POLYNOMIAL APPROXIMATION, LOCAL POLYNOMIAL CONVEXITY, AND DEGENERATE CR SINGULARITIES

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Abstract. We begin with the following question: given a closed disc $\overline{D} \Subset \mathbb{C}$ and a complex-valued function $F \in \mathcal{C}(\overline{D})$, is the uniform algebra on $\overline{D}$ generated by z and F equal to $\mathcal{C}(\overline{D})$? When $F \in \mathcal{C}^1(D)$, this question is complicated by the presence of points in the surface $\mathcal{S} := \text{graph}_{\overline{D}}(F)$ that have complex tangents. Such points are called CR singularities. Let $p \in \mathcal{S}$ be a CR singularity at which the order of contact of the tangent plane with $\mathcal{S}$ is greater than 2; i.e. a degenerate CR singularity. We provide sufficient conditions for $\mathcal{S}$ to be locally polynomially convex at the degenerate singularity p. This is useful because it is essential to know whether $\mathcal{S}$ is locally polynomially convex at a CR singularity in order to answer the initial question. To this end, we also present a general theorem on the uniform algebra generated by z and F, which we use in our investigations. This result may be of independent interest because it is applicable even to non-smooth, complex-valued F.

1. Introduction and statement of results

One of the concerns of this paper is to study the following question: given a closed disc $\overline{D} \Subset \mathbb{C}$ and a complex-valued function $F \in \mathcal{C}(\overline{D})$, when is the uniform algebra on $\overline{D}$ generated by z and F equal to $\mathcal{C}(\overline{D})$? A necessary condition for a positive answer to this question is that $\text{graph}_{\overline{D}}(F) \subset \mathbb{C}^2$ must be polynomially convex. A compact subset $K \subset \mathbb{C}^n$ is said to be **polynomially convex** if for each point $\zeta \notin K$, there exists a holomorphic polynomial P such that $P(\zeta) = 1$ and $\sup_K |P| < 1$. The compact K is said to be **locally polynomially convex** at a point $p \in K$ if there exists a closed ball $\mathbb{B}(p)$ centered at p such that $K \cap \mathbb{B}(p)$ is polynomially convex. In general, it is difficult to determine whether a given compact $K \subset \mathbb{C}^n$ is polynomially convex when $n > 1$, but questions of polynomial convexity arise repeatedly in connection with function theory. There is a considerable body of work concerning the (local) polynomial convexity of smooth surfaces in $\mathbb{C}^n$. The references associated with the smooth case are too numerous to list here; instead, the reader is referred to the survey [9]. In the instances discussed in that survey, one also obtains positive answers to the question presented above. In contrast, very little is known when F is non-smooth — either about the polynomial convexity of $\text{graph}_{\overline{D}}(F)$, or about the question asked above — beyond Mergelyan's result [7]. Mergelyan's result, however, is only applicable when F is real-valued. Part of the intention of this paper is to provide a sufficient condition on a *complex-valued* $F \in \mathcal{C}(\overline{D})$ for the question posed above to have an affirmative answer. This result is Theorem 1.1 stated below. We note

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that our sufficient condition is stated in terms of the value-distribution of F, which is easy to understand, and may be applied to concrete situations. One such application is Theorem 1.2 below.

Before stating Theorem 1.1; we need to introduce some notation. In what follows, $\overline{D}$ will denote any closed disc in $\mathbb{C}$, while $\overline{D(a;r)}$ will denote the closed disc of radius r centered at $a \in \mathbb{C}$. Unless explicitly stated otherwise, all continuous functions will be assumed to be *complex-valued*. If K is a compact subset of $\mathbb{C}$, and $\phi_1, \phi_2, \ldots, \phi_N$ are continuous functions on K, $[\phi_1, \phi_2, \ldots, \phi_N]_K$ is defined as

$$[\phi_1, \phi_2, \ldots, \phi_N]_K := \{f \in \mathcal{C}(K) : f \text{ can be approximated uniformly on } K \text{ by complex polynomials in } \phi_1, \phi_2, \ldots, \phi_N\}.$$

An **open sector with vertex at *a***, denoted by $S(a; I)$, is the set

$$S(a; I) := \{a + re^{i\theta} : r > 0, \text{and } \theta \in I\},$$

where I is an open subinterval of $[-2\pi, 2\pi)$ with $\text{length}(I) < 2\pi$, and $a \in \mathbb{C}$. Having established our notation, we can now state our first result.

Theorem 1.1. *Let F be a complex-valued continuous function on a closed disc $\overline{D} \Subset \mathbb{C}$. Suppose that there is a set $E \subset \overline{D}$ having zero Lebesgue measure such that $F^{-1}\{F(\zeta)\}$ is at most countable $\forall \zeta \in \overline{D} \setminus E$. Furthermore, suppose that for each $\zeta \in \overline{D} \setminus E$, there exists an open sector $S(0; I_\zeta)$ with vertex at $0 \in \mathbb{C}$ such that*

$$(z - \zeta)\{F(z) - F(\zeta)\} \in S(0; I_\zeta) \quad \forall z \in \overline{D} \setminus F^{-1}\{F(\zeta)\}. \tag{1.1}$$

Then, $[z, F]_{\overline{D}} = \mathcal{C}(\overline{D})$.

