RIGIDITY OF HOLOMORPHIC MAPS BETWEEN FIBER SPACES

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Abstract. In the study of holomorphic maps, the term "rigidity" refers to certain types of results that give us very specific information about a general class of holomorphic maps owing to the geometry of their domains or target spaces. Under this theme, we begin by studying when, given two compact connected complex manifolds X and Y, a degree-one holomorphic map $f : Y \longrightarrow X$ is a biholomorphism. Given that the real manifolds underlying X and Y are diffeomorphic, we provide a condition under which f is a biholomorphism. Using this result, we deduce a rigidity result for holomorphic self-maps of the total space of a holomorphic fiber space. Lastly, we consider products $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ of compact connected complex manifolds. When X_1 is a Riemann surface of genus ≥ 2, we show that any non-constant holomorphic map $F : Y \longrightarrow X$ is of a special form.

1. Introduction

The *degree* of a continuous map $f : Y \longrightarrow X$ between compact connected oriented smooth manifolds of dimension n is defined as follows:

$$f^*(1_X) \;=\; \text{degree}(f) \cdot 1_Y \,,$$

where 1_X (respectively, 1_Y) is the unique generator of $H^n(X, \mathbb{Z})$ (respectively, $H^n(Y, \mathbb{Z})$) compatible with the orientation. When X and Y are compact connected complex manifolds and $f : Y \longrightarrow X$ is holomorphic, there are natural situations in which, if f is a degree-one map, then it is automatically a biholomorphism. If $X = Y$ is a compact Riemann surface, this follows from the Riemann–Hurwitz formula. We encounter a significant obstacle when $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y \geq 2$. Note that degree$(f)$ as defined is precisely the topological degree of of f. When degree$(f) = 1$, it follows that the pre-image of a generic point in X — but *not necessarily every* point — is a singleton. Thus, it may happen (and it does: consider the case when $f : Y \longrightarrow X$ is a blow-up) that there exists a non-empty proper subvariety of Y on which f fails to be injective. Hence, one must impose some conditions on X and Y for f to be a biholomorphism. We explore this phenomenon when it is known *a priori* that the *real* manifolds underlying X and Y are diffeomorphic. We prove the following:

Theorem 1.1. *Let X and Y be compact connected complex manifolds such that the underlying real manifolds are diffeomorphic, and let $f : Y \longrightarrow X$ be a degree-one holomorphic map. If $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$, then f is a biholomorphism.*

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Remark 1.2. One expects some restrictions on the *complex* geometry of X and Y for a degree-one map $f : Y \longrightarrow X$ to be biholomorphic. The assumption on cohomology in Theorem 1.1 is a restriction of this sort. It is a rather mild condition: it is satisfied, for instance, whenever X and Y are Kähler and real diffeomorphic. Indeed, this follows immediately from the fact that for a compact Kähler manifold Z, we have $\dim H^1(Z, \mathcal{O}_Z) = \frac{1}{2}\dim H^2(Z, \mathbb{C})$ (the latter is a consequence of the Hodge decomposition). The vector space $H^1(X, \mathcal{O}_X)$ parametrizes the space of all infinitesimal deformations of any holomorphic line bundle on X (the space of holomorphic line bundles on X is a group; infinitesimal deformations of these line bundles are independent of the specific line bundle). So, the meaning of the cohomology condition in Theorem 1.1 is that the infinitesimal deformations of any holomorphic line bundle on X and Y are assumed to coincide. This assumption is used in our proof essentially in this form.

