PROPER HOLOMORPHIC MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS REVISITED

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Abstract. We prove that a proper holomorphic map between two non-planar bounded symmetric domains of the same dimension, one of them being irreducible, is a biholomorphism. Our methods allow us to give a single, all-encompassing argument that unifies the various special cases in which this result is known. We discuss an application of these methods to domains having noncompact automorphism groups that are not assumed to act transitively.

1. Introduction and statement of results

The primary objective of this paper is to prove the following result:

Theorem 1.1. *Let D_1 and D_2 be two bounded symmetric domains of complex dimension $n \geq 2$. Assume that either D_1 or D_2 is irreducible. Then, any proper holomorphic mapping of D_1 into D_2 is a biholomorphism.*

The above theorem is known in several special cases. For $D_1 = D_2 = \mathbb{B}^n$, the (Euclidean) ball in $\mathbb{C}^n$, $n \geq 2$, the result was established by Alexander [1]. This is a pioneering work that has motivated several generalizations to proper holomorphic maps between certain types of smoothly bounded pseudoconvex domains. Henkin and Novikov [6] described a method for proving the above result when $D_1 = D_2$ ($= D$, say) and D is a bounded symmetric domain that is *not* of tube type. About a decade later, Tsai [18] established the result for D_1 and D_2 as above, provided D_1 is irreducible and $\mathrm{rank}(D_1) \geq \mathrm{rank}(D_2) \geq 2$.

Tsai's result is a broad metric-rigidity theorem (under the Bergman metric) for proper holomorphic maps of D_1 into D_2, where D_1 and D_2 are as above but not necessarily equidimensional. In such a result, the condition $\mathrm{rank}(D_1) \geq \mathrm{rank}(D_2) \geq 2$ is indispensible. Adapting Tsai's ideas to the equidimensional case, Tu [19] established Theorem 1.1 in the higher-rank case, assuming D_1 *is irreducible*. In using Tsai's ideas, the assumption that D_1 is irreducible is essential — see [19, Proposition 3.3] — and it is not clear that a small mutation of those ideas allows one to weaken this assumption. In our work, we are able to assume either D_1 or D_2 to be irreducible precisely by not relying too heavily on the fine structure of these domains. Indeed, we wish to emphasize that the focus of this work is *not* on mopping up the residual cases in Theorem 1.1. The methods in [6] (and [20], on which [6] relies), [18] and [19] are tied, in a rather maximalistic way, to the fine structure of a bounded symmetric domain. In contrast, we present some ideas that make very mild use of

1991 *Mathematics Subject Classification.* Primary: 32H02, 32M15; Secondary: 32H40.

Key words and phrases. Bounded symmetric domains, Harish-Chandra realization, Jordan triple systems, proper holomorphic maps, rigidity, Schwarz lemma.

GB is supported by a UGC Centre of Advanced Study grant. JJ is supported by a UGC Centre for Advanced Study grant and by a scholarship from the IISc.

the underlying orbit structure of the bounded symmetric domains. They could therefore be applied to manifolds whose automorphism groups are not assumed to act transitively but are merely "large enough". Theorem 1.5 is an illustration of this notion. These ideas also provide a *unified* argument, irrespective of rank or reducibility, for Theorem 1.1.

We need to be more precise about the preceding remarks. This requires some elaboration on the objects of interest. A bounded symmetric domain in $\mathbb{C}^n$ is the holomorphic imbedding in $\mathbb{C}^n$ of some Hermitian symmetric space of noncompact type. It is *irreducible* if it is not a product of bounded symmetric domains of lower dimension. Cartan studied Hermitian symmetric spaces of noncompact type and classified the irreducible ones, showing that they are one of six types of homogeneous spaces. An outcome of Harish-Chandra's work in [5] is that these homogeneous spaces (and products thereof) can be imbedded in $\mathbb{C}^n$ as bounded convex balanced domains (we say that a domain $D \subset \mathbb{C}^n$ is *balanced* if, for any $z \in D$, $\zeta z \in D$ for each ζ in the closed unit disc centered at $0 \in \mathbb{C}$). This imbedding is unique up to a linear isomorphism of $\mathbb{C}^n$. Such a realization of a bounded symmetric domain is called a *Harish-Chandra realization*.

The three main features that we wish to emphasize about this work are:

a) The arguments in [1] involve many estimates showing how a proper mapping maps conical regions with vertex on $\partial\mathbb{B}^n$ into the "admissible" approach regions of Korányi [10]. Boundary approach, in a somewhat different sense, plus Chern–Moser theory [4] make an appearance in [20]. In contrast, apart from, and owing to, a result of Bell [2] on boundary behaviour, our proof involves rather "soft" methods.

b) The techniques underlying [18] and [19] rely almost entirely on the fine structure of a bounded symmetric domain. Specifically, they involve studying the effect of a proper holomorphic map on the characteristic symmetric subspaces of a bounded symmetric domain of rank ≥ 2. In contrast, our techniques rely on only a coarse distinction between the different strata that comprise the boundary of an irreducible bounded symmetric domain (e.g., see Remark 4.8 below).

c) An advantage of arguments that rely on only a coarse resolution of a bounded symmetric domain is that some of them are potentially applicable to the study of domains that have noncompact automorphism groups, but are not assumed to be symmetric. A demonstration this viewpoint is the proof of Theorem 1.5 below.

Let D be a bounded symmetric domain. The main technical tool that facilitates our study of the structure of ∂D, and describes certain elements of $\mathrm{Aut}(D)$ with the optimal degree of explicitness, is the notion of Jordan triple systems. The application of Jordan triple systems to geometry appears to have been pioneered by Koecher [9]. Our reference on this subject are the lecture notes of Loos [13], which are devoted specifically to the bounded symmetric domains. Jordan triple systems and versions of the Schwarz lemma are our primary tools. We present next an outline of how we use these tools.

An important lemma, which is inspired by Alexander's work, is the following

Key Lemma 1.2. *Let D be a realization of an irreducible bounded symmetric domain of dimension $n \geq 2$ as a bounded convex balanced domain in $\mathbb{C}^n$. For $z \in D \setminus \{0\}$, let $\Delta_z := \{\zeta z : \zeta \in \mathbb{C}$ and $\zeta z \in D\}$. Let W_1 and W_2 be two regions in D such that $0 \in W_1 \cap W_2$ and let $F : D \to D$ be a holomorphic map. Assume that:*

i) *F maps W_1 biholomorphically onto W_2 with $F(0) = 0$.*

ii) *There exists a non-empty open set* $U \subset W_1 \setminus \{0\}$ *such that, for each* $z \in U$, $\Delta_z \subset W_1$ *and* $\Delta_{F(z)} \subset W_2$.

Then, F *is an automorphism of* D.

This is a consequence of Vigué's Schwarz lemma [21] (see Result 4.6 below), and the irreducibility of D is essential to the lemma.

Our proof of Theorem 1.1 may be summarized as follows (we will assume here that D_1 and D_2 are Harish-Chandra realizations of the domains in question):

- By Bell's theorem [2, Theorem 2], F extends to a neighbourhood of $\overline{D}_1$ and we can find a point p in the Bergman–Shilov boundary of D_1, and a small ball B around it, such that $F|_B$ is a biholomorphism.
- We may assume that $F(0) = 0$. Let $\{a_k\}$ be a sequence in $D_1 \cap B$ converging to p and let $b_k := F(a_k)$. Let $\phi_k^j \in \mathrm{Aut}(D_j)$ be an automorphism that maps 0 to a_k if $j = 1$, and to b_k if $j = 2$. It turns out that both p and $F(p)$ are peak points, whence $\phi_k^j \longrightarrow p^{(j)}$ uniformly on compact subsets, where $p^{(1)} := p$ and $p^{(2)} := F(p)$.
- Using the Schwarz lemma for convex balanced domains (Result 4.5 below) we show that a subsequence of $\{(\phi_j^2)^{-1} \circ F \circ \phi_j^1\}$ converges to a linear map and that, owing to the tautness of D_1 and D_2, this map is a biholomorphism of D_1 onto D_2.
- We may now take $D_1 = D_2 = D$. We shall use our Key Lemma, with $W_1 = (\phi_k^1)^{-1}(D \cap B)$ and $W_2 = (\phi_k^2)^{-1}(D \cap F(B))$ for k sufficiently large.
- Since the analytic discs Δ_z and $\Delta_{F(z)}$ are not relatively compact in D, the mode of convergence of $\{\phi_k^j\}$ isn't *a priori* good enough to infer that appropriate families of these discs will be swallowed up by W_j, $j = 1, 2$. By Bell's theorem, each ϕ_k^j extends to some neighbourhood of D. We show that $\{\phi_k^j\}$, passing to a subsequence and relabelling if necessary, converges *uniformly* on certain special special circular subsets of D that are adherent to ∂D. This is enough to overcome the difficulty just described.

Let us define a term that we used in the sketch above, which we shall also need in stating our next theorem.

Definition 1.3. Let $D \subsetneq \mathbb{C}^n$ be a domain and let $p \in \partial D$. We say that p is a *peak point* if there exists a function $h \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D}; \mathbb{C})$ such that $h(p) = 1$ and $|h(z)| < 1\ \forall z \in \overline{D} \setminus \{p\}$. The function h is called a *peak function for p*.

When a domain D is bounded, the noncompactness of $\mathrm{Aut}(D)$ (in the compact-open topology) is equivalent to D having a boundary orbit-accumulation point; see [15].