The reader will notice that Weierstrass's approximation theorem is a special case of the above theorem: when $F(z) = \overline{z}$, (1.1) is satisfied by taking $S(0; I_\zeta)$, for each $\zeta \in \overline{D}$, to be a *fixed* sector containing the positive real axis. We remark here that the proof of the above theorem is reminiscent of the early work of Wermer — see, for instance, [12, Theorem 1] — on questions of the sort considered in this paper. The crucial difference between those results and Theorem 1.1 is that the sectors $S(0; I_\zeta)$ occurring herein are allowed to have interior angles that are *greater than* π. In results such as [12, Theorem 1], the conclusion $[z, F]_{\overline{D}} = \mathcal{C}(\overline{D})$ is obtained under the assumption that F is injective on $\overline{D}$, and the methods used in those results work only if the quantities occurring in (1.1) lie in small subsets of a half-plane. It is for these reasons that Theorem 1.1 is a more general result.

Another objective of this paper is to further investigate the smooth case. Let $\mathcal{S}$ be a smooth real surface $\mathcal{S}$ in $\mathbb{C}^n$, $n > 1$. A point $p \in \mathcal{S}$ is said to be **totally real** if the tangent plane $T_p(\mathcal{S})$ at p is not a complex line. A point on $\mathcal{S}$ that is not totally real will be called a **CR singularity**. At a totally real point $p \in \mathcal{S}$, the surface $\mathcal{S}$ is locally polynomially convex. Contrast this with a CR singularity $p \in \mathcal{S} \subset \mathbb{C}^2$ when the order of contact of $T_p(\mathcal{S})$ with $\mathcal{S}$ equals 2. Since $T_p(\mathcal{S})$ is a complex tangent, there exist holomorphic coordinates (z, w) centered at p such that $\mathcal{S}$ is locally given by an equation of the form $w = |z|^2 + \gamma(z^2 + \overline{z}^2) + G(z)$, where $\gamma \geq 0$, $G(z) = O(|z|^3)$, and three distinct situations arise. In Bishop's terminology, the CR singularity $p = (0, 0)$ is said to be elliptic if $0 \leq \gamma < 1/2$, parabolic if $\gamma = 1/2$, and hyperbolic if $\gamma > 1/2$. Bishop showed [3], among other things, that if p is elliptic, then $\mathcal{S}$ is *not* locally polynomially convex. Much later, Forstnerič & Stout [4] showed (also refer to [11] by Stout) that if $p \in \mathcal{S}$ is an isolated, hyperbolic CR singularity, then $\mathcal{S}$ is locally polynomially convex at p. Furthermore, in the hyperbolic case, writing $F(z) = |z|^2 + \gamma(z^2 + \overline{z}^2) + G(z)$, it has been shown in [4] that, for a small $\varepsilon > 0$, $[z, F]_{\{|z| \leq \varepsilon\}} = \mathcal{C}(\overline{D}(0; \varepsilon))$. The more involved case $\gamma = 1/2$ has been studied in [5].

This raises the question: what can be said about the polynomial convexity of a surface $\mathcal{S}$ if the order of contact of $T_p(\mathcal{S})$ with $\mathcal{S}$ at a CR singularity p is *greater* than 2 ? We will call such a CR singularity a **degenerate CR singularity**. Some answers to the question just asked are known when F is a globally-defined, proper branched covering $F : \mathbb{C} \to \mathbb{C}$; refer to [8]. It would, however, be useful to know what happens when F is defined locally, i.e. to deduce whether $\mathcal{S} = \text{graph}(F)$ is locally polynomially convex — as in, for instance, the Forstnerič-Stout paper — at the degenerate CR singularity $(0, F(0))$ just from local information about F. In a somewhat different direction, Wiegerinck's paper [13] studies *the failure of polynomial convexity* based on local conditions on F. In a recent paper [1], the surface expressed locally at a degenerate CR singularity as

$$\begin{aligned} \mathcal{S} \ : \quad w \ &= \sum_{\alpha+\beta=k} C_{\alpha,\beta} z^{\alpha} \overline{z}^{\beta} + G(z) \\ &\equiv \ C_{k,0} z^k + C_{0,k} \overline{z}^k + \Sigma(z) + G(z), \end{aligned} \tag{1.2}$$

where $k > 2$ and G is a smooth function satisfying $G(z) = o(|z|^k)$ as $z \to 0$, was considered. A rough statement of one of the results in [1] is that given $C_{0,k} \neq 0$, if

$$\sup_{|\zeta|=1} \frac{|\Sigma(\zeta)|}{|\zeta|^k} \ < \ |C_{0,k}| \min\left\{ \frac{\pi}{2k}, \frac{1}{2} \right\}, \tag{1.3}$$

and if $\Sigma(z)$ does not fluctuate too greatly, then $\mathcal{S}$ is locally polynomially convex at $(0, 0)$. The analytical condition (1.3) essentially says that — writing $\mathcal{S} = \text{graph}(F)$ locally — if the $\overline{z}^k$ term is in some sense the dominant term among all the leading-order terms in the Taylor expansion of F around $z = 0$, then $\mathcal{S}$ is polynomially convex in a small neighbourhood of $(0, 0)$. One can make the following observations about the result under discussion:

a) One might ask what can be deduced if some term other than the $\overline{z}^k$ term is the dominant term among all the leading-order terms in the Taylor expansion of F around $z = 0$. From that perspective, the hypothesis discussed above is somewhat restrictive.

b) A careful examination of the proof of [1, Theorem 1] reveals that even under the restrictive hypothesis that $\overline{z}^k$ be the dominant term among all the leading-order terms of F, [1, Theorem 1] can be strengthened.