The above theorem forms the key final step in the following rigidity result for a holomorphic self-map of a fiber space. Loosely speaking, if a holomorphic map of the total space sends just a single fiber of a fiber space to another fiber, then it must be a map of fiber spaces. More precisely:

Theorem 1.3. *Let X and Y be complex manifolds, let $p : Y \longrightarrow X$ be a proper holomorphic surjective submersion having connected fibers, and let X be connected. Let $F : Y \longrightarrow Y$ be a holomorphic map such that there exist points $a, b \in X$ with the property that F maps the fiber $Y_a := p^{-1}\{a\}$ into the fiber $Y_b := p^{-1}\{b\}$. Then:*

- *a) The map F is a map of fiber spaces: i.e., there exists a holomorphic map $f : X \longrightarrow X$ such that $p \circ F = f \circ p$.*
- *b) If, additionally, $F|_{Y_a}$ is a degree-one map from Y_a to Y_b and if $\dim H^1(Y_x, \mathcal{O}_{Y_x})$ is independent of $x \in X$, then F is a fiberwise biholomorphism.*

The term "rigidity" for holomorphic maps often refers to the phenomenon of a holomorphic map being structurally simple owing to the geometry of its domain or target space; see, for example, results by Remmert–Stein [6, Satz 12, 13], Kaup [3, Satz 5.2] or Kobayashi [4, Theorem 7.6.11]. In the set-up of Theorem 1.3, a rigidity result in this sense would require one to determine, for instance, conditions on (Y, X, p) that would cause any $F : Y \longrightarrow Y$ to preserve at least one fiber. This seems to be a difficult requirement. However, in the simpler set-up of certain product spaces, we do get a rigidity result of the above-mentioned style. It comes as a corollary of the following:

Proposition 1.4. *Let $Y = Y_1 \times Y_2$ be a product of compact connected complex manifolds and let $f : Y \longrightarrow X$ be a holomorphic map into a compact Riemann surface of genus ≥ 2 . Then (denoting each $y \in Y$ as (y_1, y_2), $y_j \in Y_j$, $j = 1, 2$) f depends on at most one of y_1 and y_2.*

This has the obvious corollary:

Corollary 1.5. *Let $Y = Y_1 \times Y_2$ and $X = X_1 \times X_2$ be products of compact connected complex manifolds. Assume that X_1 is a compact Riemann surface of genus ≥ 2 . Let $F = (F_1, F_2) : Y \longrightarrow X$ be a holomorphic map. Then, there is a $j \in \{1, 2\}$ such that F has the form $F(y_1, y_2) = (F_1(y_j), F_2(y_1, y_2))$.*

One may compare the above to a result by Janardhanan [2]. The hypothesis of Corollary 1.5 is weakened to allow the factors of Y to have arbitrary dimension; but with

$\dim_{\mathbb{C}} Y_j = \dim_{\mathbb{C}} X_j = 1$, Janardhanan is also able to handle the case when some of the factors are non-compact.

Remark 1.6. The assumptions in Proposition 1.4 and Corollary 1.5 cannot be weakened appreciably. Suppose $Y_1 \neq Y_2$ are two holomorphically distinct compact connected complex manifolds and suppose X' is a compact Riemann surface of genus ≥ 2 such that there are non-constant holomorphic maps $f_j : Y_j \longrightarrow X'$, $j = 1, 2$. Then, just taking $X = X' \times X'$ and $f = (f_1, f_2)$ in Proposition 1.4 shows that it cannot be true in general if $\dim_{\mathbb{C}} X \geq 2$ (more involved examples can be constructed in which $\dim_{\mathbb{C}} X \geq 2$ and is not a product). As for the requirement on the genus of X being essential: the reader is referred to [2, Remark 1.7].

2. The proof of Theorem 1.1

Let n be the complex dimension of X (also of Y). Let

$$f^* : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q}) \tag{2.1}$$

be the pullback homomorphism for f. We will show that f^* is an isomorphism. To do so, given any non-zero class $c \in H^i(X, \mathbb{Q})$, take $c' \in H^{2n-i}(X, \mathbb{Q})$ such that $c \cup c' \neq 0$. Since

$$((f^*c) \cup (f^*c')) \cap [Y] = f^*(c \cup c') \cap [Y] = \text{degree}(f) \cdot (c \cup c') \cap [X],$$

we conclude that $f^*c \neq 0$ for $c \neq 0$. Hence f^* is injective. We have $\dim H^*(X, \mathbb{Q}) = \dim H^*(Y, \mathbb{Q})$ because X and Y are diffeomorphic as real manifolds. Hence the injective homomorphism f^* is an isomorphism.