Definition 1.4. Let $D \subsetneq \mathbb{C}^n$ be a domain and let $p \in \partial D$. We say that p is a *boundary orbit-accumulation point* if there exist a point $a \in D$ and a sequence of automorphisms $\{\phi_k\}$ of D such that $\lim_{k\to\infty} \phi_k(a) = p$.

With the last two definitions, we are in a position to state our second theorem. Note that D_1 is *not* assumed to be a bounded symmetric domain. Yet, some of the techniques sketched above (versions of which have been used to remarkable effect in the literature in this field) are general enough to be applicable to the following situation.

Theorem 1.5. *Let* D_1 *be a bounded convex balanced domain in* $\mathbb{C}^n$ *whose automorphism group is noncompact and let p be a boundary orbit-accumulation point. Let* D_2 *be a realization of a bounded symmetric domain as a bounded convex balanced domain in* $\mathbb{C}^n$. *Assume* *that there is a neighbourhood U of p and a biholomorphic map $F : U \to \mathbb{C}^n$ such that $F(U \cap D_1) \subset D_2$ and $F(U \cap \partial D_1) \subset \partial D_2$. Assume that either p or $F(p)$ is a peak point. Then, there exists a* linear *map that maps D_1 biholomorphically onto D_2.*

Remark 1.6. Theorem 1.5 (together with Bell's theorem [2]) gives a very short proof of the rigidity theorem of Mok and Tsai [14] under the additional assumption that the convex domain D in their result is also circular. There is an extensive literature on rigidity theorems relating to bounded symmetric domains, but we shall not dwell any further on it.

Remark 1.7. We remark that a version of the above result can be proved without assuming that D_1 is either balanced or convex. D_1 merely needs to be complete Kobayashi hyperbolic. However, in this case, the biholomorphism of D_1 onto D_2 will not, in general, be linear. We prefer the above version: the conclusion that there exists a *linear* equivalence places Theorem 1.5 among the rigidity theorems alluded to in Remark 1.6.

The layout of this paper is as follows. Since Jordan triple systems play a vital role in describing not just the structure of the boundary of a bounded symmetric domain, but also some of its key automorphisms, we begin with a primer on Jordan triple systems. Readers who are familiar with Jordan triple systems can skip to Section 3, where we discuss the boundary geometry of bounded symmetric domains. Section 4 is devoted to stating and proving certain propositions that are essential to our proofs. Finally, in Sections 5 and 6, we present the proofs of the results stated above

2. A primer on Jordan triple systems

There is a natural connection between bounded symmetric domains and certain Hermitian Jordan triple systems. This section collects several definitions and results that are required to give a coherent description of the boundary of a bounded symmetric domain (which we shall discuss in the next section).

Unless otherwise stated, the results in this section can be found in the UC-Irvine lectures by Loos [13] describing how Jordan triple systems can be used to study the geometry of bounded symmetric domains.

Definition 2.1. A *Hermitian Jordan triple system* is a complex vector space V endowed with a triple product $(x, y, z) \longmapsto \{x, y, z\}$ that is symmetric and bilinear in x and z and conjugate-linear in y, and satisfies the Jordan identity

$$\begin{aligned}&\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} \\ &\qquad = \{\{x, y, u\}, v, w\} - \{u, \{y, x, v\}, w\} \quad \forall x, y, u, v, w \in V.\end{aligned}$$

Such a system is said to be *positive* if for each $x \in V \setminus \{0\}$ for which $\{x, x, x\} = \lambda x$ (where λ is a scalar), we have $\lambda > 0$.

As mentioned in Section 1, a bounded symmetric domain of complex dimension n has a realization D as a bounded convex balanced domain in $\mathbb{C}^n$. Let $(z, \dots, z_n)$ be the global holomorphic coordinates coming from the product structure on $\mathbb{C}^n$ and let $(\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_n)$ denote the standard ordered basis of $\mathbb{C}^n$. Let K_D denote the Bergman kernel of (the above realization of) D and h_D the Bergman metric. The function $\{\cdot, \cdot, \cdot\} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$

obtained by the requirement

$$h_D(\{\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_j, \boldsymbol{\epsilon}_k\}, \boldsymbol{\epsilon}_l) = \frac{\partial^4 \log K_D(z,z)}{\partial z_i \partial \overline{z}_j \partial z_k \partial \overline{z}_l}\bigg|_{z=0}, \tag{2.1}$$

and by extending $\mathbb{C}$-linearly in the first and third variables and $\mathbb{C}$-antilinearly in the second, has the property that $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$ is a positive Hermitian Jordan triple system (abbreviated hereafter as PHJTS). This relationship is a one-to-one correspondence between finite-dimensional PHJTSs and bounded symmetric domains — which we shall make more precise in Section 3.

Let $(V, \{\cdot,\cdot,\cdot\})$ be a HJTS. It will be convenient to work with the operators

$$\mathbf{D}(x,y)z = \mathbf{Q}(x,z)y := \{x,y,z\}. \tag{2.2}$$

We define the operator $Q : V \to \mathrm{End}(V)$ by $Q(x)y := \mathbf{Q}(x,x)y/2$. For any $x \in V$, we can define the so-called *odd powers* of x recursively by:

$$x^{(1)} := x \quad \text{and} \quad x^{(2p+1)} := Q(x)x^{(2p-1)} \text{ if } p \geq 1.$$

A vector $e \in V$ is called a tripotent if $e^{(3)} = e$.

Tripotents are important to this discussion because:

- A finite-dimensional PHJTS has plenty of non-zero tripotents.
- Given a finite-dimensional PHJTS $(V, \{\cdot,\cdot,\cdot\})$, any vector V has a certain canonical decomposition as a linear combination of tripotents.
- In a finite-dimensional PHJTS, the set of tripotents forms a real-analytic submanifold.

We refer the interested reader to [13, Chapter 3] for details of the first fact. As for the second fact, we need a couple of new notions. First: given a HJTS $(V, \{\cdot,\cdot,\cdot\})$, we say that two tripotents $e_1, e_2 \in V$ are orthogonal if $\mathbf{D}(e_1, e_2) = 0$. Second: given $x \in V$, we define the real vector space $\ll x \gg$ by

$$\ll x \gg := \mathrm{span}_{\mathbb{R}}\{x^{(2p+1)} : p = 0, 1, 2, \dots\}.$$

These two notions allows us to state the following:

Result 2.2 (Spectral decomposition theorem)**.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a finite-dimensional PHJTS. Then, each $x \in V \setminus \{0\}$ can be written uniquely as*

$$x = \lambda_1 e_1 + \cdots + \lambda_s e_s \tag{2.3}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$ and $\{e_1, \dots, e_s\}$ is a $\mathbb{R}$-basis of $\ll x \gg$ comprising pairwise orthogonal tripotents.

The decomposition of $x \in V$ as given by Result 2.2 is called the *spectral decomposition* of x. The assignment $x \longmapsto \lambda_1(x)$, where $\lambda_1(x)$ is as given by (2.3), is a well-defined function and can be shown to be a norm on V. This norm is called the *spectral norm* on V.

Next, we present another decomposition, which give us the second ingredient needed to describe the boundary geometry of a bounded symmetric domain.

Result 2.3 (Pierce decomposition)**.** *Let $(V, \{\cdot,\cdot,\cdot\})$ be a HJTS and let $e \in V$ be a tripotent. Then, the spectrum of $\mathbf{D}(e,e)$ is a subset of $\{0,1,2\}$. Let*

$$V_j = V_j(e) := \{x \in V : \mathbf{D}(e,e)x = jx\}, \quad j \in \mathbb{Z}.$$

Then:

a) $V = V_0 \oplus V_1 \oplus V_2$.
b) *If* $e \neq 0$, *then* $e \in V_2$.
c) *We have the relation* $\{V_\alpha, V_\beta, V_\gamma\} \subset V_{\alpha-\beta+\gamma}$.
d) V_0, V_1 *and* V_2 *are Hermitian Jordan subsystems of* $\{\cdot,\cdot,\cdot\}$.

The direct-sum decomposition (a) given by the above result is called the *Pierce decomposition of* V *with respect to the tripotent* e. The ideas that go into proving the Pierce decomposition theorem allow us to construct a special partial order on the set of tripotents of V. In order to avoid statements that are vacuously true, unless stated otherwise, we take $(V, \{\cdot,\cdot,\cdot\})$ to be a PHJTS. Let $e, e' \in V$ be tripotents. We say that e *is dominated by* e' $(e \preceq e')$ if there is a tripotent e_1 orthogonal to e such that $e' = e + e_1$. We say that e *is strongly dominated by* e' $(e \prec e')$ if $e \preceq e'$ and $e \neq e'$. The result of interest, in this regard, is the following:

Result 2.4. *Let* $(V, \{\cdot,\cdot,\cdot\})$ *be a HJTS. Let* $e_1, e_2 \in V$ *be orthogonal tripotents and let* $e = e_1 + e_2$. *If* $e' \in V$ *is a tripotent orthogonal to* e, *then* e' *is orthogonal to* e_1 *and* e_2.

Now suppose $\{\cdot,\cdot,\cdot\}$ *is positive. Then, the relation* $\preceq$ *is a partial order on the set of tripotents.*

Definition 2.5. A tripotent is said to be *minimal* (or *primitive*) if it is minimal for $\preceq$ among non-zero tripotents. It is said to be *maximal* if it is maximal for $\preceq$.