Item (a) above is an issue requiring care because, for example, if F were a homogeneous polynomial in z and $\overline{z}$ of degree k , and a term of the form $z^m \overline{z}^{k-m}$, with $m \geq (k - m) > 0$, significantly dominated all other terms, then $\mathcal{S} := \text{graph}(F)$ would *not* be locally polynomially convex at $(0, 0)$. This issue is resolved in Theorem 1.2 below. In the process, this theorem furnishes a considerably broader sufficient condition for local polynomial convexity. However, we need to define some further notation. If ϕ is a complex-valued function that is of class $\mathcal{C}^1$ on an open region $\Omega \subset \mathbb{C}$, we define

$$\|\nabla\phi(\zeta)\| \ := \left|\frac{\partial\phi}{\partial z}(\zeta)\right| \ + \ \left|\frac{\partial\phi}{\partial \overline{z}}(\zeta)\right| \quad \forall\zeta \in \Omega.$$

We can now state our next theorem.

Theorem 1.2. *Let $\mathcal{S}$ be a smooth surface in $\mathbb{C}^2$ that is described near $(0,0) \in \mathcal{S}$ by*

$$\begin{aligned} \mathcal{S} \ : \quad w \ &= \ \sum_{j=0}^{k} C_j z^{k-j}\overline{z}^j + G(z) \\ &\equiv \ C_0 z^k + \Sigma(z) + G(z), \end{aligned} \tag{1.4}$$

where $k > 2$, G is a function of class $\mathcal{C}^1$ around $z = 0$, and $G = O(|z|^{k+1})$. Define the set $I(\mathcal{S}) := \{j \in \mathbb{N} : k/2 < j \leq k \text{ and } C_j \neq 0\}$ and, for each $j \in I(\mathcal{S})$, define $\tau_j(z) := (\Sigma(z) - C_j z^{k-j}\overline{z}^j)/C_j z^{k-j}\overline{z}^j$ $\forall z \neq 0$. Suppose $I(\mathcal{S}) \neq \emptyset$ and that there exists an integer $M \in I(\mathcal{S})$ such that

$$\sup_{|\zeta|=1} |\tau_M(\zeta)| \ < \ \frac{\tan\{\pi/(2M-k)\}}{1 + \tan\{\pi/(2M-k)\}}, \tag{1.5}$$

and

$$k \sup_{|\zeta|=1} |\tau_M(\zeta)| + \left\{1 - \sup_{|\zeta|=1} |\tau_M(\zeta)|\right\}^{-1} \sup_{|\zeta|=1} |\zeta| \ \|\nabla\tau_M(\zeta)\| \ < \ 2M - k. \tag{1.6}$$

Then, there exists a small constant $\varepsilon > 0$ such that $\mathcal{S} \cap \{(z,w) : |z| \leq \varepsilon\}$ is polynomially convex. Furthermore, calling the function on the right-hand side of (1.4) Φ, we have $\mathcal{C}(\overline{D(0;\varepsilon)}) = [z, \Phi]_{\overline{D(0;\varepsilon)}}$.

Remark. It might seem on comparison that, owing to the condition (1.6), for the case $M = k$, the above theorem is weaker than [1, Theorem 1], where the case $M = k$ has been treated. However, the bound (1.3) used in that result is so stringent that, in fact, (1.6) is automatically implied. The conditions of the above theorem are thus more permissive.

2. Some remarks on our proof techniques

The primary purpose of this section is to state two brief lemmas that we shall need to prove the two theorems stated in §1. In doing so, we shall also make a few comments about the broad steps involved in the proofs of our theorems.

To begin with, we state a lemma due to Bishop. This lemma may be found in the first half of the proof of Theorem 4 in the paper [2]. Before stating it, we note that for the remainder of this paper, $\mathfrak{m}$ will denote the Lebesgue measure on $\mathbb{C}$. Bishop's lemma is vital to the proof of Theorem 1.1, and is as follows:

Lemma 2.1 (Bishop). *Let $\overline{D}$ be a closed disc in $\mathbb{C}$. For any measure $\mu \in \mathcal{C}(\overline{D})^\star$, define*

$$h_\mu(\zeta) \ := \ \int_{\overline{D}} \frac{d\mu(z)}{z-\zeta}.$$

Then, $|h_\mu| < \infty$ $\mathfrak{m}$-a.e. in $\mathbb{C}$. If $h_\mu = 0$ $\mathfrak{m}$-a.e. in $\mathbb{C}$, then $\mu = 0$.

With F as in Theorem 1.1, we note that by definition $[z, F]_{\overline{D}}$ is a closed $\mathbb{C}$-linear subspace of $\mathcal{C}(\overline{D})$. Therefore, there exist complex measures $\mu \in \mathcal{C}(\overline{D})^\star$ representing those continuous linear functionals on $\mathcal{C}(\overline{D})$ that annihilate $[z, F]_{\overline{D}}$. Such measures will be called **annihilating measures**. The strategy behind the proof of Theorem 1.1 is to show that any annihilating measure μ is the zero measure. To achieve this, we fix an annihilating measure μ in the proof of Theorem 1.1, and carry out the following two steps:

- For each fixed ζ lying off a certain exceptional set $\mathcal{E} \subsetneq \overline{D}$ with $\mathfrak{m}(\mathcal{E}) = 0$, we use the condition (1.1) to construct a sequence $\{f_n\}_{n\in\mathbb{N}} \subset [z, F]_{\overline{D}}$ such that $f_n(z) \longrightarrow 1/(z-\zeta)$ μ-a.e., and such that these functions are dominated by a function in $\mathbb{L}^1(d|\mu|, \overline{D})$.
- Next, we apply the dominated convergence theorem to $\{f_n\}_{\mathbb{N}}$ to show that $h_\mu(\zeta) = 0$ for each $\zeta \notin \mathcal{E}$, from which — in view of Bishop's lemma — Theorem 1.1 follows.

Details of these steps are presented in §3.

