PROPER HOLOMORPHIC MAPPINGS ONTO SYMMETRIC PRODUCTS OF A RIEMANN SURFACE

GAUTAM BHARALI, INDRANIL BISWAS, DIVAKARAN DIVAKARAN, AND JAIKRISHNAN JANARDHANAN

Abstract. We show that the structure of proper holomorphic maps between the n-fold symmetric products, $n \geq 2$, of a pair of non-compact Riemann surfaces X and Y, provided these are reasonably nice, is very rigid. Specifically, any such map is determined by a proper holomorphic map of X onto Y. This extends existing results concerning bounded planar domains, and is a non-compact analogue of a phenomenon observed in symmetric products of compact Riemann surfaces. Along the way, we also provide a condition for the complete hyperbolicity of all n-fold symmetric products of a non-compact Riemann surface.

1. Introduction

It is well known that the n-fold symmetric product of a Riemann surface, $n \geq 2$, is an n-dimensional complex manifold. One has a precise description of all proper holomorphic maps between n-fold products of bounded planar domains — provided by the Remmert–Stein Theorem [RS60] (also see [Nar71, pp. 71–78]) — and, more recently, of finite proper holomorphic maps between products of Riemann surfaces; see [Jan14], for instance. It is therefore natural to investigate the structure of such maps between symmetric products of Riemann surfaces. To this end, we are motivated by the following result of Edigarian and Zwonek [EZ05] (the notation used will be explained below):

Result 1.1 (paraphrasing [EZ05, Theorem 1]). *Let $\mathbb{G}^n$ denote the n-dimensional symmetrized polydisk and let $f : \mathbb{G}^n \to \mathbb{G}^n$ be a proper holomorphic map. Then, there exists a finite Blaschke product B such that*

$$f\big(\pi^{(n)}(z_1, \dots, z_n)\big) = \pi^{(n)}\big(B(z_1), \dots, B(z_n)\big) \;\; \forall (z_1, \dots, z_n) \in \mathbb{D}^n.$$

In this result, and in what follows, we denote the open unit disk with centre $0 \in \mathbb{C}$ by $\mathbb{D}$. Throughout this paper σ_j, $j = 1, \dots, n$, will denote the elementary symmetric polynomial of degree j in n indeterminates (when there is no ambiguity, we shall — for simplicity of notation — suppress the parameter n). The map $\pi^{(n)} : \mathbb{C}^n \to \mathbb{C}^n$ is defined as:

$$\pi^{(n)}(z_1, \dots, z_n) := \big(\sigma_1(z_1, \dots, z_n), \sigma_2(z_1, \dots, z_n), \dots, \sigma_n(z_1, \dots, z_n)\big), \;\; (z_1, \dots, z_n) \in \mathbb{C}^n.$$

The *symmetrized polydisk*, $\mathbb{G}^n$, is defined as $\mathbb{G}^n := \pi^{(n)}(\mathbb{D}^n)$. It is easy to see that $\mathbb{G}^n$ is a domain in $\mathbb{C}^n$, whence $\mathbb{G}^n$ is a holomorphic embedding of the n-fold symmetric product of $\mathbb{D}$ into $\mathbb{C}^n$.

Given a Riemann surface X, we shall denote its n-fold symmetric product by $\mathrm{Sym}^n(X)$. The complex structure on X induces a complex structure on $\mathrm{Sym}^n(X)$, which is described in brief in Section 2 below.

In this paper, we shall extend Result 1.1 — see Corollary 1.6 below — to proper holomorphic maps between the n-fold symmetric products of certain non-compact Riemann surfaces. At

2010 *Mathematics Subject Classification.* Primary 14J50, 32H35; Secondary 32Q45.

Key words and phrases. Proper holomorphic maps, Riemann surfaces, symmetric product, taut manifolds.

this juncture, the reader might ask whether there is an analogous generalization of Result 1.1 to n-fold symmetric products of *compact* Riemann surfaces. Before we answer this question, we state the following result and note that Corollary 1.6 is its non-compact analogue. Indeed, the following result (the notation therein is explained below) was among our motivations for the investigation in this paper.

Fact 1.2 (an **adaptation** of the results in [CS93] by Ciliberto–Sernesi)**.** *Let X and Y be compact Riemann surfaces with* $\mathrm{genus}(X) = \mathrm{genus}(Y) = g$*, where $g > 2$. Let $F : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ be a surjective holomorphic map, where $n = 1, 2, 3, \ldots, 2g-3$, $n \neq g-1$. Then:*

(1) *X is biholomorphic to Y;*
(2) *The map F is a biholomorphism; and*
(3) *There exists a biholomorphic map $\phi : X \to Y$ such that*

$$F(\langle x_1, \ldots, x_n\rangle) = \langle \phi(x_1), \ldots \phi_n(x_n)\rangle \ \ \forall \langle x_1, \ldots, x_n\rangle \in \mathrm{Sym}^n(X).$$

In the above, and in what follows, we denote by $\langle x_1, \ldots, x_n\rangle$ the orbit of $(x_1, \ldots, x_n) \in X^n$ under the S_n−action on X^n that permutes the entries of $(x_1, \ldots, x_n)$. The map

$$X^n \ni (x_1, \ldots, x_n) \longmapsto \langle x_1, \ldots, x_n\rangle \ \ \forall (x_1, \ldots, x_n) \in X^n$$

will be denoted by π^X_{Sym}. When there is no ambiguity, we shall drop the superscript.

Remark 1.3. The paper [CS93] does not contain a statement of Fact 1.2 in the specific form given above. Therefore, we provide a justification. We first consider the case $1 \leq n \leq g-2$. A special case of a theorem by Martens [Mar63] gives us (1) and (3) of Fact 1.2, assuming that F is a biholomorphism. In [CS93, Section 2], Ciliberto and Sernesi give a different proof of Martens's theorem. It is straightforward to check that the proof in [CS93] yields (1)–(3) above — taking $1 \leq n \leq g-2$ — even when F is just a surjective holomorphic map.

Now consider the case $g \leq n \leq 2g-3$. This time, the first two paragraphs following the heading "Proof of Theorem (1.3)" in [CS93] give us (1) and (3) of Fact 1.2, assuming again that F is a biholomorphism. Again, it is straightforward to check that the requirement that F be a biholomorphism is not essential. The argument in those paragraphs gives us (1)–(3) above — taking $g \leq n \leq 2g-3$ — even when F is just a surjective holomorphic map. ◀

The restrictions on the pair (g, n) in Fact 1.2 are essential. It is classically known that there exist nonisomorphic compact Riemann surfaces of genus 2 having isomorphic Jacobians, hence isomorphic 2-fold symmetric products. Next, consider a non-hyperelliptic compact Riemann surface X of genus 3. Given any $\langle x_1, x_2\rangle \in \mathrm{Sym}^2(X)$, there is a unique point $\langle y_1, y_2\rangle \in \mathrm{Sym}^2(X)$ such that the divisor $(x_1 + x_2 + y_1 + y_2)$ represents the holomorphic cotangent bundle. The automorphism of $\mathrm{Sym}^2(X)$ given by $\langle x_1, x_2\rangle \mapsto \langle y_1, y_2\rangle$ is *not* given by any automorphism of X (here, $g = 3$, $n = 2$, whence $n = g-1$). Furthermore, we expect any generalization of Fact 1.2 to be somewhat intricate because, among other things:

- Any generalization wherein $\mathrm{genus}(X) \neq \mathrm{genus}(Y)$ will place restrictions on the pair $(\mathrm{genus}(X), \mathrm{genus}(Y))$ owing to the Riemann–Hurwitz formula.
- The geometry of $\mathrm{Sym}^n(X)$ varies considerably depending on whether $1 \leq n \leq \mathrm{genus}(X) - 1$ or $n \geq \mathrm{genus}(X)$.

In short, any generalization of Fact 1.2 would rely on techniques very different from those involved in proving Corollary 1.6. Thus, we shall address the problem of the structure of surjective holomorphic maps in the compact case in forthcoming work.

We now focus on n-fold symmetric products of *non-compact* Riemann surfaces. We should mention here that Chakrabarti and Gorai have extended Result 1.1 to n-fold symmetric products of bounded planar domains in [CG15] Their result as well as Result 1.1 rely on an interesting adaptation — introduced in [EZ05] — of an argument by Remmert–Stein. The latter argument relies on two essential analytical ingredients:

(i) The ability to extract subsequences — given an auxiliary sequence constructed from the given proper map — that converge locally uniformly; and

(ii) A vanishing-of-derivatives argument that stems from the mean-value inequality.

These ingredients continue to be relevant when planar domains are replaced by Riemann domains and, indeed, parts of our proofs emulate the argument in [EZ05].