Result 2.6. *Consider the tripotents of* V *partially ordered by* $\preceq$.
(1) *A tripotent* e *is maximal if and only if the Pierce space* $V_0(e) = 0$.
(2) *If, for a tripotent* e, *the Pierce space* $V_2(e) = \mathbb{C}e$, *then* e *is primitive.*

Let us now also assume that $(V, \{\cdot,\cdot,\cdot\})$ is finite dimensional. Given any non-zero tripotent e, it follows from finite-dimensionality and the repeated application of Result 2.4 that e can be written as a sum of mutually orthogonal primitive tripotents. This brings us to the final concept in this primer: the *rank of a tripotent* e is the minimum number of primitive tripotents required for such a decomposition of e while the *rank of* $(V, \{\cdot,\cdot,\cdot\})$ is the highest rank that a tripotent of V can have.

3. The boundary geometry of bounded symmetric domains

In this section we describe the boundary of a bounded symmetric domain in terms of the positive Hermitian Jordan triple system associated to it. Thus, we shall follow the notation introduced in Section 2. Recall that a bounded symmetric domain D has a realization as a bounded convex balanced domain. When we say "Hermitian Jordan triple system associated to D", it is implicit that D is this realization and the association is the one given by (2.1). This is a one-to-one correspondence, described as follows:

Result 3.1 ([13], Theorem 4.1)**.** *Let* D *be a realization of a bounded symmetric domain as a bounded convex balanced domain in* $\mathbb{C}^n$ *for some* $n \in \mathbb{Z}_+$. *Then,* D *is the open unit ball in* $\mathbb{C}^n$ *with respect to the spectral norm determined by the PHJTS associated to* D. *Conversely, given a PHJTS* $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$, *the open unit ball with respect to the spectral norm determined by it is a bounded symmetric domain* D, *and the PHJTS associated to* D *by the rule* (2.1) *is* $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$.

In what follows, whenever we mention a bounded symmetric domain D, *it will be understood that D is a bounded convex balanced realization.*

The boundary of a bounded symmetric domain $D \subset \mathbb{C}^n$ has a certain stratification into real-analytic submanifolds that can be described in terms of the PHJTS associated to D. The first part of this section is devoted to describing this stratification. Fix a bounded symmetric domain $D \subset \mathbb{C}^n$ and let $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$ be the PHJTS associated to it. It turns out (see [13, Theorem 5.6]) that the set M_D of tripotents of $\mathbb{C}^n$ with respect to $\{\cdot,\cdot,\cdot\}_D$ is a disjoint union of real-analytic submanifolds of $\mathbb{C}^n$. For each $e \in M_D$, let $M_{D,e}$ denote the connected component of M_D containing e. The tangent space $T_e(M_{D,e})$, viewed *extrinsically* (i.e., so that $e + T_e(M_{D,e})$ is the affine subspace of all tangents to $M_{D,e}$ at e), is:

$$T_e(M_{D,e}) = iA(e) \oplus V_1(e),$$

where $A(e)$ is determined by the relation $V_2(e) = \{x + iy \in \mathbb{C}^n : x, y \in A(e)\}$, and $V_j(e)$ is the eigenspace of $j = 0, 1, 2$ in the Pierce decomposition of $\mathbb{C}^n$ with respect to e.

Let M_D^* be the set of all non-zero tripotents and let $\|\cdot\|_D$ denote the spectral norm determined by $\{\cdot,\cdot,\cdot\}_D$. Define

$$\begin{aligned} E_D &:= \{(e,v) \in \mathbb{C}^n \times \mathbb{C}^n : e \in M_D^* \text{ and } v \in V_0(e)\}, \\ \mathfrak{B}_D &:= \{(e,v) \in E_D : \|v\|_D < 1\}. \end{aligned}$$

We can write $\mathfrak{B}_D$ as a disjoint union of the form

$$\mathfrak{B}_D \;:=\; \bigsqcup_{\alpha \in \mathcal{C}} \mathfrak{B}_{D,\alpha}, \tag{3.1}$$

where $\mathcal{C}$ is the set of connected components of M_D^*, and each $\mathfrak{B}_{D,\alpha}$ is a connected, real-analytic submanifold of $\mathbb{C}^n \times \mathbb{C}^n$ that is a real-analytic fibre bundle whose fibres are unit $\|\cdot\|_D$-discs. The key theorem about the boundary of D is as follows:

Result 3.2 ([13], Chapter 6)**.** *Let D be a bounded symmetric domain in $\mathbb{C}^n$ and let $\mathsf{f} : \mathfrak{B}_D \to \mathbb{C}^n$ be defined by $\mathsf{f}(e,v) := e + v$. Then:*

i) $\mathsf{f}|_{\mathfrak{B}_{D,\alpha}}$ *is an imbedding for each* $\alpha \in \mathcal{C}$*;*
ii) $\partial D = \sqcup_{\alpha \in \mathcal{C}} \mathcal{M}_{D,\alpha}$*, where* $\mathcal{M}_{D,\alpha} := \mathsf{f}(\mathfrak{B}_{D,\alpha})$*;*
iii) in the above stratification of ∂D, if $\mathcal{M}_{D,\alpha}$ is of dimension d_α, then it is a closed, connected, real-analytic imbedded submanifold of the open set

$$\mathbb{C}^n \setminus \bigcup_{\beta\,:\,\dim_{\mathbb{R}}(\mathcal{M}_{D,\beta}) < d_\alpha} \mathcal{M}_{D,\beta}.$$

Furthermore, when D is an *irreducible* bounded symmetric domain in $\mathbb{C}^n$, then we can provide further information. Here, the rank of a bounded symmetric domain is the rank of the Jordan triple system $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$.

Result 3.3 ([13], Chapter 6; [21], Théorème 7.3)**.** *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$ of rank r, and let $\mathcal{C}$ denote the set of connected components of $\mathfrak{B}_D$. Then, we have the following:*

i) $\mathcal{C}$ has cardinality r.

ii) *Each connected component of the decomposition* (3.1) *is a bundle over a submanifold of non-zero tripotents of rank* j, $j \in \{1, \dots, r\}$. *Denoting this bundle as* $\mathfrak{B}_{D,j}$, $j \in \{1, 2, \dots, r\}$, *we can express the stratification of* ∂D *given by Result 3.2-(ii) as*

$$\partial D = \bigsqcup_{j=1}^{r} \mathcal{M}_{D,j},$$

where $\mathcal{M}_{D,j} := \mathsf{f}(\mathfrak{B}_{D,j})$, *and each* $\mathcal{M}_{D,j}$ *is connected.*

iii) *The stratum* $\mathcal{M}_{D,1}$ *is dense in* ∂D.

The other goal of this section is to describe the structure of the germs of complex-analytic varieties contained in the boundary of a bounded symmetric domain D. This structure can be described in extremely minute detail; see, for instance, [22] by Wolf. In fact, the papers about higher-rank bounded symmetric domains mentioned in Section 1 make extensive use of this fine structure. However, *in this work, we only need very coarse information about the complex analytic structure of* ∂D; specifically: the distinction between the Bergman–Shilov boundary of D and its complement in ∂D.

We denote the Bergman–Shilov boundary of D by $\partial_S D$. We shall not formally define here the notion of the Shilov boundary of a uniform algebra; we shall merely state that the Bergman–Shilov boundary of a bounded domain $D \Subset \mathbb{C}^n$ is the Shilov boundary of the uniform algebra $A(D) := \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$. However, we do carefully state the following definition:

Definition 3.4. Let D be a bounded domain in $\mathbb{C}^n$. An *affine* ∂D*-component* is an equivalence class under the equivalence relation $\sim_A$ on ∂D given by

$$x \sim_A y \iff x \text{ and } y \text{ can be joined by a chain of segments lying in } \partial D,$$

where a segment is a subset of $\mathbb{C}^n$ of the form $\{u + tv : t \in (0, 1)\}$, $u, v \in \mathbb{C}^n$. A *holomorphic arc component of* ∂D is an equivalence class under the equivalence relation $\sim_H$ on ∂D given by

$$x \sim_H y \iff x \text{ and } y \text{ can be joined by a chain of analytic discs lying in } \partial D.$$

Roughly speaking, given a bounded domain $D \Subset \mathbb{C}^n$ and a point $x \in \partial D$, the holomorphic arc component of ∂D containing x is the largest (germ of a) complex-analytic variety lying in ∂D that contains x. The information that we require about holomorphic boundary components is:

Result 3.5 ([13], Theorem 6.3)**.** *Let* D *be the realization of a bounded symmetric domain as a bounded convex balanced domain in* $\mathbb{C}^n$.

i) *The affine* ∂D*-components and the holomorphic arc components of* ∂D *coincide.*

ii) *A boundary component containing a point* $x \in \partial D$ *is a non-empty open region in some* $\mathbb{C}$*-affine subspace of positive dimension passing through* x unless x *is a maximal tripotent.*

Finally, we mention the following description of the Bergman–Shilov boundary of a bounded symmetric domain:

Result 3.6 ([13], Theorem 6.5)**.** *Let* $D \Subset \mathbb{C}^n$ *be as in Result 3.5. The Bergman–Shilov boundary of* D *coincides with each of the following sets:*

i) *the set of maximal tripotents of* $\mathbb{C}^n$ *with respect to* $\{\cdot, \cdot, \cdot\}_D$;

ii) the set of extreme points of $\overline{D}$*;*
iii) the set of points of $\overline{D}$ *having the maximum Euclidean distance from* $0 \in \mathbb{C}^n$*.*