The differential of f produces a holomorphic section of the holomorphic line bundle $\Omega^n_Y \otimes f^*(\Omega^n_X)^*$ on Y. This section will be denoted by s. Let

$$D := \text{Divisor}(s) \subset Y \tag{2.2}$$

be the effective divisor of this section. We note that f is a biholomorphism from $Y \setminus D$ to $X \setminus f(D)$. So to prove the theorem, it suffices to show that D is the zero divisor.

To prove that D is the zero divisor, we first note that $f(D) \subset X$ is of complex codimension at least two. Indeed, the given condition that the degree of f is one implies that if, for an irreducible component D' of D, the image $f(D')$ is a divisor in X, then f is an isomorphism on a neighborhood of D'. But this implies that D' is not contained in D. Therefore, $f(D) \subset X$ is of complex codimension at least two.

Let $c(D) \in H^2(Y, \mathbb{Q})$ be the cohomology class of D. Since $f(D) \subset X$ is of complex codimension at least two, and f^* in (2.1) is an isomorphism, it follows that

$$c(D) = 0. \tag{2.3}$$

Now consider the short exact sequence of sheaves on Y

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_Y \xrightarrow{\exp} \mathcal{O}^*_Y \longrightarrow 0.$$

Let

$$H^1(Y, \mathcal{O}_Y) \xrightarrow{\beta} H^1(Y, \mathcal{O}^*_Y) \xrightarrow{q} H^2(Y, \mathbb{Z}) \tag{2.4}$$

be the long exact sequence of cohomologies associated to it. The class in $H^1(Y, \mathcal{O}^*_Y)$ for the holomorphic line bundle $\Omega^n_Y \otimes f^*(\Omega^n_X)^*$ (this is the line bundle associated to D) will be denoted by $\gamma(D)$. For any positive integer m, the class in $H^1(Y, \mathcal{O}^*_Y)$ for the holomorphic line bundle $(\Omega^n_Y \otimes f^*(\Omega^n_X)^*)^{\otimes m}$ is $\gamma(D)^m$. From (2.3) if follows that there is a positive integer N such that

$$q(\gamma(D)^N) \,=\, 0\,,$$

where q is the homomorphism in (2.4). Therefore, there is a cohomology class $\alpha \in H^1(Y, \mathcal{O}_Y)$ such that

$$\beta(\alpha) \,=\, \gamma(D)^N. \tag{2.5}$$

Consider the pullback homomorphism

$$F\,:\, H^1(X,\, \mathcal{O}_X) \,\longrightarrow\, H^1(Y,\, f^*\mathcal{O}_X) \,=\, H^1(Y,\, \mathcal{O}_Y) \tag{2.6}$$

for f. We will show that F is injective. To prove this, first note that $f_*\mathcal{O}_Y = \mathcal{O}_X$. Using this isomorphism, the natural homomorphism

$$H^1(X,\, f_*\mathcal{O}_Y) \,\longrightarrow\, H^1(Y,\, \mathcal{O}_Y) \tag{2.7}$$

coincides with F in (2.6). But the homomorphism in (2.7) is injective. Hence F is injective. We now invoke the assumption that $\dim H^1(X, \mathcal{O}_X) \,=\, \dim H^1(Y, \mathcal{O}_Y)$. Since F is an injective homomorphism between vector spaces of same dimension, we conclude that F is an isomorphism.

Take $\alpha \in H^1(X, \mathcal{O}_X)$ such that

$$\alpha \,=\, F(\alpha')\,, \tag{2.8}$$

where α is the cohomology class in (2.5). Let L be the holomorphic line bundle on X corresponding to the element $\beta'(\alpha') \in H^1(X, \mathcal{O}^*_X)$, where

$$\beta'\,:\, H^1(X,\, \mathcal{O}_X) \,\longrightarrow\, H^1(X,\, \mathcal{O}^*_X)$$

is the homomorphism given by $\exp : \mathcal{O}_X \longrightarrow \mathcal{O}^*_X$. From (2.5) and (2.8) it follows that

$$(\Omega^n_Y \otimes f^*(\Omega^n_X)^*)^{\otimes N} \,=\, f^*L\,.$$

The holomorphic line bundle $(\Omega^n_Y \otimes f^*(\Omega^n_X)^*)^{\otimes N}$ is holomorphically trivial on $Y \setminus D$. Consequently, L is holomorphically trivial on $X \setminus f(D)$ (recall that f is a biholomorphism from $Y \setminus D$ to $X \setminus f(D)$).