However, our proofs of the theorems below do **not** reduce to a mere application of Result 1.1 to appropriate coordinate patches. An explanation of this is presented in the paragraph that follows (6.4) below. Equally significantly, we need to identify a class of Riemann surfaces X for which some form of the ingredient (i) above is available for $\mathrm{Sym}^n(X)$, $n \geq 2$. This is the objective of our first theorem — which might also be of independent interest.

Theorem 1.4. *Let X be a connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary. Then $\mathrm{Sym}^n(X)$ is Kobayashi complete, and hence taut, for each $n \in \mathbb{Z}_+$.*

We must clarify that in this paper the term *connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary* refers to a non-compact Riemann surface X obtained by excising from a compact Riemann S a finite number of closed, pairwise disjoint disks $D_1, \dots, D_m$ such that ∂D_j, $j = 1, \dots, m$, are $\mathcal{C}^2$-smooth. The complex structure on X is the one it inherits from S: i.e., a holomorphic chart of X is of the form $(\varphi, U \setminus (D_1 \sqcup \dots \sqcup D_m))$, where (ψ, U) is a holomorphic chart of S and φ is the restriction of ψ to $U \setminus (D_1 \sqcup \dots \sqcup D_m)$.

The ingredients (i) and (ii) above allow us to analyse proper holomorphic maps between a product manifold of dimension n and an n-fold symmetric product, where the factors of the product manifold need not necessarily be the same. This is formalised by our next theorem. A similar result is proved in [CG15] where the factors of the products involved are bounded planar domains. Corollary 1.6 is obtained as an easy consequence of the following:

Theorem 1.5. *Let $X = X_1 \times \dots \times X_n$ be a complex manifold where each X_j is a connected non-compact Riemann surface obtained by excising a non-empty indiscrete set from a compact Riemann surface R_j. Let Y be a connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary. Let $F : X \to \mathrm{Sym}^n(Y)$ be a proper holomorphic map. Then, there exist proper holomorphic maps $F_j : X_j \to Y$, $j = 1, \dots, n$, such that*

$$F(x_1, \dots, x_n) \,=\, \pi_{Sym} \circ \big(F_1(x_1), \dots, F_n(x_n)\big) \quad \forall (x_1, \dots, x_n) \in X.$$

The complex structure on each of the factors $X_1, \dots, X_n$ has a description analogous to the one given above for bordered Riemann surfaces.

Finally, we can state the corollary alluded to above. Observe that it is the analogue, in a non-compact setting, of Fact 1.2. It also subsumes Result 1.1: recall that the proper holomorphic self-maps of $\mathbb{D}$ are precisely the finite Blaschke products.

Corollary 1.6. *Let X be a connected non-compact Riemann surface obtained by excising a non-empty indiscrete set from a compact Riemann surface R, and let Y be a connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary. Let $F : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ be a proper holomorphic map. Then, there exists a proper holomorphic map $\phi : X \to Y$ such that*

$$F \circ \pi^X_{Sym}(x_1, \dots, x_n) \,=\, \pi^Y_{Sym}\big(\phi(x_1), \dots, \phi(x_n)\big) \quad \forall (x_1, \dots, x_n) \in X^n.$$

This corollary follows immediately from Theorem 1.5 since $F \circ \pi^X_{Sym} : X^n \to \mathrm{Sym}^n(Y)$ is proper.

We conclude this section with an amusing observation that follows from Corollary 1.6. We first make an explanatory remark. It is well known that if M_1 and M_2 are two non-compact complex manifolds of the same dimension and $F : M_1 \to M_2$ is a proper holomorphic map, then there exists a positive integer μ such that, for any generic point $p \in M_2$, $F^{-1}\{p\}$ has cardinality μ. We call this number the *multiplicity* of F, which we denote by $\mathrm{mult}(F)$.

Corollary 1.7. *Let X and Y — a pair of connected non-compact Riemann surfaces — be exactly as in Corollary 1.6. If $F : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ is a proper holomorphic map, then $\mathrm{mult}(F)$ is of the form d^n, where d is some positive integer.*

2. Preliminaries about the symmetric products

In this section we shall give a brief description, given a Riemann surface X, of the complex structure on $\mathrm{Sym}^n(X)$, $n \geq 2$, that makes it a complex manifold. We shall use the notation introduced in Section 1. Given this notation:

- Recall that $\langle x_1, \dots, x_n \rangle := \pi_{Sym}(x_1, \dots, x_n)$,
- Given subsets $V_j \subseteq X$ that are open, let us write:

$$\langle V_1, \dots, V_n \rangle := \{\langle x_1, \dots x_n \rangle : x_j \in V_j,\ j = 1, \dots, n\}.$$

Since $\mathrm{Sym}^n(X)$ is endowed with the quotient topology relative to π_{Sym}, $\langle V_1, \dots, V_n \rangle$ is, by definition, open in $\mathrm{Sym}^n(X)$.

$\mathrm{Sym}^n(X)$ is endowed with a complex structure as follows. Given a point $p \in \mathrm{Sym}^n(X)$, $p = \langle p_1, \dots p_n \rangle$, choose a holomorphic chart (U_j, φ_j) of X at p_j, $j = 1, \dots, n$, such that

$$U_j \cap U_k = \emptyset \ \text{ if } p_j \neq p_k \qquad \text{and} \qquad U_j = U_k \ \text{ if } p_j = p_k.$$

The above choice of local charts ensures that the map $\Psi_p : \langle U_1, \dots, U_n \rangle \to \mathbb{C}^n$ given by

$$\Psi_p : \langle x_1, \dots, x_n \rangle \longmapsto \big(\sigma_1(\varphi_1(x_1), \dots, \varphi_n(x_n)), \dots, \sigma_n(\varphi_1(x_1), \dots, \varphi_n(x_n))\big),$$

(where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric polynomials that were introduced in Section 1) is a homeomorphism. This follows from the Fundamental Theorem of Algebra. The collection of such charts $(\langle U_1, \dots, U_n \rangle, \Psi_p)$ produces a holomorphic atlas on $\mathrm{Sym}^n(X)$. We shall call such a chart a *model coordinate chart at* $p \in \mathrm{Sym}^n(X)$.

Finally, let Z be a compact Riemann surface, $X \subsetneq Z$ be an embedded open complex submanifold of Z, and let $\mathfrak{A}(Z)$ denote the complex structure on Z. Then, since — for any point $p \in X$ — there is a chart $(U, \varphi) \in \mathfrak{A}(Z)$ such that $U \subset X$, the above discussion shows that $\mathrm{Sym}^n(X)$ is an embedded complex submanifold of $\mathrm{Sym}^n(Z)$. We refer the reader to [Whi72] for details.

3. Hyperbolicity and its consequences

The proof of Theorem 1.5 will require several results about holomorphic mappings into Kobayashi hyperbolic spaces. We summarize the relevant results in this section. An encyclopedic reference for the results in this is section is [Kob98].

In the theory of holomorphic functions of one variable, the behaviour of holomorphic functions near an isolated singularity is well-studied. Among the important results in this area are the famous theorems of Picard. A consequence of Picard's big theorem is that any meromorphic mapping on $\mathbb{D} \setminus \{0\}$ that misses three points automatically extends to a meromorphic function defined on the whole of $\mathbb{D}$. One of the proofs of Picard's theorem relies on the fact that the sphere with three points removed is a hyperbolic Riemann surface. This perspective allows one to generalize the aforementioned extension theorem to holomorphic mappings into Kobayashi hyperbolic spaces. To this end, we need a definition.

Definition 3.1. Let Z be a complex manifold and let Y be a relatively compact complex submanifold of Z. We call a point $p \in \overline{Y}$ a *hyperbolic point* if every Z-open neighbourhood U of p contains a smaller neighbourhood V of p, $\overline{V} \subset U$, such that

$$K_Y(\overline{V} \cap Y, Y \setminus U) := \inf\{K_Y(x, y) : x \in \overline{V} \cap Y, y \in Y \setminus U\} > 0, \tag{3.1}$$

where K_Y denotes the Kobayashi pseudo-distance on Y. We say that Y is *hyperbolically embedded* in Z if every point of $\overline{Y}$ is a hyperbolic point.

The following result is an example of an extension result in higher dimensions of the type alluded to above.

Result 3.2 (Kiernan [Kie73, Theorem 3])**.** *Let X be a complex space and let $\mathscr{E} \subset X$ be a closed complex subspace. Let Y be a complex manifold that is hyperbolically embedded in a complex manifold Z. Then every holomorphic map $f : X \setminus \mathscr{E} \to Y$ extends to a meromorphic map $\widetilde{f} : X \to Z$.*

This result will play a role in the final stages of proving Theorem 1.5. To this end, we would also need — naturally, given the statement of Theorem 1.5 — conditions under which a meromorphic map between complex spaces is actually holomorphic. One situation where this happens is when the complex spaces are manifolds and the target space is Kobayashi hyperbolic.