The second lemma that plays an important role in this paper is Kallin's lemma. This is a device that is used to determine when a union of polynomially convex sets is polynomially convex. We state a certain form of Kallin's lemma that we shall use in §4; the reader is referred to [6] for Kallin's original result.

Lemma 2.2 (Kallin). *Suppose X_1 and X_2 are compact subsets of $\mathbb{C}^n$ such that $\mathcal{P}(X_j) = \mathcal{C}(X_j)$, $j = 1, 2$. Let $\phi : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic polynomial such that $\phi(X_j) \subset W_j$, $j = 1, 2$, where W_1 and W_2 are polynomially convex compact sets in $\mathbb{C}$ and $W_1 \cap W_2 = \{0\}$. Assume that $\phi^{-1}\{0\} \cap (X_1 \cup X_2) = X_1 \cap X_2$. Then $\mathcal{P}(X_1 \cup X_2) = \mathcal{C}(X_1 \cup X_2)$.*

The above version of Kallin's lemma is presented within the proof of Theorem IV in [4]. The symbol $\mathcal{P}(K)$ denotes the uniform closure on K (where K is compact) of the polynomials in z. The above lemma is useful in the context of the technique used in our proof of Theorem 1.2. This proof will essentially consist of the following three steps:

- *Step I:* We find a proper mapping $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ and a $\delta > 0$ such that $\Psi^{-1}(\mathcal{S} \cap \{(z, w) : |z| \leq \delta\})$ is a union of bordered surfaces $\mathfrak{S}_1(\delta), \dots, \mathfrak{S}_{2M-k}(\delta)$, each of which enjoys certain special properties.
- *Step II:* Each surface $\mathfrak{S}_j(\delta)$ is the graph of a function F_j, $j = 1, \dots, 2M - k$, over $\overline{D(0;\delta)}$. We show, using the properties F_j possesses, that Theorem 1.1 is applicable to each $[z, F_j]_{\overline{D(0;\delta)}}$, $j = 1, \dots, 2M - k$.
- *Step III:* Finally, we use Kallin's lemma to find an $\varepsilon \in (0, \delta)$ such that $\mathcal{P}(\mathfrak{S}_1(\varepsilon) \cup \dots \cup \mathfrak{S}_{2M-k}(\varepsilon)) = \mathcal{C}(\mathfrak{S}_1(\varepsilon) \cup \dots \cup \mathfrak{S}_{2M-k}(\varepsilon))$. Since Ψ is a proper covering, the last conclusion can be translated, using standard arguments, into the conclusion of Theorem 1.2.

Details of the above argument are presented in §4.

3. The proof of Theorem 1.1

The following lemma is central to proving Theorem 1.1.

Lemma 3.1. *Let $\mathcal{A}$ be a uniform algebra on a closed disc $\overline{D} \Subset \mathbb{C}$ that contains the function z. Fix $\zeta \in \overline{D}$. Assume that there is a function $W \subset \mathcal{A}$ with the property that the set*

$$S_\zeta \ := \ \{(z-\zeta)W(z) : z \in \overline{D} \setminus (\{\zeta\} \cup W^{-1}\{0\}) \ \}$$

is contained in some open sector $S(0;I)$ with vertex at $0 \in \mathbb{C}$. Then, there exists a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\lim_{n\to\infty} f_n(z) \ = \ \frac{1}{z-\zeta} \quad \forall z \in \overline{D} \setminus (\{\zeta\} \cup W^{-1}\{0\}),$$

and

$$|f_n(z)| \ \leq \ \frac{4}{|z-\zeta|} \quad \forall z \in \overline{D} \setminus (\{\zeta\} \cup W^{-1}\{0\}), \ \textit{and} \ \forall n \in \mathbb{N}.$$

Proof. We can find a $\phi \in [-\pi, \pi)$ and an integer $\nu \in \{1,2\}$ such that

$$\mathrm{Re}[(e^{i\phi}w)^{1/\nu}] \ > \ 0 \quad \forall w \in S(0;I),$$

where the ν^{th}-root above is the appropriate branch of the ν^{th}-root that is analytic on $\mathbb{C}\setminus(-\infty,0]$ and achieves the above inequality. Notice that it either suffices to choose $\nu = 1$, or that $\nu = 2$ necessarily, depending on whether the interior angle of $S(0;I)$ is at most π, or is strictly greater than π. Define the holomorphic functions:

$$P_n(w) \ := \ \left\{1 - \frac{1}{[1+(e^{i\phi}w)^{1/\nu}]^n}\right\} \frac{1}{(e^{i\phi}w)^{1/\nu}} \quad \forall w \in S(0;I), \ \forall n \in \mathbb{N}.$$

Notice that P_n extends to a continuous function on $\overline{S}(0;I)$. Therefore, defining

$$Q_n(w) \ := \ \begin{cases} e^{i\phi}P_n(w)^\nu \ = \ \left\{1 - \dfrac{1}{[1+(e^{i\phi}w)^{1/\nu}]^n}\right\}^\nu \dfrac{1}{w}\ , & \text{if } w \in \overline{S}(0;I) \setminus \{0\}, \\ \\ e^{i\phi}n, & \text{if } w = 0, \end{cases}$$

we conclude that $Q_n \in \mathcal{O}(S(0;I)) \cap \mathcal{C}(\overline{S}(0;I))$ for each $n \in \mathbb{N}$.