Since L is holomorphically trivial on $X \setminus f(D)$, and $f(D) \subset X$ is of complex codimension at least two, the holomorphic line bundle L on X is holomorphically trivial. Therefore, the holomorphic line bundle $(\Omega^n_Y \otimes f^*(\Omega^n_X)^*)^{\otimes N} \,=\, f^*L$ on Y is holomorphically trivial.

Finally, consider the section s in (2.2). Note that $s^{\otimes N}$ is a holomorphic section of $(\Omega^n_Y \otimes f^*(\Omega^n_X)^*)^{\otimes N}$ vanishing on D and nonzero elsewhere. On the other hand, any holomorphic section of the holomorphically trivial line bundle on Y is either nowhere zero, or it is identically zero. Therefore, we conclude that D is the zero divisor. This completes the proof of Theorem 1.1. $\square$

3. The proof of Theorem 1.3

Given the conditions on p, it follows from Ehresmann's theorem [1] that the triple (Y, X, p) is a $\mathcal{C}^\infty$-smooth fiber bundle with fiber F. Thus, for each $x \in X$, there is a connected open neighborhood U^x of x and a diffeomorphism $\varphi_x : p^{-1}(U^x) \longrightarrow U^x \times \mathsf{F}$ such that the diagram

$$\begin{tikzcd} p^{-1}(U^x) \arrow[r, "\varphi_x"] \arrow[dr, "p"'] & U^x \times \mathsf{F} \arrow[d, "\mathsf{proj}_1"] \\ & U^x \end{tikzcd}$$

commutes (here, proj_1 denotes the projection of $U^x \times \mathsf{F}$ onto the first factor).

Write $S := \{x \in X : \exists\, x' \in X \text{ such that } F(Y_x) \subset Y_{x'}\}$. We shall first show that S is an open set. Suppose $x_0 \in S$, whence there is a point $x' \in X$ such that $F(Y_{x_0}) \subset Y_{x'}$. Let $B_{x'}$ be a neighborhood around x' that is biholomorphic to a ball. By continuity of F and p, and due to compactness of Y_{x_0}, we can find an open neighborhood V_{x_0} of x_0 such that $p \circ F(p^{-1}(V_{x_0})) \subset B_{x'}$. Recall that, by hypothesis, the fibers of p are connected. Thus, for each $x \in V_{x_0}$:

- Y_x is a connected, compact complex manifold.
- $p \circ F$ maps each Y_x into an Euclidean open set.

It follows from the maximum modulus theorem that $p \circ F|_{Y_x}$ is constant for each $x \in V_{x_0}$. This means: $x_0 \in S \Rightarrow V_{x_0} \subset S$. In other words, S is an open set.

We now argue that S is closed. Write $d = \dim_{\mathbb{C}} X$ and $k = \dim_{\mathbb{C}} Y$. There is a $\mathcal{C}^\infty$-atlas $\{(W^y; x_1^y, \dots, x_{2d}^y, \Phi_1^y, \dots, \Phi_{2k}^y) : y \in Y\}$ of Y, where W^y is a coordinate patch centered at y such that

$$Y_{p(z)} \cap W^y \;=\; \{x_1^y = C_1^{p(z)}, \dots, x_{2d}^y = C_{2d}^{p(z)}\} \quad \forall\, z \in W^y,$$