4. Some essential propositions

This section contains several lemmas and propositions — some being simple consequences of known results, and some requiring substantial work — that will be needed to prove our theorems. We begin with the following result of Bell:

Result 4.1 ([2], Theorem 2)**.** *Suppose* $f : D_1 \to D_2$ *is a proper holomorphic map between bounded circular domains. Suppose further that* D_2 *contains the origin and that the Bergman kernel* $K(w, z)$ *associated to* D_1 *is such that for each compact subset* G *of* D_1*, there is an open set* $U = U(G)$ *containing* $\overline{D}_1$ *such that* $K(\cdot, z)$ *extends to be holomorphic on* U *for each* $z \in G$*. Then* f *extends holomorphically to a neighbourhood of* $\overline{D}_1$*.*

Now let D be any bounded balanced domain (not necessarily convex) in $\mathbb{C}^n$. If D is not convex, it will not be a unit ball with respect to some norm on $\mathbb{C}^n$. However, we do have a function that has the same homogeneity property as a norm, with respect to which D is the "unit ball". The function $M_D : \mathbb{C}^n \to [0, \infty)$ defined by

$$M_D(z) := \inf\{t > 0 : z/t \in D\}$$

is called the *Minkowski functional for* D. Assume that the intersection of each complex line passing through $0 \in \mathbb{C}^n$ with ∂D is a circle. Let G be a compact subset of D. Then, as M_D is upper semicontinuous, $\exists r_G \in (0, 1)$ such that $G \subset \{z \in \mathbb{C}^n : M_D(z) < r_G\}$ and the latter is an open set. Hence $z/r_G \in D$ $\forall z \in G$. Clearly, $r_G w \in D$ $\forall w \in \{z \in \mathbb{C}^n : M_D(z) < 1/r_G\} =: U(G)$. By our assumptions, $\overline{D} \subset U(G)$. Let K_D be the Bergman kernel of D. We recall that:

$$K_D(w, z) = \sum_{\nu \in \mathbb{N}} \psi_\nu(w)\overline{\psi_\nu(z)} \quad \forall (w, z) \in D \times D,$$

where the right-hand side converges absolutely and uniformly on any compact subset of $D \times D$ and $\{\psi_\nu\}_{\nu \in \mathbb{N}}$ is *any* complete orthonormal system for the Bergman space of D . Then — owing to the fact that the collection $\{C_\alpha z^\alpha : \alpha \in \mathbb{N}^n\}$ (where $C_\alpha > 0$ are suitable normalization constants) is a complete orthonormal system for the Bergman space of D — we can infer two things. First: the functions

$$\phi_z(w) := K_D(r_G w, z/r_G), \quad w \in U(G), \tag{4.1}$$

are well-defined by power series for each $z \in G$. Secondly:

$$K_D(r_G w, z/r_G) = K_D(w, z) \ \ \forall (w, z) \in D \times G.$$

Comparing this with (4.1), we see that each ϕ_z extends $K_D(\cdot, z)$ holomorphically. In view of Result 4.1, we have just deduced:

Lemma 4.2. *Let* $f : D_1 \to D_2$ *be a proper holomorphic map between bounded circular domains. Suppose* D_1 *and* D_2 *are both balanced. Assume that the intersection of every complex line passing through* 0 *with* ∂D_1 *is a circle. Then* f *extends holomorphically to a neighbourhood of* $\overline{D}_1$*.*

We remark that the above conclusion also follows from a later work [3] of Bell.

Let D be a bounded symmetric domain in $\mathbb{C}^n$. Let $(\mathbb{C}^n, \{\cdot,\cdot,\cdot\}_D)$ be the Jordan triple system associated to D (as in other places in this paper, we assume that D is a Harish-Chandra realization). Let $\mathbf{D}_D$ and Q_D be the maps (2.2) for the triple product $\{\cdot,\cdot,\cdot\}_D$. We define the linear operators $\mathbf{B}_D(x,y) : \mathbb{C}^n \to \mathbb{C}^n$:

$$\mathbf{B}_D(x,y) := \mathsf{id}_D - \mathbf{D}_D(x,y) + Q(x)Q(y), \quad x, y \in \mathbb{C}^n.$$

Consider the sesquilinear form $(x,y) \longmapsto \mathrm{Tr}[\mathbf{D}_D(x,y)]$ on $\mathbb{C}^n$. It turns out that the positivity of $\{\cdot,\cdot,\cdot\}_D$ is equivalent to the above sesquilinear form being an inner product on $\mathbb{C}^n$; see [13, Chapter 3]. Furthermore with respect to this inner product, we have:

$$\mathbf{B}_D(x,y)^* = \mathbf{B}_D(y,x) \ \ \forall x, y \in \mathbb{C}^n.$$

It is now easy to deduce that $\mathbf{B}_D(a,a)$ is a self-adjoint, positive semi-definite linear operator. Consequently, $\mathbf{B}_D(a,a)$ admits a unique positive semi-definite square root, which we denote by $\mathbf{B}_D(a,a)^{1/2}$. Having made these two definitions, we can state the following useful facts about the geometry of D.

Result 4.3 ([13], Proposition 9.8; [16], Proposition III.4.1)**.** *Let D be the realization of a bounded symmetric domain as a convex balanced domain in $\mathbb{C}^n$. Fix a point $a \in D$ and let*

$$g_a(z) := a + \mathbf{B}_D(a,a)^{1/2}(\mathsf{id}_D + \mathbf{D}_D(z,a))^{-1}(z) \ \ \forall z \in D.$$

Then, $g_a \in \mathrm{Aut}(D)$, $g_a(0) = a$, and $g_a'(z) = \mathbf{B}_D(a,a)^{1/2} \circ \mathbf{B}_D(z,-a)^{-1}$. Furthermore, $g_a^{-1} = g_{-a}$.

Various versions of the following lemma have been known for a long time. We refer the reader to [17, Lemma 15.2.2] for a proof.

Lemma 4.4. *Let D be a bounded domain in $\mathbb{C}^n$ and let $p \in \partial D$. Assume that there exists a ball B centered at p and a function $h \in \mathcal{O}(B \cap D) \cap \mathcal{C}(\overline{B \cap D}; \mathbb{C})$ such that $h(p) = 1$ and $|h(z)| < 1 \ \forall z \in \overline{B \cap D} \setminus \{p\}$. Let $a_0 \in D$ and $\{\phi_k\}$ be a sequence of automorphisms of D such that $\phi_k(a_0) \longrightarrow p$ as $k \to \infty$. Then, $\{\phi_k\}$ converges uniformly on compact subsets of D to const_p — the map that takes the constant value p.*

We now state a version of Schwarz's lemma for convex balanced domains and then a version of Schwarz's lemma for irreducible bounded symmetric domains, both of which are needed in the proof of our Key Lemma (see Section 1).

Result 4.5 ([17], Theorem 8.1.2)**.** *Let Ω_1 and Ω_2 be balanced regions in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and let $F : \Omega_1 \to \Omega_2$ be a holomorphic map. Suppose Ω_2 is convex and bounded. Then:*

i) $F'(0)$ maps Ω_1 into Ω_2; and
ii) $F(r\Omega_1) \subseteq r\Omega_2$ $(0 < r \leq 1)$ if $F(0) = 0$.

Result 4.6 ([21], Théorème 7.4)**.** *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$ in its Harish-Chandra realization (whence it is the unit ball in $\mathbb{C}^n$ for the associated spectral norm $\|\cdot\|$). Let $F : D \to D$ be a holomorphic map such that $F(0) = 0$. Assume that for some non-empty open set $U \subset D$, we have $\|F(z)\| = \|z\| \ \forall z \in U$. Then F is an automorphism of D.*

With these two results, we can now give a proof of our Key Lemma:

The proof of the Key Lemma. Let $z \in U$, and set $w := F(z)$. By hypothesis, F maps Δ_z into D and $(F|_{W_1})^{-1}$ maps Δ_w into D. Applying Result 4.5 to $F|_{\Delta_z}$ and to $(F|_{W_1})^{-1}\big|_{\Delta_w}$, we have $\|F(z)\| = \|z\|$ for every $z \in U$. Thus by the Schwarz lemma for irreducible bounded symmetric domains, F is an automorphism of D. □

We now state and prove a technical proposition regarding the invertibility of the operator $\mathbf{B}_D$ at certain off-diagonal points in $\partial D \times \partial D$, where D is an *irreducible* bounded symmetric domain of dimension ≥ 2. Here $\mathcal{M}_{D,1}$ denotes the stratum of ∂D described by Result 3.3. This result and our Key Lemma are the central ingredients in the proof of our Main Theorem.

Proposition 4.7. *Let D be the realization of an irreducible bounded symmetric domain of dimension n as a bounded convex balanced domain in $\mathbb{C}^n$, $n \geq 2$. Let $p \in \partial D$. For each $z_0 \in \mathcal{M}_{D,1}$ and each $\mathcal{M}_{D,1}$-open neighbourhood $U \ni z_0$, there exists a point $w \in U$ such that* $\det \mathbf{B}_D(\cdot, p)$ *is non-zero on the set $\{\zeta w : \zeta \in \mathbb{C}, |\zeta| = 1\}$.*

Remark 4.8. In the following proof, we argue by assuming that the conclusion above is false. We can instantly arrive at a contradiction at the point (•) in the proof below if we invoke results on the fine structure of ∂D; see [11] or [22], for instance. However, we provide an elementary argument beyond (•) to complete the proof in the hope that appropriate analogues of the above may be formulated in other contexts.