Result 3.3 (Kodama [Kod79])**.** *Let $f : X \to Y$ be a meromorphic map, where X is a complex manifold and Y is a Kobayashi hyperbolic manifold. Then f is holomorphic.*

The following lemma enables us — as we shall see in Section 6 — to use the preceding results in our specific set-up.

Lemma 3.4. *Let Y, a non-compact Riemann surface, be as in Theorem 1.5 and let S be the compact connected Riemann surface from which Y is obtained by excising a finite number of closed disks. Then* $\mathrm{Sym}^n(Y)$ *is hyperbolically embedded in* $\mathrm{Sym}^n(S)$*.*

Proof. Let $\mathcal{W} \subset S$ be another connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary such that $\overline{Y} \subset \mathcal{W}$. By Theorem 1.4, $\mathrm{Sym}^n(Y)$ and $\mathrm{Sym}^n(\mathcal{W})$ are both Kobayashi complete. In particular, K_Y and $K_{\mathcal{W}}$ are **distances**. It follows from the discussion at the end of Section 2 that $\mathrm{Sym}^n(Y)$ and $\mathrm{Sym}^n(\mathcal{W})$ are embedded submanifolds of $\mathrm{Sym}^n(S)$. Observe that it suffices to show that each $p \in \partial\mathrm{Sym}^n(Y)$ is holomorphically embedded in S. Fix a point $p \in \partial\mathrm{Sym}^n(Y)$. Given any S-open neighbourhood U of p, we choose a neighbourhood V of p such that $\overline{V} \subset U$ and $\overline{V} \subset \mathrm{Sym}^n(\mathcal{W})$. For any $x \in \overline{V} \cap Y$ and $y \in Y \setminus U$, we have $K_Y(x, y) \geq K_{\mathcal{W}}(x, y) > 0$. We know that $\overline{V} \cap \overline{Y}$ and $\overline{Y \setminus U}$ are compact in $\mathcal{W}$. Thus, the inequality in (3.1) follows from the last inequality. $\square$

As the proof of the above lemma shows, Theorem 1.4 is an essential ingredient in the proof of Theorem 1.5. In the remainder of this section, we shall present some prerequisites for proving Theorem 1.4. We begin with a couple of definitions.

Definition 3.5. Let Z be a complex manifold and $Y \subset Z$ be a connected open subset of Z. Let $p \in \partial Y$. We say that *p admits a weak peak function for Y* if there exists a continuous function $f_p : \overline{Y} \to \mathbb{C}$ such that $f|_Y$ is holomorphic,

$$f_p(p) = 1 \qquad \text{and} \qquad |f_p(y)| < 1 \ \ \forall y \in Y.$$

We say that p *admits a local weak peak function for* Y if p admits a weak peak function for $Y \cap U_p$, where U_p is some open neighbourhood (in Z) of p.

In what follows, given a complex manifold X, C_X will denote the Carathéodory pseudo-distance on X. The term *Carathéodory hyperbolic* has a meaning analogous to that of the term Kobayashi hyperbolic. Furthermore, we say that X is *strongly C_X-complete* if X is Carathéodory hyperbolic and each closed ball in X, with respect to the distance C_X, is compact.

We shall also need the following:

Result 3.6. *Let Z be a Stein manifold and $Y \subset Z$ a relatively compact connected open subset of Z. If each point of ∂Y admits a weak peak function for Y, then Y is strongly C_Y-complete.*

The above result has been established with $Z = \mathbb{C}^n$ and Y a bounded domain in $\mathbb{C}^n$ in [Kob98, Theorem 4.1.7]. Its proof applies *mutatis mutandis* for Y and Z as in Result 3.6 (that the class of bounded holomorphic functions on Y separates points is routine to show with our assumptions on the pair (Y, Z)).

4. The proof of Theorem 1.4

Before we provide a proof, some remarks on notation are in order. For simplicity of notation, in this section (**unlike** in subsequent sections), the symbol D_j, $j \in \mathbb{N}$, will denote a closed topological disk. Given non-empty open subsets A and B of the Riemann surface S (explained below), we shall denote the relation $\overline{A} \subset B$ (especially when there is a sequence of such relations) as $A \Subset B$, where the closure is taken in S.

The proof of Theorem 1.4. We begin with the following

Claim. *Each $y \in \partial X$ admits a weak peak function for X.* Each of the individual ingredients in this construction is classical, so we shall be brief. Fix $y \in \partial X$. Let S be the compact Riemann surface such that

$$X = S \setminus \big(D_1 \sqcup \dots \sqcup D_m\big),$$

where each D_j is a **closed** topological disk with $\mathcal{C}^2$-smooth boundary. We may assume without loss of generality that $y \in \partial D_1$. Let us write $X^* = S \setminus (\Delta_1 \sqcup \dots \sqcup \Delta_m)$, where each Δ_j, $j = 1, \dots, m$, is a closed topological disk such that

$$\Delta_j \subsetneq (D_j)^\circ, \ \ j = 2, \dots, m.$$

Before we describe Δ_1, let us choose a holomorphic chart (U, ψ) centered at y such that:

- $\psi : (U, y) \longrightarrow (\mathbb{D}, 0)$,
- $\psi^{-1}((0, 1]\,) \subset S \setminus \overline{X}$, and
- U is **so small** that $\psi^{-1}(\mathbb{D} \cap \overline{D(1;1)}\,) \cap \overline{X} = \{y\}$.

The last requirement is possible because ∂X is of class $\mathcal{C}^2$. It is easy to construct a *local* peak function ϕ at y for X that is, in fact, holomorphic on U and such that

$$\begin{aligned} &|\phi(x)| < 1 \ \ \forall x \in U \setminus \psi^{-1}(\mathbb{D} \cap \overline{D(1;1)}\,), \\ &|\phi(x)| > 1 \ \ \forall x \in \psi^{-1}(\mathbb{D} \cap D(1;1)), \\ &|\phi(x)| = 1 \ \ \forall x \in \psi^{-1}(\mathbb{D} \cap \partial D(1;1)) \ \text{ with } \ \phi^{-1}\{1\} = \{y\} \end{aligned} \tag{4.1}$$

(the interested reader is referred to the proof of Proposition 5.2 for further details). Let $\Delta_1 \subsetneq (D_1)^\circ$ and be such that $\Delta_1 \cap U \neq \emptyset$ and $\partial\Delta_1$ intersects ∂U at exactly two points. In fact, we can choose Δ_1 such that, in addition to these properties, $\partial\Delta_1$ also intersects $\psi^{-1}(\{\zeta \in \mathbb{C} : |\zeta| = 1-\varepsilon\})$ in exactly two points for some positive $\varepsilon \ll 1$. Pick two S-open neighbourhoods, V_1 and V_2, of y such that

$$V_1 \Subset V_2 \Subset U \quad \text{and} \quad \psi^{-1}((1-\varepsilon)\mathbb{D} \cap \overline{D(1;1)}\,) \cap X^* \subset V_1.$$

Let $\chi_1, \chi_2 \longrightarrow [0,1]$ be two functions in $\mathcal{C}^\infty(X^*)$ with

$$\chi_1|_{V_1\cap X^*} \equiv 1 \quad \text{and} \quad \chi_1|_{X^*\setminus V_2} \equiv 0,$$
$$\chi_2|_{V_2\cap X^*} \equiv 1 \quad \text{and} \quad \chi_2|_{X^*\setminus U} \equiv 0.$$

Finally, consider the function:

$$G(x) := \begin{cases} (1-\phi(x))\chi_2(x), & \text{if } x \in (X^* \cap U), \\ 0, & \text{if } x \in (X^* \setminus U). \end{cases}$$

In what follows, it will be understood that any expression of the form Ψ/G is **0 by definition** outside $\mathsf{supp}(\Psi)$. Define the $(0,1)$-form $\omega \in \Gamma\big(T^{*\,(0,1)}S|_{X^*}\big)$ as

$$\omega = \frac{\overline{\partial}\chi_1}{G}$$

By construction, ω is of class $\mathcal{C}^\infty$, vanishes on $(X^* \cap \overline{V}_1) \cup (X^* \setminus V_2)$, and

$$x \in X \cap (V_2 \setminus \overline{V}_1) \implies \omega(x) = \overline{\partial}\left(\frac{\chi_1}{1-\phi}\right)\bigg|_x. \tag{4.2}$$

By the Behnke–Stein theorem [BS49], X^* is Stein. Thus, it admits a solution to the $\overline{\partial}$-problem

$$\overline{\partial}u \;=\; \omega \;\text{ on } X^*. \tag{4.3}$$

Furthermore, it is a classical fact that there exists a solution, say u^∞, to (4.3) of class $\mathcal{C}^\infty(X^*)$. Write $u_y := u^\infty|_{\overline{X}}$. As $X \Subset X^*$, u_y is bounded. Thus — by subtracting a large positive constant if necessary — we may assume that $\mathsf{Re}(u_y) < 0$ on $\overline{X}$. Observe that, by (4.2), (4.3) and the construction of G,

$$\big(-(\chi_1/G) + u_y\big)^{-1} \in \mathscr{O}(X).$$

By (4.1) and by our adjustment of $\mathsf{Re}(u_y)$, we have

$$\mathsf{Re}\big((-(\chi_1/G) + u_y)^{-1}\big)(x) < 0 \;\; \forall x \in X.$$

From this, it is easy to check that $f_y(x) := e^{(1/(-(\chi_1/G)+u_y))(x)}$, $x \in \overline{X}$, is a weak peak function at y for X. Hence our claim.