Since $|1+(e^{i\phi}w)^{1/\nu}| > 1 \ \forall w \in S(0;I)$, we have

$$\lim_{n\to\infty} Q_n(w) \ = \ 1/w \quad \forall w \in S(0;I) \tag{3.1}$$

and, for the same reason

$$|Q_n(w)| \ \leq \ \left|1 + \frac{1}{|1+(e^{i\phi}w)^{1/\nu}|^n}\right|^\nu \frac{1}{|w|} \ \leq \ \frac{4}{|w|} \quad \forall w \in S(0;I), \ \forall n \in \mathbb{N}. \tag{3.2}$$

Recall that:

- Since W is continuous, there exists an $R > 0$ such that $\overline{S}_\zeta \subseteq \overline{S}(0;I) \cap \overline{D}(0;R)$;
- $Q_n \in \mathcal{O}[S(0;I) \cap D(0;R)] \cap \mathcal{C}[\overline{S}(0;I) \cap \overline{D}(0;R)]$ for each $n \in \mathbb{N}$.

Since $\overline{S}(0;I)\cap\overline{D}(0;R)$ is simply connected, by Mergelyan's theorem each Q_n is uniformly approximable on the set $\overline{S}(0;I)\cap\overline{D}(0;R)$ by polynomials in z. Therefore, if we define $q_n(z) := Q_n[(z-\zeta)W(z)]$, then $q_n\in\mathcal{A}$ for each $n\in\mathbb{N}$. Now define

$$f_n(z) \ := \ W(z)q_n(z) \quad \forall z\in\overline{D}.$$

Clearly $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$. Observe that

$$\lim_{n\to\infty} f_n(z) \ = \ W(z)\Big[\lim_{n\to\infty} Q_n((z-\zeta)W(z))\Big] \ = \ \frac{1}{z-\zeta} \quad \forall z\in\overline{D}\setminus(\{\zeta\}\cup W^{-1}\{0\}),$$

which follows from (3.1), since $(z-\zeta)W(z)\in S(0;I)$ $\forall z\in\overline{D}\setminus(\{\zeta\}\cup W^{-1}\{0\})$. For the same reason, (3.2) implies that

$$|f_n(z)| \ \leq \ |W(z)|\times\frac{4}{|(z-\zeta)W(z)|} \ = \ \frac{4}{|z-\zeta|} \quad \forall z\in\overline{D}\setminus(\{\zeta\}\cup W^{-1}\{0\}). \quad \square$$

We now have all the tools necessary to provide the

The proof of Theorem 1.1. Let $\mu\in\mathcal{C}(\overline{D})^\star$ be a measure that annihilates $[z,F]_{\overline{D}}$ (see §2 for a definition). In view of Bishop's lemma, i.e. Lemma 2.1, we need to show that for the chosen annihilating measure μ, $h_\mu = 0$ $\mathfrak{m}$-a.e. So, we first consider $\zeta\notin\overline{D}$. Then, there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ that approximates the function $z\longmapsto(z-\zeta)^{-1}$ uniformly on $\overline{D}$. Clearly, $\{p_n\}_{n\in\mathbb{N}}\subset[z,F]_{\overline{D}}$. Thus, owing to uniform convergence:

$$0 \ = \ \lim_{n\to\infty}\int_{\overline{D}} p_n\,d\mu \ = \ \int_{\overline{D}}\frac{d\mu(z)}{z-\zeta} \qquad (\zeta\notin\overline{D}). \tag{3.3}$$

Now define $A := \{a\in\overline{D}\setminus E : \mu(\{a\})\neq 0\}$. Since μ is a finite, regular measure, A is countable. Hence the set

$$\widetilde{E} \ := \ \bigcup_{a\in A} F^{-1}\{F(a)\}$$

is, by hypothesis, a countable union of countable sets. Since $z\longmapsto|z|^{-1}$ is locally integrable with respect to the Lebesgue measure, and since μ is a finite measure supported in $\overline{D}$, if we define

$$H_\mu(\zeta) \ := \ \int_{\overline{D}}\frac{d|\mu|(z)}{|z-\zeta|}\ ,$$

then $H_\mu<\infty$ $\mathfrak{m}$-a.e. on $\overline{D}$. Let $E^* = \{\zeta\in\overline{D} : H_\mu(\zeta)=\infty\}$. Define

$$\mathcal{E} \ := \ E\bigcup\widetilde{E}\bigcup E^*.$$

By the discussion just concluded, $\mathfrak{m}(\mathcal{E}) = 0$. Now pick a $\zeta\in\overline{D}\setminus\mathcal{E}$. Define $W(z) := (z-\zeta)\{F(z)-F(\zeta)\}$. By (1.1), Lemma 3.1 is applicable with this choice of W and with $\mathcal{A}=[z,F]_{\overline{D}}$. Thus, there exists a sequence of functions $\{f_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ such that:

$$\lim_{n\to\infty} f_n(z) \ = \frac{1}{z-\zeta} \quad \forall z\in\overline{D}\setminus F^{-1}\{F(\zeta)\}, \tag{3.4}$$

and

$$|f_n(z)| \ \leq \ \frac{4}{|z-\zeta|} \quad \forall z\in\overline{D}\setminus F^{-1}\{F(\zeta)\}, \text{ and } \forall n\in\mathbb{N}. \tag{3.5}$$

Note that since $\zeta \notin (E \cup \widetilde{E})$, $\mu(F^{-1}\{F(\zeta)\}) = 0$. Thus

$$(3.4) \implies f_n(z) \longrightarrow \frac{1}{z-\zeta} \quad \mu\text{-a.e.}$$

Furthermore, as $\zeta \notin E^*$, $H_\mu(\zeta) < \infty$. Thus, in the present context:

$$(3.5) \implies \text{The functions } f_n \text{ are dominated } \mu\text{-a.e. by a function in } \mathbb{L}^1(d|\mu|; \overline{D}).$$

In view of the last two assertions, we may apply the dominated convergence theorem as follows:

$$0 \;=\; \lim_{n\to\infty} \int_{\overline{D}} f_n \, d\mu \;=\; \int_{\overline{D}} \frac{d\mu(z)}{z-\zeta} \qquad (\zeta \in \overline{D} \setminus \mathcal{E}). \tag{3.6}$$

From (3.3) and (3.6), we conclude that $h_\mu(\zeta) = 0 \; \forall \zeta \in \mathbb{C} \setminus \mathcal{E}$. This means that $h_\mu = 0$ $\mathfrak{m}$-a.e., whence $\mu = 0$. Since this is true for any annihilating measure, $[z, F]_{\overline{D}} = \mathcal{C}(\overline{D})$.