Proof. Let us denote $\det \mathbf{B}_D(z, p)$ as $h(z)$, where $z \in \mathbb{C}^n$. Let us assume that the result is false. Then, there exists a point $z_0 \in \mathcal{M}_{D,1}$ and an $\mathcal{M}_{D,1}$-open neighbourhood $U \ni z_0$ such that for each $w \in U$, there exists a $\zeta_w \in \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ with $h(\zeta_w w) = 0$. Let q denote the quotient map $q : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$. Also write

$$Z_h := h^{-1}\{0\}, \quad Z := Z_h \cap \mathcal{M}_{D,1}.$$

Our assumption implies that $q(Z)$ contains a non-empty open set $\mathcal{V} \subset \mathbb{CP}^{n-1}$. Let $\mathcal{A} := \{z \in \mathbb{C}^n : 1 - \varepsilon < \|z\| < 1 + \varepsilon\}$, where $\|\cdot\|$ denotes the spectral norm relative to which D is the unit ball, and ε is a *fixed* number in $(0, 1)$. As $\mathcal{V} \subset q(\mathcal{A})$, it is easy to see that $\mathcal{V}$ can be covered by finitely many holomorphic coordinate patches $(U_1, \psi_1), \dots, (U_M, \psi_M)$ such that the maps

$$q_j := \psi_j \circ q|_{q^{-1}(U_j) \cap \mathcal{A}} : q^{-1}(U_j) \cap \mathcal{A} \to \mathbb{C}^{n-1}$$

are Lipschitz maps. Since Lipschitz maps cannot increase Hausdorff dimension (see [17, Proposition 14.4.4], for instance) and $\dim_{\mathbb{R}}(\mathcal{V}) = 2n - 2$, the preceding discussion shows that the Hausdorff dimension of Z (and hence the dimension of Z as a real-analytic set) is $2n - 2$. As Z_h is a complex analytic subvariety, its singular locus is of complex dimension $\leq n - 2$. Thus, we can find a point $x_0 \in Z$ that is a regular point of Z_h, and an open ball B around x_0 that is so small that

- $\mathcal{M}_{D,1} \cap B$ is a submanifold of B;
- $B \cap Z_h$ is an $(n-1)$-dimensional complex submanifold of B;
- the dimension of $B \cap Z$ is $2n - 2$.

These three facts imply that $M := B \cap Z_h \subset \mathcal{M}_{D,1}$. We can deduce this by considering a local defining function $\rho_B : B \to \mathbb{R}$ for $\mathcal{M}_{D,1}$ and observing that, by Łojasiewicz's theorem [12], $\rho_B|_M \equiv 0$. *If $D = \mathbb{B}^n$, we already have a contradiction* and, hence, the proof.

Since $\mathcal{M}_{D,1}$ is a real-analytic submanifold of $\mathbb{C}^n \setminus \bigsqcup_{j\geq 2}\mathcal{M}_{D,j}$, where $\mathcal{M}_{D,j}$ are the strata of ∂D discussed in Section 3, we can define the Levi-form of $\mathcal{M}_{D,1}$ — denoted by $\mathfrak{L}(z,V)$, $z \in \mathcal{M}_{D,1}$, $V \in H_z(\mathcal{M}_{D,1})$. A few words about notation: in this proof, we shall work with the tangent bundle of $\mathcal{M}_{D,1}$ defined extrinsically. So, when referring to vectors in $T_z(\mathcal{M}_{D,1})$, we shall view them either as real or as complex vectors, as convenient, such that $z + T_z(\mathcal{M}_{D,1})$ is the hyperplane tangent to $\mathcal{M}_{D,1}$ at $z \in \mathcal{M}_{D,1}$. In this scheme:

$$H_z(\mathcal{M}_{D,1}) := T_z(\mathcal{M}_{D,1}) \cap iT_z(\mathcal{M}_{D,1}).$$

As $\dim_{\mathbb{C}}(M) = n-1$, $\mathfrak{L}(z,\boldsymbol{\cdot}) \equiv 0\ \forall z \in M$. The curve $\gamma : (-\varepsilon,\varepsilon) \to \mathcal{M}_{D,1}$ (for $\varepsilon > 0$ suitably small) $\gamma(t) := \exp(it)z$ is transverse to M at z. This is because if $\gamma'(0) = iz$ were in $H_z(\mathcal{M}_{D,1})$, then

$$i\gamma'(0) = -z \in H_z(\mathcal{M}_{D,1}) \subset T_z(\mathcal{M}_{D,1}),$$

which contradicts the convexity of D. Consequently, for $\varepsilon_0 > 0$ sufficiently small, the set $\{\exp(it)z : t \in (-\varepsilon_0,\varepsilon_0), z \in M\}$ contains an $\mathcal{M}_{D,1}$-open neighbourhood of x_0. Thus, $\mathcal{M}_{D,1}$ is Levi-flat at x_0. As $\mathcal{M}_{D,1}$ is real-analytic, it is a Levi-flat hypersurface.

We shall now show that Levi-flatness of $\mathcal{M}_{D,1}$ leads to a contradiction. Let us pick an $x \in \mathcal{M}_{D,1}$. Owing to Levi-flatness, we can find a ball B_x, centered at x, such that

$$D_x^- := D \cap B_x, \quad D_x^+ := B_x \setminus \overline{D}$$

are both pseudoconvex. Let $\boldsymbol{n}_x$ denote the unit outward normal vector to ∂D at x ($x \in \mathcal{M}_{D,1}$). Owing to convexity of D, we can find an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that

$$H_x(\varepsilon_0;\delta) := x + \delta\boldsymbol{n}_x + \{V \in H_x(\mathcal{M}_{D,1}) : |V| < \varepsilon_0\} \subset D_x^+$$

for each $\delta \in (0,\delta_0)$. Here, $|\boldsymbol{\cdot}|$ denotes the Euclidean norm. As $H_x(\varepsilon_0;\delta)$ is a copy of a complex $(n-1)$-dimensional ball and as D_x^+ is taut — see [7, Proposition 2.1] — it follows that $H_x(\varepsilon_0;0) \subset \mathcal{M}_{D,1}$. To summarize, $\mathcal{M}_{D,1}$ has the following property:

($\bullet$) At each $x \in \mathcal{M}_{D,1}$, a germ of the set $(x + H_x(\mathcal{M}_{D,1}))$ lies in $\mathcal{M}_{D,1}$.

Let us now pick and fix a point $y^0 \in \mathcal{M}_{D,1}$. Let $(z_1,\dots,z_n)$ be global holomorphic coordinates in $\mathbb{C}^n$, associated to an appropriate rigid motion of D, such that $y^0 = (0,\dots,0)$, $D \subset \{\mathsf{Re} z_1 > 0\}$ and $H_{y^0}(\mathcal{M}_{D,1}) = \{z_1 = 0\}$ relative to these coordinates. Let W be a non-zero vector in $H_{y^0}(\mathcal{M}_{D,1})$ and let $D_W := D \cap \operatorname{span}_{\mathbb{C}}\{W, \boldsymbol{n}_{y^0}\}$. Clearly, D_W is convex and by ($\bullet$) $\mathcal{M}_{D,1} \cap \operatorname{span}_{\mathbb{C}}\{W,\boldsymbol{n}_{y^0}\} =: \mathcal{M}_W$ has the property that for each point $y \in \mathcal{M}_W$, the germ of a complex line through y, call it $\Lambda_{y,W}$, lies in $\mathcal{M}_W$. Let us view D_W as lying in $\mathbb{C}^2$, whence a portion of $\mathcal{M}_W$ near $(0,0)$ can be parametrized by three real variables as follows:

$$r(t,u,v) = \rho(t) + a(t)(u+iv), \quad |t| < \varepsilon_1, \ |u|, \ |v| < \varepsilon_2,$$

where $\rho : (-\varepsilon_1,\varepsilon_1) \to \mathcal{M}_W$ is a smooth curve through $(0,0)$ such that $\rho'(t)$ is orthogonal to $\Lambda_{\rho(t),W}$ for each t, and $a : (-\varepsilon_1,\varepsilon_1) \to \mathbb{C}^2$ is such that $a(t)$ is parallel to $\Lambda_{\rho(t),W}$ for each t. For the remainder of this paragraph, $\boldsymbol{n}(t,u,v)$ will denote the *inward* unit normal to ∂D_W at $r(t,u,v)$, and $\boldsymbol{\cdot}$ will denote the standard inner product on $\mathbb{R}^4$. Define the matrix-valued function $\Gamma : (-\varepsilon_1,\varepsilon_1) \times (-\varepsilon_2,\varepsilon_2)^2 \to \mathbb{R}^{3\times 3}$ by

$$\Gamma(\tau,U,V) := \mathsf{Hess}_{t,u,v}\left(r(t,u,v) \boldsymbol{\cdot} \boldsymbol{n}(\tau,U,V)\right)\big|_{(t,u,v)=(\tau,U,V)}.$$