and where each $C_j^{p(z)} \in \mathbb{R}$ is a constant that depends only on $p(z)$. More descriptively: $(x_1^y, \dots, x_{2d}^y, \Phi_1^y, \dots, \Phi_{2k}^y)$ imposes a smooth product structure upon W^y in such a way that each $\mathbb{R}$-affine slice of the domain $(x_1^y, \dots, x_{2d}^y, \Phi_1^y, \dots, \Phi_{2k}^y)(W^y) =: G^y \subset \mathbb{R}^{2(d+k)}$ by a translate of the "$\Phi_1^y \dots \Phi_{2k}^y$-plane" parametrizes a patch of some fiber of p. Let

$$\Omega_y \;:=\; \text{the connected component of } W^y \cap F^{-1}(W^{F(y)}) \text{ containing } y.$$

We claim that

$$p^{-1}(S) \cap \Omega_y \;=\; \bigcap_{i=1}^{2d} \; \bigcap_{\alpha \in \mathbb{N}^{2k} \setminus \{(0,\dots,0)\}} \left\{ z \in \Omega_y : \frac{\partial^\alpha \big(x_i^{F(y)} \circ F\big)}{\partial \Phi^{y,\alpha}}(z) = 0 \right\}. \tag{3.1}$$

Our notation $\partial^\alpha g / \partial \Phi^{y,\alpha}$ — for any $\mathcal{C}^\infty$-smooth function $g : \Omega_y \longrightarrow \mathbb{R}$ — perhaps needs a little clarification. Denote by ψ^y the coordinate map $\psi^y := (x_1^y, \dots, x_{2d}^y, \Phi_1^y, \dots, \Phi_{2k}^y) : \Omega_y \longrightarrow G^y$ described above. Let us write $\psi^y(z) = (x_1, \dots, x_{2d}, \Phi_1, \dots, \Phi_{2k})$, which are just points varying through G^y. Then, $\partial^\alpha g / \partial \Phi^{y,\alpha}$ is defined as:

$$\frac{\partial^\alpha g}{\partial \Phi^{y,\alpha}}(z) \;:=\; \frac{\partial^{|\alpha|} g \circ (\psi^y)^{-1}}{\partial \Phi_1^{\alpha_1} \dots \partial \Phi_{2k}^{\alpha_{2k}}}(\psi^y(z)), \quad z \in \Omega_y.$$

That $p^{-1}(S) \cap \Omega_y$ is a subset of the set on the right-hand side of (3.1) (call it K_y) is clear. Now, if $z \in K_y$, it implies that there is a small open neighborhood N_z of z such that the holomorphic map $p \circ F|_{Y_{p(z)}} : Y_{p(z)} \longrightarrow X$ is constant on the set $N_z \cap Y_{p(z)}$, since the Taylor coefficient (relative to local coordinates) of $p \circ F|_{Y_{p(z)}}$ at z corresponding to each $\alpha \neq (0, \dots, 0)$ is zero. By the principle of analytic continuation, $p \circ F$ must be constant on $Y_{p(z)}$. So, $p(z) \in S$ and hence $K_y \subset p^{-1}(S) \cap \Omega_y$. This establishes (3.1).

By (3.1), the intersection $p^{-1}(S) \cap \Omega_y$ is closed relative to each Ω_y. As $\{\Omega_y : y \in Y\}$ is an open cover of Y, we see that $p^{-1}(S)$ is closed. It is now easy to see, as (Y, X, p) is locally trivial, that S is closed. By hypothesis, $S \neq \varnothing$. As X is connected, it follows that $S = X$.

Since $S = X$, the map $f : X \longrightarrow X$ given by

$$f(x) \ := \ p \circ F(y) \text{ for any } y \in Y_x$$

is well-defined. Clearly $p \circ F = f \circ p$. Let us now *fix* a point $p \in \mathsf{F}$. For any $x_0 \in X$,

$$f|_{U^{x_0}}(x) \ = \ p \circ F \circ \varphi_{x_0}^{-1}(x, p) \quad \forall x \in U^{x_0},$$

whence f is $\mathcal{C}^\infty$-smooth. Therefore, relative to any local holomorphic coordinate system, we can apply the Cauchy–Riemann operator to the equation $p \circ F = f \circ p$ to conclude that f is holomorphic. This establishes part (a) of Theorem 1.3.