$\square$

4. The proof of Theorem 1.2

Before proceeding with the proof of our second theorem, we clarify two pieces of notation that we shall use in the following proof. The expression $\phi \in \mathcal{C}^1(\overline{D(0;\varepsilon)})$ will signify that ϕ has continuous first-order derivatives at all points in some neighbourhood of the closed disc $\overline{D(0;\varepsilon)}$. On the other hand, the expression $\phi \in \mathcal{C}^1(\overline{D(0;\varepsilon)}^*)$ will mean that ϕ has continuous first-order derivatives at all points in some neighbourhood of $\overline{D(0;\varepsilon)}$ *except at* $0 \in \mathbb{C}$.

Next, we define a couple of concepts that will be used in the proof below. Firstly, if K is a compact subset of $\mathbb{C}^n$, the **polynomially convex hull of $\boldsymbol{K}$**, written $\widehat{K}$, is defined by

$$\widehat{K} \;:=\; \{\zeta \in \mathbb{C}^n \mid |P(\zeta)| < \sup_K |P|, \text{ for every holomorphic polynomial } P\}.$$

Given a uniform algebra $\mathcal{A}$, the **maximal ideal space of $\boldsymbol{\mathcal{A}}$** is the space of all unit-norm algebra-homomorphisms of $\mathcal{A}$ to $\mathbb{C}$, viewed as a subset of the dual space $\mathcal{A}^\star$ with the weak* topology (it is a standard fact that every complex homomorphism of $\mathcal{A}$ is in fact continuous). Recall that for a compact subset K, the maximal ideal space of $\mathcal{C}(K)$ is homeomorphically identified with K. We will need this fact in the following proof.

Having established these preliminaries, we are in a position to give

The proof of Theorem 1.2. We begin by introducing a new system of global holomorphic coordinates $(\mathfrak{z}, \mathfrak{w})$ defined by

$$\mathfrak{z} \;:=\; z,$$
$$\mathfrak{w} \;:=\; w - C_0 z^k.$$

Relative to these new coordinates, $\mathcal{S}$ is expressed as

$$\mathcal{S} \;:\quad \mathfrak{w} \;=\; \Sigma(\mathfrak{z}) + G(\mathfrak{z}).$$

For simplicity of notation, we shall denote the new coordinates by (z, w), and work with the following presentation of $\mathcal{S}$:

$$\mathcal{S} \;:\quad w \;=\; \Sigma(z) + G(z), \tag{4.1}$$

where the meanings of Σ and G remain unchanged from those in (1.4). Moreover, the reader may check that neither of the hypotheses (1.5) or (1.6) are affected by this change of coordinate. Let us refer to the right-hand side of (4.1) by $\mathfrak{F}(z)$.

Let $I(\mathcal{S})$ be as in Theorem 1.2, and let $M \in I(\mathcal{S})$ be such that the associated τ_M satisfies the conditions (1.5) and (1.6). Define $\varDelta := 2M - k$. Observe that $\varDelta > 0$. Define the map $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\Psi(z,w) := (z, w^{\varDelta})$. This is a proper, holomorphic, $\varDelta$-to-1 covering map. We now present the first step of this proof.

Step I: *We show that there exists a small constant $\delta > 0$ such that $\Psi^{-1}(\mathcal{S} \cap \{(z,w) : |z| \leq \delta\}) = \bigcup_{j=1}^{\varDelta} \mathfrak{S}_j(\delta)$, where, for $0 < r \leq \delta$, $\mathfrak{S}_j(r)$ represent graphs of the form*

$$\mathfrak{S}_j(r) \ := \ \{(z,w) : w = C_*\omega_j\{|z|^{k/\varDelta}e^{-i\theta} + f(z) + R(z)\},\ |z| \leq r\}, \quad j = 1, \dots, \varDelta, \tag{4.2}$$

and where

- *We write $z := |z|e^{i\theta}$;*
- *$C_* := |C_M|^{1/\varDelta}e^{i\mathrm{Arg}(C_M)/\varDelta}$, and $\omega_j = e^{2\pi i(j-1)/\varDelta}$, i.e. a $\varDelta^{th}$-root of unity;*
- *$f \in \mathcal{C}^1(\overline{D(0;\delta)})$ if $M \neq k$, but $f \in \mathcal{C}^1(\overline{D(0;\delta)^*})$ if $M = k$; and*
- *$R \in \mathcal{C}^1(\overline{D(0;\delta)})$ and $R(z) = O(|z|^{1+(k/\varDelta)})$.*

To see this, we first note that if $\varDelta = 1$ then $f(z) = (\Sigma(z) - C_M z^{k-M}\overline{z}^M)/C_M$ and $R(z) = G(z)/C_M$, and the stated properties of f and G are obvious from our hypotheses. Therefore, we may focus on the case $\varDelta \neq 1$. In this situation, we first write

$$\mathfrak{F}(z) \ := \ C_M z^{k-M}\overline{z}^M\left\{1 + \tau_M(z) + \frac{G(z)}{C_M z^{k-M}\overline{z}^M}\right\}, \quad z \neq 0. \tag{4.3}$$

Observe that

- Owing to the estimate (1.5), $|\tau_M(z)| < 1$; and
- $G(z)/C_M z^{k-M}\overline{z}^M \longrightarrow 0$ uniformly as $z \to 0$.