The convexity of D_W implies that $\Gamma(\tau,U,V)$ is *positive* semidefinite at each (τ,U,V) (recall that $\boldsymbol{n}(\tau,U,V)$ is the inward normal at $r(\tau,U,V)$). By choosing $\varepsilon_1,\varepsilon_2 > 0$ small enough, we can ensure that $(n_1^2 + n_2^2)(t,u,v) \neq 0$ for every (t,u,v), where we write $\boldsymbol{n} = (n_1,n_2,n_3,n_4)$, and that a is of the form $a(t) = (\alpha(t) + i\beta(t), 1)$. We compute to observe that two of the principal minors of Γ turn out to be $-(n_1\alpha' + n_2\beta')^2$ and $-(n_2\alpha' - n_1\beta')^2$, which must be non-negative. This gives us the system of equations

$$\begin{aligned} n_1\alpha' + n_2\beta'\big|_{(\tau,U,V)} &= 0 \\ -n_1\beta' + n_2\alpha'\big|_{(\tau,U,V)} &= 0 \ \ \forall(\tau, U, V). \end{aligned}$$

By our assumption on $\boldsymbol{n}$, this implies that $\alpha' = \beta' \equiv 0$. Restating this geometrically, there is a small $\mathcal{M}_W$-open neighbourhood of $0 \in \partial D_W$ such that, for every y in this neighbourhood, $\Lambda_{y,W}$ is parallel to the vector W. This holds true for each non-zero $W \in H_{y^0}(\mathcal{M}_{D,1})$. Thus, there is an $\mathcal{M}_{D,1}$-open patch $\omega \ni y^0$ such that

$$x + H_x(\mathcal{M}_{D,1}) \text{ is parallel to } \{z_1 = 0\} \text{ for every } x \in \omega. \tag{4.2}$$

By Result 3.3, $\mathcal{M}_{D,1}$ is connected. Thus, if $y^0 \neq y \in \mathcal{M}_{D,1}$, then y can be joined to y^0 by a chain of $\mathcal{M}_{D,1}$-open patches $\omega_0, \dots, \omega_N$, where ω_0 equals the patch ω in (4.2), $\omega_{j-1}j \cap \omega_j \neq \varnothing$, $j = 1, \dots, N$, and $\omega_N \ni y$. By a standard argument of real-analytic continuation, we deduce that (4.2) holds with ω_N replacing ω (where z_1 comes from the global system of coordinates fixed at the beginning of the previous paragraph). Hence, $x + H_x(\mathcal{M}_{D,1})$ is parallel to $\{z_1 = 0\}$ for each $x \in \mathcal{M}_{D,1}$. As $\mathcal{M}_{D,1}$ is dense in ∂D, and D is bounded, we can find a $\xi \in D$ and a vector $W = (W_1, \dots, W_n)$ with $W_1 = 0$ such that the ray $\{\xi + tW; t \geq 0\}$ intersects ∂D at a point in $\mathcal{M}_{D,1}$. Then, this ray must be tangential to $\mathcal{M}_{D,1}$ at the point of intersection, which is absurd as D is convex. Hence, our initial assumption must be false. ☐

5. The proof of Theorem 1.1

Before we proceed further, we clarify our notation for the different norms that will be used in the proof of Theorem 1.1. With D_1 and D_2 as in Theorem 1.1, $\|\cdot\|_j$ will denote the spectral norms such that D_j is the unit $\|\cdot\|_j$-ball in $\mathbb{C}^n$, $j = 1, 2$. The Euclidean norm on $\mathbb{C}^n$ will be denoted by $|\cdot|$. We will also need to impose norms on certain linear operators on $\mathbb{C}^n$. We shall use the operator norm induced by the Euclidean norm: for a $\mathbb{C}$-linear operator A on $\mathbb{C}^n$, we set

$$\|A\|_{op} := \sup_{|x|=1} |Ax|.$$

The proof of Theorem 1.1. We shall take D_1 and D_2 to be Harish-Chandra realizations of the given bounded symmetric domains. We may assume, composing F with suitable automorphisms if necessary, that $F(0) = 0$.

By Lemma 4.2, F extends to a holomorphic map defined on a neighbourhood N of $\overline{D_1}$. For simplicity of notation, we shall denote this extension also as F. The complex Jacobian $\mathsf{Jac}_{\mathbb{C}}F$ is holomorphic on N and $\mathsf{Jac}_{\mathbb{C}}F \not\equiv 0$ on D_1. Hence, by the maximum principle, $\mathsf{Jac}_{\mathbb{C}}F \not\equiv 0$ on ∂D_1. By definition, we can find a point $p \in \partial_S D_1$ such that

$$\sup_{\overline{D_1}} |\mathsf{Jac}_{\mathbb{C}}F| = |\mathsf{Jac}_{\mathbb{C}}F(p)| \ \neq 0.$$

By the inverse function theorem, we can find a ball $B(p, r) \subset N$ such that $F|_{B(p,r)}$ is injective. Let us write

$$\Omega_1 := B(p, r) \cap D_1, \ \ \Omega_2 := F(B(p, r)) \cap D_2.$$

We shall use our Key Lemma (see Section 1, and Section 4 for its proof) to deduce the result. The regions W_1 and W_2 of that Lemma will be constructed by applying suitable automorphisms to Ω_1 and Ω_2.

Claim. $F(p) \in \partial_S D_2$.
Suppose $F(p) \notin \partial_S D_2$. It follows from Result 3.5 and Result 3.6 that there is a vector $V \in \mathbb{C}^n \setminus \{0\}$ and neighbourhood ω of $0 \in \mathbb{C}$ such that $\psi(\omega) \subset F(B(p,r)) \cap \partial D_2$, where $\psi : \omega \ni \zeta \longmapsto F(p) + \zeta V$. Next, define

$$\widetilde{\psi} := (F|_{B(p,r)})^{-1} \circ \psi.$$

Since $F|_{D_1}$ is proper and $F|_{B(p,r)}$ is injective,

$$F(z) \in F(B(p,r)) \cap \partial D_2 \iff z \in B(p,r) \cap \partial D_1.$$

Thus $\widetilde{\psi}(\omega) \subset \partial D_1$. Furthermore, $\widetilde{\psi}$ is non-constant and $\widetilde{\psi}(0) = p$. By definition, each point of $\widetilde{\psi}(\omega) \setminus \{p\}$ lies in the holomorphic arc component of ∂D_1 containing p. This is a contradiction since p, being an extreme point, is a one-point affine ∂D_1-component and thus, by Result 3.5, a one-point holomorphic arc component of ∂D_1. Hence the claim.

Let us now take a sequence $\{a_k\} \subset \Omega_1$ such that $a_k \to p$, and let $b_k := F(a_k)$. Let $\phi^1_k \in \mathrm{Aut}(D_1)$ denote an automorphism that maps 0 to a_k. Let $\phi^2_k \in \mathrm{Aut}(D_2)$ be an automorphism that maps 0 to b_k. Owing to Result 3.6 and to convexity, we can construct a peak function for p on $\overline{D}_1$. Likewise (in view of the last claim) $F(p)$ is a peak point of D_2. By Lemma 4.4, we get:

$$\phi^j_k \longrightarrow \mathsf{const}_{p^j} \text{ uniformly on compacts}, \;\; j = 1, 2, \tag{5.1}$$

where $p^j, p^1 := p, p^2 := F(p)$.

We now define

$$\Omega^k_j := (\phi^j_k)^{-1}(\Omega_j), \;\; j = 1, 2, \;\; k \in \mathbb{Z}_+.$$

Given any $r > 0$, write $rD_j := \{z \in \mathbb{C}^n : \|z\|_j < r\}$, $j = 1, 2$. By (5.1), there exists a sequence $k_1 < k_2 < k_3 < \dots$ in $\mathbb{Z}_+$ such that

$$\phi^1_{k_\nu}\left((1 - 1/s)\overline{D}_1\right) \subset \Omega_1 \;\; \forall \nu \geq s, \;\; s \in \mathbb{Z}_+.$$

By (5.1) again, we can extract a sequence of indices $\nu(1) < \nu(2) < \nu(3) < \dots$ such that

$$\phi^2_{k_{\nu(t)}}\left((1 - 1/s)\overline{D}_2\right) \subset \Omega_2 \;\; \forall t \geq s, \;\; s \in \mathbb{Z}_+.$$

In the interests of readability of notation, let us re-index $\{k_{\nu(s)}\}_{s \in \mathbb{Z}_+}$ as $\{k_m\}_{m \in \mathbb{Z}_+}$. Then, the above can be summarized as:

- $(*)$ With the sequences of maps $\{\phi^1_k\} \subset \mathrm{Aut}(D_1)$ and $\{\phi^2_k\} \subset \mathrm{Aut}(D_2)$ as described above, there is a sequence $\{k_m\}_{m \in \mathbb{Z}_+} \subset \mathbb{Z}_+$ and a strictly increasing $\mathbb{Z}_+$-valued function ν^* such that

$$(1 - 1/s)\overline{D}_1 \subset \Omega^{k_m}_1 \;\; \forall m \geq s, \;\; s \in \mathbb{Z}_+, \tag{5.2}$$

$$(1 - 1/\nu^*(s))\overline{D}_2 \subset \Omega^{k_m}_2 \;\; \forall m \geq s, \;\; s \in \mathbb{Z}_+. \tag{5.3}$$