To prove part (b), recall that given any set $\Sigma \subset \mathcal{C}(\mathsf{F}; \mathsf{F})$ — where the latter denotes the space of all continuous self-maps of F endowed with the compact-open topology — the function from Σ to $\mathbb{Z}$ defined by $\psi \longmapsto \text{degree}(\psi)$ is locally constant. Hence, as X is connected and, by hypothesis, $\text{degree}(F|_{Y_a}) = 1$, we have

$$\text{degree}(F|_{Y_x}) \ = \ 1 \quad \forall x \in X. \tag{3.2}$$

From our hypothesis, it follows that

$$\dim H^1(Y_x, \mathcal{O}_{Y_x}) \ = \ \dim H^1(Y_{f(x)}, \mathcal{O}_{Y_{f(x)}}) \quad \forall x \in X.$$

In view of this, (3.2), and the fact that fibers are connected, we may apply Theorem 1.1 to conclude that F is a fiberwise biholomorphism. This completes the proof of Theorem 1.3.

□

4. Concerning Proposition 1.4 and Corollary 1.5

Corollary 1.5 is an absolutely obvious consequence of Proposition 1.4. Thus, we shall only discuss the latter.

The proof of Proposition 1.4 relies upon the following result of Kobayashi–Ochiai:

Result 4.1 (Theorem 7.6.1, [4]). *Let X and Y be two compact complex-analytic spaces. If Y is of general type, then the set of all dominant meromorphic maps of X to Y is finite.*

A couple of remarks are in order. First: we shall not define the term *general type* here as it is somewhat involved. We refer the reader to [4, § 7.4]. The fact that we need from [4, § 7.4] is that any compact Riemann surface with genus ≥ 2 is of general type.

Secondly: in their original announcement and proof of the above result in [5], Kobayashi and Ochiai require X to be Moišezon. However, in a footnote to [5], the authors observe that this restriction on X can be removed. The relevant argument is presented in the proof of [4, Theorem 7.6.1].

The proof of Proposition 1.4 . Since there is nothing to prove if f is constant, we shall assume that f is non-constant. For each $y = (y_1, y_2) \in Y$, we define:

$$f^{y_2} \ := \ f(\,\cdot\,, y_2), \qquad f^{y_1} \ := \ f(y_1, \,\cdot\,).$$

Assume that both f^{y_2} and f^{y_1} are non-surjective for each $y \in Y$. As f^{y_2} is a holomorphic map between *compact* complex manifolds, for each fixed $y_2 \in Y_2$, $f^{y_2}(Y_1) \subsetneq X$ is a complex-analytic subvariety. Thus, f^{y_2} is a constant map for each $y_2 \in Y_2$; call this constant $c(y_2)$. Fix a point $a \in Y_1$. By the same argument as above, we have

$$c(y_2) \;=\; f(a, y_2) \;=\; C \quad \forall y_2 \in Y_2,$$

where C is a constant. This contradicts the fact that f is non-constant. Hence, there exists a $j \in \{1, 2\}$ and a point $a = (a_1, a_2) \in Y$ such that f^{a_j} is surjective.

There is no loss of generality in taking $j = 1$. By continuity of f and compactness of Y_2, it follows that there is an open neighborhood $U \subset Y_1$ of a_1 such that the maps $f(z, \bullet) : Y_2 \longrightarrow X$ are surjective for each $z \in U$. Since X has genus ≥ 2, it is of general type. It follows from Result 4.1 (which is a refinement of an argument of Kobayashi–Ochiai [5]) that the surjective holomorphic maps from Y_2 to X form a finite set. Hence, f restricted to the open set $\pi_{Y,1}^{-1}(U)$ (where $\pi_{Y,k}$ denotes the projection of Y onto its kth factor) is independent of y_1. By the principle of analytic continuation, it follows that f is independent of y_1. Hence the desired result. $\square$

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