In this paragraph, we assume that $n \geq 2$. Let us pick a point $\langle y_1, \dots, y_n\rangle \in \partial\mathrm{Sym}^n(X)$. It is routine to see that π_{Sym} is a proper map. Thus, we may assume without loss of generality that $y_1 \in \partial X$. Our Claim above gives us a weak peak function for X at y_1: call it f. Set

$$\mathsf{h}(z) \;:=\; \frac{1+z}{1-z},$$

which maps $\mathbb{D}$ biholomorphically to the open right half-plane $\mathbf{H}_+$ and maps $(0,1) \longmapsto (1,+\infty)$. Let $(\boldsymbol{\cdot})^{1/n}$ denote the holomorphic branch on $\mathbf{H}_+$ of the n-th root such that

$$z^{1/n} \in \big\{w \in \mathbb{C} : \mathsf{Re}(w) > 0,\; |\mathsf{Im}(w)| < \arctan(\pi/2n)\mathsf{Re}(w)\big\} \;\; \forall z \in \mathbf{H}_+\,. \tag{4.4}$$

Furthermore, note that

($*$) $(\cdot)^{1/n}$ extends to $\partial\mathsf{H}_+$ as a continuous function such that $\lim_{\overline{\mathsf{H}}_+\ni z\to\infty} z^{1/n} = \infty$.

If $n = 1$ then set $F := f$. If $n \geq 2$, then define

$$F(\langle x_1, \dots, x_n\rangle) := \mathsf{h}^{-1}\left(\prod\nolimits_{1\leq j\leq n}\left(\frac{1+f(x_j)}{1-f(x_j)}\right)^{1/n}\right) \quad \forall\langle x_1,\dots,x_n\rangle \in \overline{\mathrm{Sym}^n(X)}.$$

By (4.4) we see that $F \in \mathcal{C}\big(\overline{\mathrm{Sym}^n(X)}\big) \cap \mathscr{O}(\mathrm{Sym}^n(X))$. By ($*$) and the properties of f it follows that F is a weak peak function for $\mathrm{Sym}^n(X)$ at $\langle y_1, \dots, y_n\rangle \in \partial\mathrm{Sym}^n(X)$.

We have just shown that, whether $n = 1$ or $n \geq 2$, each point in $\partial\mathrm{Sym}^n(X)$ admits a weak peak function for $\mathrm{Sym}^n(X)$. Recall that X^* is Stein. It follows from Result 3.6, by taking $Z = \mathrm{Sym}^n(X^*)$, that $\mathrm{Sym}^n(X)$ is strongly Carathéodory complete. In particular, $\mathrm{Sym}^n(X)$ is Kobayashi complete.

By a result of Kiernan [Kie70], it follows that $\mathrm{Sym}^n(X)$ is taut. $\square$

5. Technical propositions

In proving Theorem 1.5, we will need to understand the behaviour of holomorphic maps $f : Z \to \overline{\mathrm{Sym}^n(Y)}$, $n \geq 2$ — where Z is connected and Y is as in Theorem 1.5 — in the event that $\mathsf{range}(f) \not\subset \mathrm{Sym}^n(Y)$.

To this end, we shall use the notation introduced in Sections 1 and 2. Thus, given a Riemann surface Y and $(y_1, \dots, y_n) \in Y^n$, π_{Sym} is as introduced in Section 1, and

$$\langle y_1, \dots, y_n\rangle := \pi_{Sym}(y_1, \dots, y_n).$$

For a point $y \in Y$, a presentation of y having the form of the left-hand side of the above equation will be called the *quotient representation of y*.

We require one further observation. For a Riemann surface Y, let $D_1, \dots, D_n$ be non-empty subsets of Y such that $D_1 \times \dots \times D_n$ is not necessarily closed under the S_n-action on Y^n, $n \geq 2$. In any circumstance, we shall use $\pi_{Sym}(D_1 \times \dots \times D_n)$ to denote the image of the set $D_1 \times \dots \times D_n$ under the map $\pi_{Sym} : Y^n \to \mathrm{Sym}^n(Y)$.

We begin with the following simple lemma:

Lemma 5.1. *Let X be a Riemann surface, $n \geq 2$, and let $D_1, \dots, D_n \subset X$ be open subsets. Write $\mathcal{D} := \bigcup_{j=1}^n D_j$. Define $H := \pi_{Sym}(D_1 \times \dots \times D_n)$. Suppose $\phi : \mathcal{D} \to \mathbb{C}$ is a bounded holomorphic map and $\mathscr{S}$ a symmetric polynomial in n indeterminates. Then, the relation $\Gamma \subset H \times \mathbb{C}$ defined by*

$$\Gamma := \big\{\big(\langle v_1, \dots, v_n\rangle, w\big) \in H \times \mathbb{C} : w = \mathscr{S}(\phi(x_1), \dots, \phi(x_n)) \text{ and } (x_1, \dots, x_n) \in (\pi_{Sym}^{-1}\{\langle v_1, \dots, v_n\rangle\} \cap D_1 \times \dots \times D_n)\big\}.$$

is the graph of a holomorphic function defined on H.

Proof. Let π_1 (resp., π_2) denote the projection onto the first (resp., second) factor of $H \times \mathbb{C}$. Consider any point $\langle v_1, \dots, v_n\rangle \in H$. That $\pi_1^{-1}\{\langle v_1, \dots, v_n\rangle\} \cap \Gamma$ is a singleton follows clearly from the fact that $\mathscr{S}$ is a symmetric polynomial. It is thus the graph of a function Φ. Consider the mapping $\Phi' : D_1 \times \dots \times D_n \to \mathbb{C}$ given by

$$(x_1, \dots, x_n) \mapsto \mathscr{S}(\phi(x_1), \dots, \phi(x_n)).$$

Let us write $\Delta := D_1 \times \dots \times D_n$. By construction, $\Phi' = \Phi \circ \big(\pi_{Sym}\big|_{\Delta}\big)$. Since π_{Sym} admits holomorphic branches of local inverses around any of its regular values, the above construction shows that Φ is holomorphic outside the set of critical values of π_{Sym} in H. Since ϕ, and thus Φ', are bounded, Riemann's removable singularities theorem implies that Φ is holomorphic on H. $\square$

The key result needed is the following, which generalizes Lemma 5 of [EZ05].

Proposition 5.2. *Let Y be a connected bordered Riemann surface with $\mathcal{C}^2$-smooth boundary and let S be the compact Riemann surface from which Y is obtained by excising a finite number of closed disks. Let Z be a connected complex manifold and let $f : Z \to \mathrm{Sym}^n(Y)$, $n \geq 2$, be a holomorphic map such that $f(Z) \subset \overline{\mathrm{Sym}^n(Y)}$ (where the closure is in $\mathrm{Sym}^n(S)$). Suppose there exists a $z_0 \in Z$ such that $f(z_0)$ is of the form $\langle y_1, *\rangle$, $y_1 \in \partial Y$. Then*

$$f(z) \text{ is of the form } \langle y_1, *\rangle \text{ for all } z \in Z.$$

Moreover, if y_1 appears k times, $1 \leq k \leq n$, in the quotient representation of $f(z_0)$ then the same is true for $f(z)$ for all $z \in Z$.