Step 1. *Analysing the family* $\{(\phi^2_{k_m})^{-1} \circ F \circ \phi^1_{k_m}\}_{m \in \mathbb{Z}_+}$
Consider the maps $G_m : D_1 \to D_2$ defined by

$$G_m := (\phi^2_{k_m})^{-1} \circ F \circ \phi^1_{k_m}.$$

By Montel's theorem, and passing to a subsequence and relabelling if necessary, we get a map $G \in \mathcal{O}(D_1; \mathbb{C}^n)$ such that $G_m \to G$ uniformly on compact subsets. Let us fix an $s \in \mathbb{Z}_+$. By $(*)$, we infer that $\exists M_s \in \mathbb{Z}_+$ such that $(1-1/s)\overline{D}_j \subset \Omega_j^{k_m}$ $\forall m \geq M_s$, $j = 1, 2$. Note that $G_m|_{\Omega_1^{k_m}}$ is a biholomorphism, whence $G'_m(0)$ is invertible for each m. Hence, by the Schwarz lemma for convex balanced domains (i.e. Result 4.5 above) $G'_m(0)$ maps $(1-1/s)D_1$ into D_2 and $G'_m(0)^{-1}$ maps $(1-1/s)D_2$ into D_1 $\forall m \geq M_s$. We claim that this implies that $G'(0)$ is invertible. Suppose not. Then we would find a z_0 with $\|z_0\|_1 = (1-2/s)$ such that $G'(0)z_0 = 0$. Note that $G'_m(0) \to G'(0)$ in norm, whence, given any $\varepsilon > 0, \|G'_m(0)z_0\|_2 < \varepsilon$ for every sufficiently large m. If we now choose $\varepsilon \leq (1-2/s)^2$, we see that

$$G'_m(0)^{-1}\left(\{\|w\|_2 = (1-2/s)\}\right) \not\subset D_1$$

for all sufficiently large m. This is a contradiction. Hence the claim.

Now that it is established that $G'(0)$ is invertible, it follows that $G'_m(0)^{-1} \to G'(0)^{-1}$ in norm. Hence, $G'(0)^{-1}$ maps $(1-1/s)D_2$ into D_1. Recall that $s \in \mathbb{Z}_+$ was arbitrarily chosen and that the function ν^* in $(*)$ is strictly increasing. Thus, $G'(0)^{-1}$ maps D_2 into D_1. By construction, $G(D_1) \subset \overline{D}_2$. Now, D_2 is complete (Kobayashi) hyperbolic. Hence D_2 is taut; see [8]. As $G(0) = 0 \in D_2, G$ maps D_1 to D_2. So, the holomorphic map $G'(0)^{-1} \circ G : D_1 \to D_1$ satisfies all the conditions of Cartan's uniqueness theorem. Thus,

$$G'(0)^{-1} \circ G = \mathsf{id}_{D_1},$$

which means that $G = G'(0)|_{D_1}$.

Step 2. *Showing that D_1 and D_2 are biholomorphically equivalent*
We have shown in Step 1 that $G'(0)^{-1}$ maps $(1-1/s)D_2$ into D_1. As $G'(0)$ is injective, this means that $G'(0)(D_1)$ contains $(1-1/s)D_2$ for arbitrarily large $s \in \mathbb{Z}_+$. Thus G maps D_1 onto D_2. It follows that D_1 is biholomorphic to D_2.

It would help to simplify our notation somewhat. By the nature of the argument in Step 1, it is clear that we can assume that the sequences $\{a_k\} \subset \Omega_1$ and $\{b_k\} \subset \Omega_2$ are so selected that $(*)$ is true with $\{k_m\}_{m\in\mathbb{Z}_+} = \{1, 2, 3, \dots\}$. Owing to Step 2, we may now assume $D_1 = D_2 := D$. The argument we will make in Step 3 below is valid regardless of the specific sequence $\{a_k\}$ or $\{b_k\}$. Hence, in the next three paragraphs following this, the sequence $\{A_k\}$ will stand for either $\{a_k\}$ or $\{b_k\}$, and the point q will stand for either p or $F(p)$. Also, we will abbreviate $\phi^j_{A_k}$ to ϕ_k.

Step 3. *Producing subsequences of $\{\phi_k\}$ that converge on "large" subsets of ∂D.*
By Result 4.3 we may take $\phi_k = g_{A_k}$, whence

$$\phi'_k(z) = \mathbf{B}_D(A_k, A_k)^{1/2} \circ \mathbf{B}_D(-z, A_k)^{-1}. \tag{5.4}$$

In the argument that follows, it is implicit that each ϕ_k is defined as a holomorphic map on some neighbourhood (which depends on ϕ_k) of $\overline{D}$; see Lemma 4.2. By Proposition 4.7 we can find a point $\xi_0 \in \mathcal{M}_{D,1}$ such that

$$\det \mathbf{B}_D(e^{i\theta}\xi_0, q) \neq 0 \ \ \forall\theta \in \mathbb{R}.$$

By continuity, there exists a $\overline{D}$-open neighbourhood Γ of q, an $\mathcal{M}_{D,1}$-open neighbourhood W of ξ_0, and a $\overline{D}$-open set V with the following properties:

(a) $z \in V \implies e^{i\theta}z \in V \ \forall\theta \in \mathbb{R}$;
(b) $V \cap \partial D = S^1 \cdot W$;

(c) $z \in V \implies tz \in V \ \forall t \in [1, 1/\|z\|]$

(now $\|\cdot\|$ is the spectral norm associated to D); such that

$$\det \mathbf{B}_D(z,w) \neq 0 \ \ \forall (z,w) \in \overline{V} \times \Gamma. \tag{5.5}$$

Here, given a set $X \subset \mathbb{C}^n, S^1 \cdot X$ stands for the set $\{e^{i\theta}x : x \in X, \theta \in \mathbb{R}\}$. Let us call any pair (V,W), where V is a $\overline{D}$-open set and W is an $\mathcal{M}_{D,1}$-open set, a *truncated prism with base* $S^1 \cdot W$ if (V,W) satisfies properties (a)-(c) above.

We can find V' and W', with $\overline{W'} \subset W$, such that (V', W') is a truncated prism with base $S^1 \cdot W'$ with the properties:

- $\overline{V'} \subset V$;
- There exists a $\delta_0 \ll 1$ such that for $z_1, z_2 \in \overline{V'}$, the segment $[z_1, z_2] \subset V$ whenever $|z_1 - z_2| < \delta_0$.

Owing to holomorphicity and convexity,

$$\phi_k(z_1) - \phi_k(z_2) = \int_0^1 \phi_k'\left(z_1 + t(z_2 - z_1)\right)(z_2 - z_1)dt, \ \ z_1, z_2 \in \overline{D}. \tag{5.6}$$

We can find a $K \equiv K(W)$ such that, in view of (5.5), $\{\mathbf{B}_D(z, A_k) : k \geq K(W), z \in \overline{V}\}$ is a compact family in $GL(n, \mathbb{C})$. Hence, in view of (5.4) (and since $\{\mathbf{B}_D(A_k, A_k) : k \in \mathbb{Z}_+\}$ is a relatively compact family in $\mathbb{C}^{n \times n}$), there exists a constant $C > 0$ such that

$$\|\phi_k'(z)\|_{op} \ \leq \ C \ \forall z \in \overline{V}, \ \ \forall k \geq K.$$

By our construction of V', and from (5.6), we conclude:

$$|\phi_k(z_1) - \phi_k(z_2)| \leq C|z_1 - z_2| \ \ \forall z_1, z_2 \in \overline{V'}, \ |z_1 - z_2| < \delta_0, \text{ and } \forall k \geq K.$$

In short, $\{\phi_k|_{\overline{V'}}\} \subset \mathcal{C}(\overline{V'}; \mathbb{C}^n)$ is an equicontinuous family.

By the Arzela-Ascoli theorem, we can find a subsequence of $\{\phi_k\}$ that converges uniformly to q on $\overline{V'}$. For simplicity of notation, let us continue to denote this subsequence as $\{\phi_k\}$. Then there exists a $K_1 \in \mathbb{Z}_+$ such that $\phi_k(\overline{V'}) \subset \Omega$ (which denotes either Ω_1 or Ω_2) $\forall k \geq K_1$. Furthermore, we may assume that K_1 is so large that, thanks to $(*)$,

$$(1 - 1/s)\overline{D} \subset \phi_k^{-1}(\Omega) \ \ \forall k \geq K_1,$$

where s is so large that $(1 - 1/s)\overline{D} \cap V'$ is a non-empty open set. By construction:

$$z \in V' \cap D \implies \Delta_z \subset (1 - 1/s)\overline{D} \cup V'.$$

Hence $\Delta_z \subset \phi_k^{-1}(\Omega) \ \forall k \geq K_1$. We summarize the content of this paragraph as follows:

$(**)$ Given any truncated prism (V,W) with base $S^1 \cdot W$ such that $\mathbf{B}_D(z, A_k) \neq 0$ on $\overline{V}$ for all k sufficiently large, we can find a $K_1 \in \mathbb{Z}_+$ and a truncated prism (V', W') with $\overline{V'} \subset V$ such that $\Delta_z \subset \phi_k^{-1}(\Omega)$ for each $z \in V' \cap D$ and each $k \geq K_1$.

