integrand in (5.12) to be 0 when $x \notin B_d(0;\varrho)$, whence $B_d(0;\varrho)$ may be replaced by $\mathbb{R}^d$ in (5.12), but we will continue to refer to this extension to $\mathbb{R}^d$ by the expression of the integrand given above. Making a change of variable $x = y^z + \delta v$, we get

$$h_\delta(z) = \int_{\mathbb{R}^d} \frac{f(y^z + \delta v)/G(y^z + \delta v)\ dv}{\left\{1 + \delta^{-2} \sum_{j=1}^{n} \partial_j \rho(\gamma(y^z + \delta v))[\gamma_j(y^z + \delta v) - z_j]\right\}^d}. \tag{5.16}$$

Observe that

$$\left|1+\delta^{-2}\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right|$$
$$\geq\ 1+\delta^{-2}\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\}. \quad (5.17)$$

We compute

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\} \quad (5.18)$$
$$=\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-\gamma(y^z)]+\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z)-z_j]\right\}$$
$$=\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-\gamma(y^z)]+\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z))[\gamma_j(y^z)-z_j]\right.$$
$$\left.\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y^z+\delta v))-\partial_j\rho(\gamma(y^z))\right][\gamma_j(y^z)-z_j]\right\}$$

From the fact that $\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z))[\gamma_j(y^z)-z_j]\right\}\geq 0\ \forall z\in\overline{\Omega}$, and from Lemma 5.3 and (5.15), we have

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\}\ \geq\ \frac{C(\gamma)}{2}\,\delta^2|v|^2\ \forall z\in\overline{\Omega}\cap\mathcal{U}(\varrho), \quad (5.19)$$

whenever $(y^z+\delta v)\in\mathrm{supp}(f)$. From (5.17) and (5.19) we have

$$|h_\delta(z)|\ \leq\ \|f/G\|_\infty\int_{\mathbb{R}^d}\left\{1+\frac{C(\gamma)}{2}|v|^2\right\}^{-d}dv\quad\forall z\in\overline{\Omega}\cap\mathcal{U}(\varrho). \quad (5.20)$$

We note here that [29, Lemma 2.4] makes it clear that $\|1/G\|_\infty<\infty$.

We now consider the case when $z\in\overline{\Omega}\backslash\mathcal{U}(\varrho)$. Due to the fact that $\gamma[\mathbb{B}_d(0;\varrho)]\subsetneq\mathcal{U}(\varrho)$, we can find a uniform constant $c'>0$ such that

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-z_j]\right\}\ \geq\ c'\ \mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-\gamma_j(0)]\right\}\quad\forall x\in B_d(0;\varrho). \quad (5.21)$$

This time, we make a change of variable $x = \delta v$. For $z \in \overline{\Omega} \setminus \mathcal{U}(\varrho)$, this results in

$$h_\delta(z) = \int_{\mathbb{R}^d} \frac{f(\delta v)/G(\delta v)\ dv}{\left\{1 + \delta^{-2} \sum_{j=1}^n \partial_j \rho(\gamma(\delta v))[\gamma_j(\delta v) - z_j]\right\}^d}. \tag{5.22}$$

From Lemma 5.3 and (5.21) we can deduce that

$$|h_\delta(z)| \ \leq \ \|f/G\|_\infty \int_{\mathbb{R}^d} \{1 + c'|v|^2\}^{-d}\ dv \quad \forall z \in \overline{\Omega} \setminus \mathcal{U}\varrho. \tag{5.23}$$

Claim (i) follows from (5.20) and (5.23).

The above argument actually yields the following observation, which we record.

Fact. *There exists a uniform constant* $\kappa > 0$ *such that the integrands occuring in* (5.16) *and* (5.22) *are dominated by the* $\mathbb{L}^1$ *function*

$$\mathbb{R}^d \ni v \mapsto \|f/G\|_\infty \{1 + \kappa |v|^2\}^{-d}.$$

Claim (ii) $\lim_{\delta \to 0} h_\delta(z) = 0$ if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$.