Proof. Let $y_2, \dots, y_l$ be the other *distinct* points that appear in the quotient representation of $f(z_0)$. Let $D_1, \dots, D_l$ be small coordinate disks in S centered at $y_1, \dots, y_l$, respectively, whose closures are pairwise disjoint. Let (D_j, ψ_j), $j = 1, \dots, l$, denote the coordinate charts. By "coordinate disks centered at y_j", we mean that $\psi_j(D_j) = \mathbb{D}$ and $\psi_j(y_j) = 0$, $j = 1, \dots, l$. Furthermore, as Y has $\mathcal{C}^2$-smooth boundary, we can (by shrinking D_1 and scaling ψ_1 if necessary) ensure that

- $\psi_1(\partial Y \cap D_1) \cap \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta) - 1|^2 + |\mathsf{Im}(\zeta)|^2 = 1\} = \{\psi_1(y_1)\} = \{0\}$; and
- $\psi_1(Y \cap D_1) \subset \{\zeta \in \mathbb{D} : |\mathsf{Re}(\zeta) - 1|^2 + |\mathsf{Im}(\zeta)|^2 > 1\}$.

Let us define $\phi \in \mathscr{O}(D_1)$ by

$$\phi(y) := \exp\left\{\frac{\psi_1(y)}{2 - \psi_1(y)}\right\} \quad \forall y \in D_1.$$

Using the fact that the Möbius transformation $\zeta \mapsto \zeta/(2-\zeta)$ maps the circle $\{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta) - 1|^2 + |\mathsf{Im}(\zeta)|^2 = 1\}$ onto $\{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) = 0\}$, it is routine to verify that

$$\phi(y_1) = 1 \quad \text{and} \quad |\phi(y)| < 1 \quad \forall y \in D_1 \cap (\overline{Y} \setminus \{y_1\}), \tag{5.1}$$

and that ϕ is a bounded function.

Write $\mathcal{D} := \sqcup_{j=1}^{l} D_j$ and define a function $\widetilde{\phi} \in \mathscr{O}(\mathcal{D})$ as follows

$$\widetilde{\phi}(y) := \begin{cases} \phi(y), & \text{if } y \in D_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $H := \pi_{Sym}(D_1^k \times D_2^{k_2} \times \dots \times D_l^{k_l})$, where k_j is the number of times y_j appears in the quotient representation of $f(z_0)$, $j = 2, \dots, l$. Now consider the following relation $\Gamma \subset H \times \mathbb{C}$ defined by

$$\begin{aligned}\Gamma := \big\{ \big(\langle v_1, \dots, v_n\rangle, w\big) \in H \times \mathbb{C} : w = \widetilde{\phi}(x_1) + \dots + \widetilde{\phi}(x_n) \\ \text{and } (x_1, \dots, x_n) \in (\pi_{Sym}^{-1}\{\langle v_1, \dots, v_n\rangle\} \cap D_1^k \times D_2^{k_2} \times \dots \times D_l^{k_l})\big\}. \end{aligned} \tag{5.2}$$

It follows from Lemma 5.1 that Γ is the graph of a function, say Φ, that is holomorphic on H. Now as $f(z_0) \in H$ and H is an open neighborhood of $f(z_0)$, we can find a small connected open set $U \subsetneq Z$ around z_0 such that $f(U) \subset H$. Consider the holomorphic map $\Phi \circ \big(f|_U\big)$. As $f(U) \subset \overline{\mathrm{Sym}^n(Y)}$, we have, by construction:

$$\Phi \circ \big(f|_U\big)(z_0) = k = \sup_{z \in U} \big|\Phi \circ \big(f|_U\big)(z)\big|.$$

By the maximum modulus theorem, $\Phi \circ \big(f|_U \big) \equiv k$. By the definition of the function Φ, we deduce that the conclusion of our proposition holds true on the open set U.

Let E be the set of points of Z for which the conclusion of the proposition holds true. By hypothesis, E is non-empty. The above argument shows that E is an open set. Let $z \in Z \setminus E$. If y_1 does not appear in the quotient representation $f(z)$ at all, then, by continuity, there exists a neighbourhood U_z of z such that the same is true for every point in U_z. On the other hand, if y_1 does appear in the quotient representation of $f(z)$ but not k times, then the argument given prior to this paragraph shows that we can find a neighbourhood U_z of z such that the same is true for every point in U_z. In either case, therefore, $U_z \subset (Z \setminus E)$. This shows that E is closed. Therefore $E = Z$. $\square$

6. The proof of Theorem 1.5

In proving Theorem 1.5 we will find it convenient to use a certain expression, which we now define.

Definition 6.1. Let $M_1, \dots, M_n$ and N be complex manifolds, and $\mathfrak{V}$ a proper (possibly empty) analytic subvariety of $M_1 \times \dots \times M_n$. Let $f : (M_1 \times \dots \times M_n) \setminus \mathfrak{V} \to N$ be a holomorphic map. We say that f *depends only on the j-th coordinate on* $(M_1 \times \dots \times M_n) \setminus \mathfrak{V}$, $1 \le j \le n$, if for each $x \in M_j$ lying outside some proper (possibly empty) analytic subvariety of M_j,

$$f(x_1, \dots, x_{j-1}, x, x_j, \dots, x_{n-1}) = f(y_1, \dots, y_{j-1}, x, y_j, \dots, y_{n-1}) \\ \text{for all } (x_1, \dots, x_{n-1}) \neq (y_1, \dots, y_{n-1}) \in \prod_{i \neq j} M_i \quad (6.1)$$

such that $(x_1, \dots, x_{j-1}, x, x_j, \dots, x_{n-1}), (y_1, \dots, y_{j-1}, x, y_j, \dots, y_{n-1}) \notin \mathfrak{V}$.

Before we give the proof of Theorem 1.5, we ought to point out to the reader a convention that will be used below. Given a product space, π_j will denote the projection onto the j-th coordinate. If several product spaces occur in a discussion, we **shall not add** additional labels to π_j to indicate the domain of this projection unless there is scope for ambiguity.

The proof of Theorem 1.5. Let S be a compact Riemann surface such that Y is obtained from S by excising a finite number of closed disks such that ∂Y is $\mathcal{C}^2$-smooth. Theorem 1.5 is a tautology when $n = 1$, so it will be understood here that $n \ge 2$. Let R_j, $j = 1, \dots, n$, be as in the statement of Theorem 1.5.

Let $p = (p_1, \dots, p_n)$ be a point in $R_1 \times \dots \times R_n$ such that, for each $1 \le j \le n$, p_j is a limit point of $R_j \setminus X_j$. Also by hypothesis, we can choose p_j to belong to ∂X_j. Let (U_j, ψ_j) be holomorphic coordinate charts of R_j chosen in such a way that:

- $p_j \in U_j$; and
- Each U_j is biholomorphic to a disk.

Let $W_j := U_j \cap X_j$ and $V_j := \psi_j(W_j)$. For $(z_1, \dots, z_n) \in V_1 \times \dots \times V_n$, let

$$g(z_1, \dots, z_n) := F(\psi_1^{-1}(z_1), \dots, \psi_n^{-1}(z_n)). \quad (6.2)$$

Fix a point $q \in \partial X_n \cap U_n$. Consider a sequence $\{w_\nu\} \subset V_n$ such that $w_\nu \to \psi_n(q)$. Let

$$\phi_\nu : V_1 \times \dots \times V_{n-1} \to \mathrm{Sym}^n(Y) := g(z_1, \dots, z_{n-1}, w_\nu).$$

We claim that we can extract a subsequence $\{w_{\nu_m}\} \subset \{w_\nu\}$ such that $\{\phi_{\nu_m}\}$ converges uniformly on compacts to a holomorphic mapping $h : V_1 \times \dots \times V_{n-1} \to \mathrm{Sym}^n(S)$. To this end, fix another connected bordered Riemann surface, $Y^* \subset S$, with $\mathcal{C}^2$-smooth boundary such that $Y \Subset Y^*$. By Theorem 1.4, both $\mathrm{Sym}^n(Y)$ and $\mathrm{Sym}^n(Y^*)$ are taut. Owing to the tautness of $\mathrm{Sym}^n(Y)$, and as F is proper, we can extract a subsequence $\{w_{\nu_m}\} \subset \{w_\nu\}$ such that ϕ_{ν_m} is compactly divergent — i.e., given compacts $K_1 \subset V_1 \times \cdots \times V_{n-1}$ and $K_2 \subset \mathrm{Sym}^n(Y)$, there exists an integer $M(K_1, K_2)$ such that

$$\phi_{\nu_m}(K_1) \cap K_2 = \emptyset \quad \forall m \geq M(K_1, K_2). \tag{6.3}$$

We now view each ϕ_{ν_m} as a map into $\mathrm{Sym}^n(Y^*)$. This time, owing the tautness of $\mathrm{Sym}^n(Y^*)$, there exists a holomorphic map $h : V_1 \times \cdots \times V_{n-1} \to \mathrm{Sym}^n(S)$ such that — passing to a subsequence of $\{\phi_{\nu_m}\}$ and **relabelling** if necessary — $\{\phi_{\nu_m}\}$ converges uniformly on compacts to h. This establishes our claim. From this and (6.3) it follows that $h(V_1 \times \cdots \times V_{n-1}) \subset \partial \mathrm{Sym}^n(Y)$. It follows from Proposition 5.2 that there exists a point $\xi \in \partial Y$ such that

$$h(z) \text{ is of the form } \langle \xi, * \rangle \text{ for all } z \in V_1 \times \cdots \times V_{n-1}. \tag{6.4}$$