Step 4. *Completing the proof.*
By Proposition 4.7 and $(**)$, we can find a truncated prism (V', W') with base $S^1 \cdot W'$ which has all the properties stated in $(**)$. Let $s \in \mathbb{Z}_+$ be so large that $(1 - 1/s)D \cap V' := U'$ is a non-empty open set. As $G_k \to G$ uniformly on U' (by Step 1), there exists a point $w_0 \in G(U')$, $K_2 \in \mathbb{Z}_+$ and a $c > 0$ such that the ball

$$B(w_0, c) \subset G(U') \cap G_k(U') \ \text{ and } \ B(w_0, c) \subset \Omega_2^k \ \ \forall k \geq K_2.$$

Write $\|\cdot\|$ for the spectral norm associated to D. Let $R : \mathbb{C}^n \setminus 0 \to \partial D$ be given by $R(w) := w/\|w\|$. By Proposition 4.7 and $(**)$, we can find a $\mathcal{M}_{D,1}$-open subset ω_2 such that

$$\omega_2 \subset R(B(w_0, c)),$$

a truncated prism (V_2, ω_2) with base $S^1 \cdot \omega_2$, and a $K_3 \in \mathbb{Z}_+$ such that $\Delta_w \subset \Omega_2^k$ for each $w \in V_2 \cap D$ and each $k \geq K_3$. Let us now set $U := G^{-1}\left(R^{-1}(\omega_2) \cap B(w_0, c)\right)$, and $K^* := \max(K_1, K_2, K_3)$. Finally, we set

$$W_j := \left(\phi^j_{K^*}\right)^{-1}(\Omega_j), \ \ j = 1, 2,$$

with the understanding that $\phi_k^1 = g_{a_k}$ and $\phi_k^2 = g_{b_k}$.

As $U \subset V', \Delta_z \subset W_1$ for each $z \in U$. By construction

$$G_{K^*}(z) \in B(w_0, c) \subset W_2 \ \ \forall z \in U.$$

Finally, by construction, for each $z \in U$, there exists a point $w_z \in \Delta_{G_{K^*}(z)}$ that belongs to $V_2 \cap D$. Thus, $\Delta_{G_{K^*}(z)} \subset W_2$. Recall that $G_{K^*}|_{W_1} : W_1 \to W_2$ is a biholomorphism and $G_{K^*}(0) = 0$. By our Key Lemma, G_{K^*}, and consequently F, must be a biholomorphism. $\square$

6. The proof of Theorem 1.5

As p is an orbit accumulation point, there is a point $a_0 \in D_1$ and a sequence $\{\phi_k\} \subset \mathrm{Aut}(D_1)$ such that $\phi_k(a_0) \to p$. Regardless of whether p is a peak point or $F(p)$ is a peak point, let us denote the relevant peak function as H. Let B denote a small ball centered at p, with $B \Subset U$, if p is a peak point, and centered at $F(p)$, with $B \Subset F(U)$, if $F(p)$ is a peak point. Depending on whether p or $F(p)$ is a peak point, set $G := F^{-1}$ or $G := F$, respectively. Finally, set

$$h := \begin{cases} H \circ G|_{\overline{B \cap D_2}}, & \text{if } p \text{ is a peak point,} \\ H \circ G|_{\overline{B \cap D_1}}, & \text{if } F(p) \text{ is a peak point.} \end{cases}$$

By our hypothesis on F, it follows that h satisfies all the conditions required of the function h in Lemma 4.4 for the appropriate choice of (D, p) depending on whether p or $F(p)$ is a peak point.

Let us now denote the automorphisms discussed above as $\phi_k^1, \ k = 1, 2, 3, \dots$ Then, using H or the function h constructed above, depending on whether p or $F(p)$ is a peak point, we deduce by Lemma 4.4 that $\phi_k^1 \longrightarrow \mathsf{const}_p$ uniformly on compact subsets of D_1. Set $a_k := \phi_k^1(0)$. As $a_k \to p$, we may assume without loss of generality that $a_k \in U$. Let $b_k := F(a_k)$, and let $\phi_k^2 \in \mathrm{Aut}(D_2)$ be an automorphism that maps 0 to b_k (which is possible as $\mathrm{Aut}(D_2)$ acts transitively on D_2). Repeating the above argument, $\phi_k^2 \longrightarrow \mathsf{const}_{F(p)}$ uniformly on compact subsets of D_2. We have arrived at the same result as in (5.1). Thereafter, if we define

$$\Omega_j^k := (\phi_k^j)^{-1}(\Omega_j), \ \ j = 1, 2, \ k \in \mathbb{Z}_+,$$

where $\Omega_1 := U$ and $\Omega_2 := F(U)$, then, reasoning exactly as in the passage following (5.1), we deduce that $(*)$ from Section 5 holds true for our present set-up.

With $\{k_m\}_{m \in \mathbb{Z}_+}$ as given by $(*)$, let us define the maps $G_m : \Omega_1^{k_m} \to \Omega_2^{k_m}$ by

$$G_m := (\phi^2_{k_m})^{-1} \circ F \circ \phi^1_{k_m}.$$

By construction, each G_m is a biholomorphic map. In particular,

$$G_m(0) = 0, \text{ and } G_m'(0) \in GL(n, \mathbb{C}). \tag{6.1}$$

We may assume, owing to (5.1), that *the sequences* $\{\Omega_j^{k_m}\}_{m \in \mathbb{Z}_+}$ *are increasing sequences.* By Montel's theorem, and arguing by induction, we can find sequences $\{G_{l,m}\}$ and holomorphic maps $\Gamma_l : \Omega_1^{k_l} \to \overline{D}_2$ such that:

- $\{G_{1,m}\}_{m\in\mathbb{Z}_+}$ is a subsequence of $\{G_\nu\}_{\nu\in\mathbb{Z}_+}$ and $\{G_{l+1,m}\}_{m\in\mathbb{Z}_+}$ is a subsequence of $\{G_{l,\nu}\}_{\nu\in\mathbb{Z}_+}$;
- $G_{l,m}|_{\Omega_1^{k_l}} \longrightarrow \Gamma_l$, as $m \to \infty$, uniformly on compact subsets of $\Omega_1^{k_l}$;

for $l = 1, 2, 3, \dots$ Owing to this construction, the rule

$$\Gamma(z) := \Gamma_l(z) \text{ if } z \in \Omega_1^{k_l},$$

gives a well-defined holomorphic map $\Gamma : D_1 \to \overline{D}_2$.

Let us define $H_l := G_{l,l}$. Now suppose $\Gamma(D_1) \cap \partial D_2 \neq \varnothing$. Then, $\exists \xi \in D_1$ such that $\Gamma(\xi) \in \partial D_2$. Let $M \in \mathbb{Z}_+$ be so large that $\Omega_1^{k_M} \ni \xi$. As D_2 is a bounded symmetric domain, it is taut. Thus, by focusing attention on the sequence

$$\{H_l|_{\Omega_1^{k_M}} : l = M, M+1, M+2, \dots\} \subset \mathcal{O}(\Omega_1^{k_M}; D_2),$$

we must conclude, by assumption, that $\Gamma(\Omega_1^{k_M}) \subset \partial D_2$. But, by (6.1), $\Gamma(0) = 0 \notin \partial D_2$. This is a contradiction, from which we infer:

(a) The range of Γ is a subset of D_2.

Now observe that, by $(*)$, we have:

(b) The sequence $\{H_l : l = s, s+1, s+2, \dots\}$ converges uniformly to Γ on $(1 - 1/s)\overline{D}_1$, $s \in \mathbb{Z}_+$.

(c) H_l^{-1} maps 0 to 0 and $(1 - 1/\nu^*(l))D_2$ into D_1 (since $\mathsf{dom}(H_l^{-1}) = \mathsf{range}(H_l) \supseteq \Omega_2^{k_l}$).

In view of (6.1) and the fact that D_1 and D_2 are balanced, (a)-(c) are precisely the ingredients required to to repeat the argument in Step 1 of the proof of Theorem 1.1 to infer that $\Gamma'(0)$ is invertible, $\Gamma'(0)^{-1} : D_2 \to D_1$ and

$$\Gamma'(0)^{-1} \circ \Gamma = \mathsf{id}_{D_1}.$$

Thus, by (a), $\Gamma'(0)(D_1) \subset D_2$. One of the consequences of repeating the argument contained in Step 1 in Section 1.1 is, in view of (c), that $\Gamma'(0)^{-1}$ maps $(1 - 1/\nu^*(l))D_2$ into D_2 for every $l \in \mathbb{Z}_+$. As ν^* is strictly increasing and $\mathbb{Z}_+$-valued, and as $\Gamma'(0)$ is injective, this means that $\Gamma'(0)(D_1)$ contains $(1 - 1/s)D_2$ for *arbitrarily large* $s \in \mathbb{Z}_+$, whence $\Gamma'(0)$ maps D_1 onto D_2. Hence, $\Gamma'(0)|_{D_1}$ is a biholomorphism of D_1 onto D_2. $\square$

Acknowledgements. Jaikrishnan Janardhanan would like to thank his colleagues and friends G.P. Balakumar, Dheeraj Kulkarni, Divakaran Divakaran and Pranav Haridas for many interesting discussions. Gautam Bharali contributed to this work while on sabbatical at the Norwegian University of Science and Technology (NTNU) in Trondheim. He would like to acknowledge the support and the hospitality of the Department of Mathematical Sciences at NTNU. He also thanks John Erik Fornæss for his helpful comments on this work.

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