Notice that if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$, then $\mathrm{Re}\left\{\sum_{j=1}^n \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]\right\} > 0\ \forall x \in B_d(0;\varrho)$. Thus

$$\lim_{\delta \to 0} \frac{\delta^d f(x)/G(x)}{\left\{\delta^2 + \sum_{j=1}^n \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]\right\}^d} = 0.$$

In view of the fact recorded above, we can apply the dominated convergence theorem to (5.12). This results in Claim (ii).

Claim (iii) $\lim_{\delta \to 0} h_\delta[\gamma(s)] = f(s)\ \forall s \in B_d(0;\varrho)$.

Refer to Lemma 5.1. When $z = \gamma(s)$ in that lemma, $y^z = \gamma(s)$. Equation (5.16) reads as

$$h_\delta[\gamma(s)] = \int_{\mathbb{R}^d} \frac{f(s + \delta v)/G(s + \delta v)\ dv}{\left\{1 + \delta^{-2} \sum_{j=1}^n \partial_j \rho(\gamma(s + \delta v))[\gamma_j(s + \delta v) - \gamma_j(s)]\right\}^d}.$$

In view of 5.11, the integrands occuring above tend to

$$\frac{f(s)}{G(s)}\left\{1+\frac{1}{2}\sum_{j,k=1}^{n}\left(\partial^2_{j\bar{k}}\rho(\gamma(s))[d\gamma(s)v]_j\ \overline{[d\gamma(s)v]}_k+\partial^2_{jk}\rho(\gamma(s))[d\gamma(s)v]_j\ [d\gamma(s)v]_k\right)\right\}^{-d},$$

as $\delta\to 0$.

Once again, Claim (iii) follows from the dominated convergence theorem.

■

The proof of Theorem 1.5 : For a $p\in\boldsymbol{M}$, let $\mathcal{M}$ stand for an arbitrary d-dimensional stratum, $d\geq 1$, of the local stratification (4.5) of $\boldsymbol{M}$ near p. Let $q\in\mathcal{M}$ and let $\gamma:(B_d(0;R),0)\to(\mathcal{M},q)$ be a non-singular, real-analytic parametrization of $\mathcal{M}$ near q. Notice that by the definition of $\mathcal{M}$, γ satisfies the hypotheses of Theorem 5.5. Choose and fix a neighbourhood $V_q\ni q$ such that

$$\mathcal{M}\cap V_q=\gamma[B_d(0;\varrho)],$$

where $\varrho>0$ is as given by Theorem 5.5. In the notation of Lemma 2.2, let $\chi\in\mathcal{C}^\infty_c(V_q;[0,1])$ be a bump function. Define $f\in\mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$ by the equation

$$f(x)=\chi|_{\text{Image}(\gamma)}[\gamma(x)],\quad x\in B_d(0;\varrho)$$

and let $\{h_\delta\}_{\delta>0}\subset A(\Omega)$ be the sequence obtained by using the above f in the equation (5.12). Then, the sequence $\{h_{1/k}\}_{k\in\mathbb{N}}\subset A(\Omega)$ satisfies (i)–(iii) in the hypothesis of Lemma 2.2.

We have just shown that each $\mathcal{M}$ is a countable union of peak-interpolation sets for $A(\Omega)$. Since each of the finitely many points in M_0 (M_0 as given by (4.5)) are peak points for $A(\Omega)$ (since Ω is convex) , $\boldsymbol{M}$ is a compact subset of $\partial\Omega$ that is a countable union of peak-interpolation sets for $A(\Omega)$. Using Bishop's theorem again, we conclude that $\boldsymbol{M}$ (and hence, any compact subset of $\boldsymbol{M}$) is a peak-interpolation set for $A(\Omega)$.

$\blacksquare$

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