It is a classical fact — see [Jos06, Chapter 5], for instance — that there exists a bounded, non-constant function χ that is holomorphic on some open connected set $\mathcal{W}$ that contains $\overline{Y}$. Let $\Psi : \mathrm{Sym}^n(\mathcal{W}) \to \mathbb{C}^n$ be defined by

$$\langle z_1, \ldots, z_n \rangle \longmapsto \big(\sigma_1(\chi(z_1), \ldots, \chi(z_n)), \sigma_2(\chi(z_1), \ldots, \chi(z_n)), \ldots, \sigma_n(\chi(z_1), \ldots, \chi(z_n))\big).$$

A remark on the purpose of the map Ψ is in order. If we could, by shrinking each U_j if necessary, find a single model coordinate chart $(\Omega, \mathit{\Psi})$ on $\mathrm{Sym}^n(Y)$ (refer to Section 2 for some remarks on the term "model coordinate chart") so that

i) $F(W_1 \times \cdots \times W_n) \subset \Omega$, and
ii) $\overline{W}_j \cup \partial X_j$ is indiscrete for each $j = 1, \ldots, n$,

then the principal part of our proof would reduce to an application of [CG15, Theorem 1.2] by Chakrabarti–Gorai. However, it is far from clear that one can find coordinate charts that satisfy **both** (*i*) and (*ii*). The role of the map Ψ is to compensate for this difficulty.

Step 1. *Finding local candidates for $F_1, \ldots, F_n$*

The argument at this stage of our proof closely follows that of Edigarian–Zwonek [EZ05] and Chakrabarti–Gorai [CG15]. But since we must modify the map g (see (6.2) above) in order to use the latter argument — which has consequences on what follows — we shall present parts of this argument in some detail. We begin by defining $G := \Psi \circ g$ (the need for this map is hinted at by our preceding remarks). By the definition of the map Ψ, its holomorphic derivative is non-singular on an open dense subset of $\mathrm{Sym}^n(\mathcal{W})$. Therefore — since F is a proper holomorphic map — the complex Jacobian of G does not vanish identically on $V_1 \times \cdots \times V_n$. We expand this latter determinant along the last column to conclude that there exists $\mu \in \{1, \ldots, n\}$ such that

$$\det \left[\frac{\partial G_i}{\partial z_j} \right]_{i=1,\ldots,n,\, i \neq \mu,\, j=1,\ldots,n-1} \not\equiv 0 \text{ on } V_1 \times \cdots \times V_n. \tag{6.5}$$

Let us write $\theta := \Psi \circ h$, $\theta^{(m)} := \Psi \circ \phi_{\nu_m}$, $m = 1, 2, 3, \ldots$, and $\mathcal{V} := V_1 \times \cdots \times V_{n-1}$. Owing to (6.4), there exists a $C \in \mathbb{C}$ such that

$$C^n - C^{n-1}\theta_1 + \cdots + (-1)^{n-1} C \theta_{n-1} + (-1)^n \theta_n \equiv 0 \text{ on } \mathcal{V}, \tag{6.6}$$

where $\theta = (\theta_1, \ldots, \theta_n)$. Differentiating with respect to z_j, $j = 1, \ldots, n-1$, we get

$$-C^{n-1} \frac{\partial \theta_1}{\partial z_j} + \cdots + (-1)^{n-1} C \frac{\partial \theta_{n-1}}{\partial z_j} + (-1)^n \frac{\partial \theta_n}{\partial z_j} \equiv 0 \text{ on } \mathcal{V}. \tag{6.7}$$

Rearranging (6.7), we get the following system of $(n-1)$ equations:

$$\sum_{k=1,\dots,n,\,k\neq\mu} (-1)^k C^{n-k}\frac{\partial\theta_k}{\partial z_j} \;=\; (-1)^{\mu+1}C^{n-\mu}\frac{\partial\theta_\mu}{\partial z_j} \;\text{ on } \mathcal{V},\;\; j=1,\dots,n-1. \tag{6.8}$$

Given an $(n-1)\times n$ matrix B and $l\in\{1,\dots,n\}\setminus\{\mu\}$, denote by $\Delta_l(B)$ the determinant of the $(n-1)\times(n-1)$ matrix obtained by:

- deleting the μ-th column of B; and
- replacing the l-th column by the μ-th column of B.

Denote by $\Delta_\mu(B)$ the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the μ-th column of B. Note that each of the functions Δ_j is a polynomial in the entries of the matrix B.

We now introduce the $(n-1)\times n$ matrices

$$D_{n-1}\theta(z') \;:=\; \left[\frac{\partial\theta_k}{\partial z_j}(z')\right]_{1\leq j\leq n-1,\,1\leq k\leq n} \quad\text{and}\quad {}^mD_{n-1}\theta(z') \;:=\; \left[\frac{\partial\theta_k^{(m)}}{\partial z_j}(z')\right]_{1\leq j\leq n-1,\,1\leq k\leq n},$$

where $z' := (z_1,\dots,z_{n-1})$. We also set:

$$\mathfrak{A} \;:=\; \big\{z'\in\mathcal{V} : \Delta_\mu\big(D_{n-1}\theta(z')\big)=0\big\}.$$

Depending on $\mathfrak{A}$, we need to consider two cases.

Case 1. $\mathfrak{A}\subsetneq\mathcal{V}$.

By applying Cramer's rule to the system described by (6.8), we get:

$$(-1)^l C^{n-l} \;=\; (-1)^\mu C^{n-\mu}\frac{\Delta_l(D_{n-1}\theta)}{\Delta_\mu(D_{n-1}\theta)} \;\text{ on } (\mathcal{V}\backslash\mathfrak{A}) \text{ and } \; l\in\{1,\dots,n\}\backslash\{\mu\}.$$

If $\mu\neq 1$, we shall argue by taking $l=\mu-1$ in the above. If $\mu=1$, we shall take $l=2$. We shall first consider the case $\mu\neq 1$. In this case, the above equation gives

$$-\,C\Delta_\mu(D_{n-1}\theta) = \Delta_{\mu-1}(D_{n-1}\theta) \;\text{ on } \mathcal{V}. \tag{6.9}$$

Case 2. $\mathfrak{A}=\mathcal{V}$.

As in Case 1, we assume $\mu\neq 1$. Since the system (6.8) — treating C as the indeterminate — admits a solution, $\Delta_\mu(D_{n-1})\equiv 0$ forces on us the conclusion (6.9) for trivial reasons.

So, in each of the above cases, we get the identity (6.9). Differentiating this identity with respect to z_j and eliminating C, we get the relations

$$\Delta_\mu(D_{n-1}\theta)\frac{\partial\Delta_{\mu-1}(D_{n-1}\theta)}{\partial z_j} - \Delta_{\mu-1}(D_{n-1}\theta)\frac{\partial\Delta_\mu(D_{n-1}\theta)}{\partial z_j} \;\equiv\; 0 \;\text{ on } \mathcal{V},\;\; j=1,\dots,n-1.$$

The left-hand sides of the above relations are constituted of polynomial expressions involving

$$\lim_{m\to\infty} g_s\left(z_1,\dots,z_{n-1},w_{\nu_m}\right),\;\; s=1,\dots,n,$$

their compositions with the function χ, and their partial derivatives (with respect to $z_1,\dots,z_{n-1}$) up to order two. Hence, by Weierstrass's theorem on the derivatives of holomorphic functions, we have

$$\begin{aligned}
&\lim_{m\to\infty}\Delta_\mu({}^mD_{n-1}\theta)(z')\frac{\partial\Delta_{\mu-1}({}^mD_{n-1}\theta)}{\partial z_j}(z')-\Delta_{\mu-1}({}^mD_{n-1}\theta)(z')\frac{\partial\Delta_\mu({}^mD_{n-1}\theta)}{\partial z_j}(z')\\
&=\Delta_\mu(D_{n-1}\theta)(z')\frac{\partial\Delta_{\mu-1}(D_{n-1}\theta)}{\partial z_j}(z')-\Delta_{\mu-1}(D_{n-1}\theta)(z')\frac{\partial\Delta_\mu(D_{n-1}\theta)}{\partial z_j}(z')\\
&=0\quad \forall z'\in\mathcal{V},\;\; j=1,\dots,n-1. \qquad (6.10)
\end{aligned}$$

Consider the functions $\tau_j : V_1\times\dots\times V_n\to\mathbb{C}$ defined as follows:

$$\begin{aligned}
\tau_j(z',z_n) \;:=\; &\Delta_\mu(D_{n-1}G)(z',z_n)\frac{\partial\Delta_{\mu-1}(D_{n-1}G)}{\partial z_j}(z',z_n)\\
&-\Delta_{\mu-1}(D_{n-1}G)(z',z_n)\frac{\partial\Delta_\mu(D_{n-1}G)}{\partial z_j}(z',z_n),
\end{aligned}$$

for each $j=1,\dots,n-1$. Here, $D_{n-1}G(\bullet,z_n)$ is an $(n-1)\times n$ matrix that is defined in the same way as $D_{n-1}\theta$. Observe that (6.10) holds for any subsequence $\{w_{\nu_m}\}\subset\{w_\nu\}$ with the properties discussed right after (6.2), where $V_n\ni w_\nu\to q$. Finally, as $q\in\partial X\cap U_n$ was picked arbitrarily, (6.10) implies that

$$\tau_j(z',\zeta)\longrightarrow 0 \;\text{ as } \zeta\to\psi(U_n)\cap\partial V_n \text{ for each } z'\in\mathcal{V},$$

and for each $j=1,\dots,n-1$. Thus we can extend each τ_j to a continuous function $\widetilde{\tau}_j$ defined on $V_1\times\dots\times V_{n-1}\times\psi(U_n)$ by setting $\widetilde{\tau}_j(z',z_n)=0$ whenever $z_n\in\psi(U_n)\setminus V_n$. By Rado's theorem — see [Nar71, Chapter 4] — $\widetilde{\tau}_j$ is holomorphic on $V_1\times\dots\times V_{n-1}\times\psi(U_n)$. Let us now fix $z'\in\mathcal{V}$ and $j : 1\le j\le n-1$. By construction, $\psi(U_n)\setminus V_n$ has at least one limit point in $\psi(U_n)$. Thus, by the identity theorem, $\widetilde{\tau}_j(z',\bullet)$ is identically 0. As this holds true for every z' and j, it follows that each τ_j is identically 0.

Set

$$\gamma_n := -\frac{\Delta_{\mu-1}(D_{n-1}G)}{\Delta_\mu(D_{n-1}G)}.$$

The function γ_n is well-defined on the set $(V_1\times\dots\times V_n)\setminus\mathcal{A}$, where

$$\mathcal{A} \;:=\; \{z\in V_1\times\dots\times V_n : \Delta_\mu\big(D_{n-1}G(z)\big)=0\}.$$

By (6.5), $\mathcal{A}$ is a proper analytic subvariety of $V_1\times\dots\times V_n$. Observe that $\tau_j|_{(V_1\times\dots\times V_n)\setminus\mathcal{A}}$ is the numerator of $\frac{\partial\gamma_n}{\partial z_j}$, whence

$$\frac{\partial\gamma_n}{\partial z_j}\;\equiv\;0 \;\text{ on } (V_1\times\dots\times V_n)\setminus\mathcal{A},$$

for $j=1,\dots,n-1$. Since $\mathcal{A}$ is a proper analytic subvariety, this implies that γ_n depends only on z_n — in the sense of Definition 6.1 — on each set of the form $\mathscr{M}\setminus\mathcal{A}$, where $\mathscr{M}$ is a connected component of $V_1\times\dots\times V_n$.

Appealing to (6.9), and arguing in the same manner as above, we get

$$\gamma_n(z',\zeta)\longrightarrow C \;\text{ as } \zeta\to\psi(U_n)\cap\partial V_n \text{ for each } z'\in\mathcal{V},$$

where, we now recall, C satisfies the equation (6.6). Again, by an argument involving Rado's theorem — see [EZ05] or [CG15] — that is analogous to the one above, it follows that

$$\begin{aligned}
\gamma_n^n(z)-\gamma_n^{n-1}(z)G_1(z)+\dots+(-1)^{n-1}\gamma_n(z)G_{n-1}(z)&\\
+(-1)^nG_n(z)\;\equiv\;0\;\;\forall z\in(V_1\times\dots\times V_n)\setminus\mathcal{A},&\quad (6.11)
\end{aligned}$$

where we write $G = (G_1, \dots, G_n)$. This shows that $\gamma_n((V_1 \times \dots \times V_n) \setminus \mathcal{A}) \subset \chi(Y)$ which, by the choice of χ, is bounded. By Riemann's removable singularities theorem, γ_n extends to be holomorphic on $V_1 \times \dots \times V_n$.

A completely analogous argument can be given — which results in a slightly different expression for γ_n — when $\mu = 1$ (in which case, we take $l = 2$, l as introduced at the beginning of Step 1).

Repeating this argument with some i replacing n above yields us maps $\gamma_i : V_1 \times \dots \times V_n \to \mathbb{C}$ that satisfy equations analogous to (6.11). What we have at this stage is summarized by the following commutative diagram:

$$\begin{tikzcd}
X_1 \times \dots \times X_n \arrow[r, "F"] & \mathrm{Sym}^n(Y) \arrow[r, "\Psi"] & \mathbb{C}^n \\
 & Y^n \arrow[u, "\pi_{Sym}"] \arrow[r, "(\chi\circ\pi_1, \dots, \chi\circ\pi_n)"] & \mathbb{C}^n \arrow[u, "\pi^{(n)}"'] \\
V_1 \times \dots \times V_n \arrow[uu, "(\psi_1^{-1}\circ\pi_1, \dots, \psi_n^{-1}\circ\pi_n)"] \arrow[urr, "(\gamma_1, \dots, \gamma_n)"] & &
\end{tikzcd}$$

where we use π_j, $j = 1, \dots, n$, to denote the projection onto the j-th coordinate (where the product domain in question is understood from the context). Let us write:

$$\begin{aligned}
\mathscr{C} &:= \text{the set of critical points of } \pi_{Sym} : Y^n \to \mathrm{Sym}^n(Y), \\
\mathscr{C}^* &:= \text{the set of critical points of } \pi^{(n)} : \mathbb{C}^n \to \mathbb{C}^n.
\end{aligned}$$

Here $\pi^{(n)}$ is as introduced in Section 1. We now find connected open sets $W_j^* \subset W_j$, $j = 1, \dots, n$, that are so small that:

a) $F(W_1^* \times \dots \times W_n^*) \cap \pi_{Sym}(\mathscr{C}) \cap \Psi^{-1}\big(\pi^{(n)}(\mathscr{C}^*)\big) = \emptyset$;

b) $\pi^{(n)}$ is invertible on $\Psi\big(F(W_1^* \times \dots \times W_n^*)\big)$; and

c) The map $(\chi \circ \pi_1, \dots, \chi \circ \pi_n)$ is invertible on each image of $\Psi\big(F(W_1^* \times \dots \times W_n^*)\big)$ under a branch of a local inverse of $\pi^{(n)}$ that intersects the image of $(\chi \circ \pi_1, \dots, \chi \circ \pi_n)$.

Let $\big(\pi^{(n)}\big)_s^{-1}$, $s = 1, \dots, n!$, denote the branches introduced in (c). The definition of the map Ψ ensures that, in fact, the images of $\Psi\big(F(W_1^* \times \dots \times W_n^*)\big)$ under each $\big(\pi^{(n)}\big)_s^{-1}$ are contained in $(\chi \circ \pi_1, \dots, \chi \circ \pi_n)(Y)$. From this and a routine diagram-chase — since, by construction, the arrow representing $\pi^{(n)}$ can be reversed on $\Psi\big(F(W_1^* \times \dots \times W_n^*)\big)$ — we see that there exists a number s^0, $1 \le s^0 \le n!$ such that

$$(\gamma_1, \dots, \gamma_n) \circ \big(\psi_1(W_1^*) \times \dots \times \psi_n(W_n^*)\big) \;=\; \big(\pi^{(n)}\big)_{s^0}^{-1}\big(\Psi(F(W_1^* \times \dots \times W_n^*))\big).$$

Thus, by (c), there is a local holomorphic inverse — call it $\mathscr{I} \equiv (\mathscr{I}_1, \dots, \mathscr{I}_n)$ — of $(\chi \circ \pi_1, \dots, \chi \circ \pi_n)$ such that the maps

$$\mathsf{f}_j := \mathscr{I}_j \circ (\gamma_1, \dots, \gamma_n) \circ (\psi_1 \circ \pi_1, \dots, \psi_n \circ \pi_n)$$

are well-defined on $W_1^* \times \dots \times W_n^*$ and holomorphic, $j = 1, \dots, n$. From the above commutative diagram, we see that

$$F|_{W_1^* \times \dots \times W_n^*} = \pi_{Sym} \circ (\mathsf{f}_1, \dots, \mathsf{f}_n).$$

Since, by construction, $\psi_1(W_1^*)\times\cdots\times\psi_n(W_n^*)$ lies in a connected component of $V_1\times\cdots\times V_n$, γ_j depends only on z_j on $\psi_1(W_1^*)\times\cdots\times\psi_n(W_n^*)$ for each $j=1,\dots,n$. Then, owing to the structure of the map $(\chi\circ\pi_1,\dots,\chi\circ\pi_n)$, of which $\mathscr{I}$ is a local inverse, it follows that

$$\text{for each } j,\ j=1,\dots,n, \text{ the map } \mathsf{f}_j : W_1^*\times\cdots\times W_n^*\to Y_j \text{ depends only on the } j\text{-th coordinate on } W_1^*\times\cdots\times W_n^*. \tag{6.12}$$

Step 2. *Establishing the (global) existence of $F_1,\dots,F_n$*
We abbreviate $X_1\times\cdots\times X_n$ to X. Let $\mathscr{E} := F^{-1}\big(\pi_{Sym}(\mathscr{C})\big)$ ($\mathscr{C}$ is as introduced above), which is a proper analytic subset of X. If $x\in X\setminus\mathscr{E}$, then we can find a connected product neighbourhood Ω_x of x such that the map $F|_{\Omega_x}$ lifts to Y^n (i.e., it admits a holomorphic map $f:\Omega_x\to Y^n$ such that $\pi_{Sym}\circ f = F|_{\Omega_x}$).

Fix an $x_0\in X\setminus\mathscr{E}$. Consider any path $\Gamma:[0,1]\to X\setminus\mathscr{E}$ such that $\Gamma(0)$ is in $W_1^*\times\cdots\times W_n^*$ and $\Gamma(1)=x_0$. Here, $W_j^*\subset X_j$, $j=1,\dots,n$, are the domains introduced towards the end of the argument in Step 1. We can cover $\Gamma([0,1])$ by finitely many product neighbourhoods — call them $\Omega^0,\Omega^1,\dots\Omega^s$ — on which the map F lifts to Y^n. Moreover, it is easy to see that we can find $\Omega^0,\Omega^1,\dots\Omega^s$ and lifts $(f_1^i,\dots,f_n^i):\Omega^i\to Y^n$ of $F|_{\Omega^i}$ to Y^n for each i such that:

- $\Omega^0 = W_1^*\times\cdots\times W_n^*$;
- $(f_1^0,\dots,f_n^0):\Omega^0\to Y^n$ is the map $(\mathsf{f}_1,\dots,\mathsf{f}_n)$ provided by Step 1;
- $\Omega^i\cap\Omega^{i-1}\neq\emptyset$ for $i=1,\dots,s$;
- For each $i=1,\dots,s$, $f_j^i\big|_{K^i}\equiv f_j^{i-1}\big|_{K^i}$ for each $j=1,\dots,n$, where K^i is some connected component of $\Omega^i\cap\Omega^{i-1}$.

Then, owing to (6.12), it follows from the identity theorem and induction that each f_j^s depends only on the j-th coordinate on Ω^s.

In short, given any $x_0\in X\setminus\mathscr{E}$, we can find a product neighborhood $N=N_1\times\cdots\times N_n\ni x_0$ and maps $f_j:N\to Y$ that depend only on the j-th coordinate on N such that $F|_N = \pi_{Sym}\circ(f_1,\dots,f_n)$.

Claim. *This $(f_1,\dots,f_n)$ does not depend on the choice of path Γ joining x_0 to $W_1^*\times\cdots\times W_n^*$ or the choice of Ω^i, $i=1,\dots,s$, covering $\Gamma([0,1])$.*
To see this, suppose $(\varphi_1,\dots,\varphi_n)$ is a lift of F to Y^n on a neighbourhood of x_0 obtained by carrying out the above procedure along some different path or via a different cover of $\Gamma([0,1])$. Then, there exists a permutation ρ of $\{1,\dots,n\}$ such that

$$(\varphi_1,\dots,\varphi_n)\equiv(f_{\rho(1)},\dots,f_{\rho(n)}) \text{ on a neighbourhood of } x_0.$$

Now, φ_j depends **only** on the j-coordinate. The above equation implies that φ_j depends only on the $\rho(j)$-th coordinate, $j=1,\dots,n$. This is impossible unless ρ is the identity permutation. Hence the claim.

Since the $x_0\in X\setminus\mathscr{E}$ mentioned above is completely arbitrary, it follows from the above Claim that we have holomorphic maps $\widetilde{F}_j : X\setminus\mathscr{E}\to Y$, $j=1,\dots n$, such that $\widetilde{F}_j$ depends only on the j-th coordinate on $X\setminus\mathscr{E}$ (in the sense of Definition 6.1) and such that

$$F|_{X\setminus\mathscr{E}} = \pi_{Sym}\circ(\widetilde{F}_1,\dots,\widetilde{F}_n). \tag{6.13}$$

By Lemma 3.4, Y is hyperbolically embedded in S (S is as introduced at the beginning of this proof). Then, by Results 3.2 and 3.3, each $\widetilde{F}_j$ extends to a holomorphic map on X_j, $j=1,\dots,n$. By continuity, we can now view these extended maps as holomorphic maps

$F_j : X_j \to Y$. In view of (6.13), we have our result. The properness of each of the maps F_j is straightforward. $\square$

Acknowledgements

A part of the research in this paper was conducted during a visit by Gautam Bharali and Indranil Biswas to the International Centre for Theoretical Sciences (ICTS) to participate in the meeting *Complex Geometry* in March 2017 (programme code: **ICTS/Prog-compgeo/2017/03**). They thank the ICTS for its support. Gautam Bharali is supported by a Swarnajayanti Fellowship (grant no. DST/SJF/MSA-02/2013-14). Indranil Biswas is supported by a J.C. Bose Fellowship. Divakaran Divakaran is supported by a project grant from the Indian Institute of Science Education and Research Bhopal (grant no. IISERB/INS/MATH/2016091). Jaikrishnan Janardhanan is supported by a DST-INSPIRE fellowship from the Department of Science and Technology, India.

References

[BS49] Heinrich Behnke and Karl Stein, *Entwicklung analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. **120** (1949), 430–461.

[CG15] Debraj Chakrabarti and Sushil Gorai, *Function theory and holomorphic maps on symmetric products of planar domains*, J. Geom. Anal. **25** (2015), no. 4, 2196–2225.

[CS93] Ciro Ciliberto and Edoardo Sernesi, *On the symmetric products of a curve*, Arch. Math. (Basel) **61** (1993), no. 3, 285–290.

[EZ05] Armen Edigarian and Włodzimierz Zwonek, *Geometry of the symmetrized polydisc*, Arch. Math. (Basel) **84** (2005), no. 4, 364–374.

[Jan14] Jaikrishnan Janardhanan, *Proper holomorphic mappings between hyperbolic product manifolds*, Internat. J. Math. **25** (2014), no. 04, 1450039.

[Jos06] Jürgen Jost, *Compact Riemann Surfaces*, 3rd ed., Universitext, Springer-Verlag, Berlin, 2006.

[Kie70] Peter Kiernan, *On the relations between taut, tight and hyperbolic manifolds*, Bull. Amer. Math. Soc. **76** (1970), 49–51.

[Kie73] ______, *Extensions of holomorphic maps*, Trans. Amer. Math. Soc. **172** (1973), 347–355.

[Kob98] Shoshichi Kobayashi, *Hyperbolic Complex Spaces*, Springer, Berlin, 1998.

[Kod79] Akio Kodama, *On bimeromorphic automorphisms of hyperbolic complex spaces*, Nagoya Math. J. **73** (1979), 1–5.

[Mar63] Henrik H. Martens, *A new proof of Torelli's theorem*, Ann. of Math. (2) **78** (1963), 107–111.

[Nar71] Raghavan Narasimhan, *Several Complex Variables*, The University of Chicago Press, Chicago, Ill.-London, 1971, Chicago Lectures in Mathematics.

[RS60] R. Remmert and K. Stein, *Eigentliche holomorphe Abbildungen*, Math. Z. **73** (1960), no. 2, 159–189.

[Whi72] Hassler Whitney, *Complex Analytic Varieties*, Addison-Wesley Publishing Co., Reading (Mass.)–London–Don Mills (Ont.), 1972.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: `bharali@iisc.ac.in`

School of Mathematics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Mumbai 400005, India
E-mail address: `indranil@math.tifr.res.in`

Department of Mathematics, Indian Institute of Science Education and Research Bhopal, Bhopal 462066, India
E-mail address: `divakaran@iiserb.ac.in`

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India
E-mail address: `jaikrishnan@iitm.ac.in`