For these reasons, we can find a small constant $\delta > 0$ such that

$$\left|\tau_M(z) + \frac{G(z)}{C_M z^{k-M}\overline{z}^M}\right| \ < \ 1 \quad \forall z : 0 < |z| \leq \delta.$$

Given this fact, $\mathfrak{F}(z)$ has $\varDelta$ distinct $\varDelta^{th}$-roots — call them $F_j(z)$, $j = 1, \dots, \varDelta$ — when $0 < |z| \leq \delta$, which are obtained by applying the Binomial Theorem with exponent $1/\varDelta$ to the expression in (4.3) that is enclosed in brackets. This results in the expression

$$\begin{aligned} F_j(z) \ &= \ C_*\omega_j\left\{|z|^{k/\varDelta}e^{-i\theta} + \sum_{\nu=1}^{\infty}\alpha_\nu|z|^{k/\varDelta}e^{-i\theta}\tau_M(z)^\nu + R(z)\right\} \\ &\equiv \ C_*\omega_j\{|z|^{k/\varDelta}e^{-i\theta} + f(z) + R(z)\}, \quad \forall z : |z| \leq \delta. \end{aligned}$$

Note that in the above expression, the quantity $|z|^{k/\varDelta}e^{-i\theta}\tau_M(z)$ is interpreted as

$$|z|^{k/\varDelta}e^{-i\theta}\tau_M(z) \ = \ \begin{cases} |z|^{k/\varDelta}e^{-i\theta}\tau_M(z), & \text{if } 0 < |z| \leq \delta, \\ 0, & \text{if } z = 0, \end{cases}$$

and the α_ν's are the coefficients occurring in the Taylor expansion of $(1+x)^{1/\varDelta}$ around $x = 0$.

Note that since the series expansion that produces $F_j(z)$ converges absolutely and uniformly for the specified range of z, rearrangement is permissable; and it is by rearrangement that the quantities $f(z)$ and $R(z)$ are constructed. Furthermore, term-by-term differentiation is possible. But note that

$$|z|^{k/\Delta}e^{-i\theta}\tau_M(z) \text{ is } \begin{cases} \text{continuously differentiable at } z=0 \text{ if } k>\Delta, \\ \text{not differentiable at } z=0 \text{ if } k=\Delta. \end{cases}$$

The degrees of regularity claimed for f and R readily follow from the last two statements. This concludes the first step of our proof.

Step II: *We show that there exists an* $\varepsilon \in (0,\delta]$ *such that* $\mathcal{P}(\mathfrak{S}_j(\varepsilon)) = \mathcal{C}(\mathfrak{S}_j(\varepsilon)),\ j=1,\dots,\Delta$. Note that each $\mathfrak{S}_j(\delta)$ is a complex linear image of the set $\{(z,w) : w = |z|^{k/\Delta}e^{-i\theta} + f(z) + R(z),\ |z| \leq \delta\}$. Therefore, to accomplish this step, it suffices to find an $\varepsilon > 0$ such that $[z, F_0]_{\overline{D(0;\varepsilon)}} = \mathcal{C}(\overline{D(0;\varepsilon)})$, where $F_0(z) := |z|^{k/\Delta}e^{-i\theta} + f(z) + R(z)$. We will use Theorem 1.1 to accomplish this.

We shall first need a few computations. Writing $z = |z|e^{i\theta} = re^{i\theta}$, recall that

$$\frac{\partial}{\partial z} = \frac{e^{-i\theta}}{2}\left\{\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right\}, \qquad \frac{\partial}{\partial\overline{z}} = \frac{e^{i\theta}}{2}\left\{\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right\}.$$

Therefore, we have

$$\begin{aligned} \frac{\partial}{\partial z}(\ |z|^{k/\Delta}e^{-i\theta}) &= \frac{e^{-2i\theta}}{2}\left(\frac{k}{\Delta}-1\right)|z|^{(k/\Delta)-1}, \\ \frac{\partial}{\partial\overline{z}}(\ |z|^{k/\Delta}e^{-i\theta}) &= \frac{1}{2}\left(\frac{k}{\Delta}+1\right)|z|^{(k/\Delta)-1}. \end{aligned} \tag{4.4}$$

Another computation that we will find useful is the following estimate for the quantity

$$\left|\frac{\partial f}{\partial z}(\zeta)\right| + \left|\frac{\partial f}{\partial\overline{z}}(\zeta)\right|, \quad \zeta \neq 0.$$

Using the expressions (4.4), we compute:

$$
\begin{aligned}
\left|\frac{\partial f}{\partial z}(\zeta)\right| &+ \left|\frac{\partial f}{\partial \bar z}(\zeta)\right| \\
&\leq \frac{1}{2}\left(\frac{k}{\varDelta}-1\right)|\zeta|^{(k/\varDelta)-1}\left|\sum_{\nu=1}^{\infty}\alpha_\nu\tau_M(\zeta)^\nu\right| + |\zeta|^{k/\varDelta}\sum_{\nu=1}^{\infty}\nu|\alpha_\nu||\tau_M(\zeta)|^{\nu-1}\left|\frac{\partial\tau_M}{\partial z}(\zeta)\right| \\
&\quad + \frac{1}{2}\left(\frac{k}{\varDelta}+1\right)|\zeta|^{(k/\varDelta)-1}\left|\sum_{\nu=1}^{\infty}\alpha_\nu\tau_M(\zeta)^\nu\right| + |\zeta|^{k/\varDelta}\sum_{\nu=1}^{\infty}\nu|\alpha_\nu||\tau_M(\zeta)|^{\nu-1}\left|\frac{\partial\tau_M}{\partial \bar z}(\zeta)\right| \\
&\leq \frac{k}{\varDelta}|\zeta|^{(k/\varDelta)-1}\sum_{\nu=1}^{\infty}|\alpha_\nu||\tau_M(\zeta)|^\nu + |\zeta|^{k/\varDelta}\|\nabla\tau_M(\zeta)\|\sum_{\nu=1}^{\infty}\nu|\alpha_\nu||\tau_M(\zeta)|^{\nu-1} \\
&= \frac{k}{\varDelta}|\zeta|^{(k/\varDelta)-1}\Big[1-(1-|\tau_M(\zeta)|\,)^{1/\varDelta}\Big] \\
&\qquad + |\zeta|^{k/\varDelta}\|\nabla\tau_M(\zeta)\|\,\frac{d}{dx}\Big[1-(1-x)^{1/\varDelta}\Big]\Big|_{x=|\tau_M(\zeta)|} \qquad (0<|\zeta|\leq\delta)
\end{aligned}
$$

$$
\leq \frac{k}{\varDelta}|\zeta|^{(k/\varDelta)-1}|\tau_M(\zeta)| + \frac{|\zeta|^{k/\varDelta}}{\varDelta}\|\nabla\tau_M(\zeta)\|\,(1-|\tau_M(\zeta)|\,)^{-1} \quad (0<|\zeta|\leq\delta) \tag{4.5}
$$

The last line of the above estimate follows from the fact that as $|\tau_M(\zeta)|<1$ for the relevant range of ζ, we have the inequalities $(1-|\tau_M(\zeta)|\,)<(1-|\tau_M(\zeta)|\,)^{1/\varDelta}<1$.

In the language of Theorem 1.1, define $E:=\{0\}$. The above computations will allow us to determine the distribution of values of the quantity

$$(z-\zeta)\{F_0(z)-F_0(\zeta)\} \quad \zeta\notin E,\ z\notin F_0^{-1}\{F_0(\zeta)\},$$

for each $\zeta\in\overline{D}(0;\varepsilon)\setminus E$ and for all $z\in\overline{D}(0;\varepsilon)\setminus F_0^{-1}\{F_0(\zeta)\}$, for some sufficiently small $\varepsilon\in(0,\delta]$. For this purpose, we:

- *Fix* a $\zeta : 0<|\zeta|\leq\delta$;
- Define $A:=\sup_{|\zeta|=1}|\tau_M(\zeta)|$; and
- Define

$$B := \frac{k}{\varDelta}\sup_{|\zeta|=1}|\tau_M(\zeta)| + \frac{1}{\varDelta}\left\{1-\sup_{|\zeta|=1}|\tau_M(\zeta)|\right\}^{-1}\sup_{|\zeta|=1}|\zeta|\,\|\nabla\tau_M(\zeta)\|.$$

Our task will have to be taken up under two different cases.

Case 1: *EITHER $k/2<M<k$, OR $M=k$ and the line joining z to ζ does not contain the origin in its interior.*

We will explain the reason behind this unusual division into cases in a moment. First, however, we define the real-valued function $\psi_{\zeta,z}:[0,1]\longrightarrow\mathbb{R}$ by

$$\psi_{\zeta,z}(t) := \mathrm{Re}[\,(z-\zeta)F_0(tz+(1-t)\zeta)\,].$$

Note that when $k/2<M<k$, then for $|z|$ sufficiently small, the $\psi_{\zeta,z}$ thus defined would be of class $\mathcal{C}^1$ in a small neighbourhood of the unit-interval. In the present setting, even when $M=k$, $\psi_{\zeta,z}$ is of class $\mathcal{C}^1$ on the open unit-interval. Note that since f is not differentiable at the origin, *the preceding statements would not be true if $M=k$ and the line joining z to*

ζ *contained the origin in its interior.* This explains the necessity of the present division into cases. By this discussion, we see that the Mean Value Theorem is applicable to $\psi_{\zeta,z}$. Thus, there exists a $t_* \in (0,1)$ such that

$$\text{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ = \ \psi_{\zeta,z}(1)-\psi_{\zeta,z}(0) \ = \ \psi'_{\zeta,z}(t_*). \tag{4.6}$$

Define $\xi_* := t_*z + (1-t_*)\zeta$. It *may be possible* that $\xi_* = 0$. In that case

$$\xi_* = 0 \implies \begin{cases} \text{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ = \ 0, \\ (z-\zeta)\{F_0(z)-F_0(\zeta)\} \ \neq \ 0. \end{cases} \tag{4.7}$$

The second statement above follows from the fact that $z \notin F_0^{-1}\{F_0(\zeta)\}$. Now, we consider the case $\xi_* \neq 0$. We use the calculations (4.4) and (4.5) to compute

$$\begin{aligned} \psi'_{\zeta,z}(t_*) \ &= \ \text{Re}\left[\ (z-\zeta)^2\frac{\partial F_0}{\partial z}(\xi_*) + |z-\zeta|^2\frac{\partial F_0}{\partial \overline{z}}(\xi_*)\right] \\ &\geq \ \frac{1}{2}\left(\frac{k}{\varDelta}+1\right)|\xi_*|^{(k/\varDelta)-1}|z-\zeta|^2 - \frac{1}{2}\left(\frac{k}{\varDelta}-1\right)|\xi_*|^{(k/\varDelta)-1}|z-\zeta|^2 \\ &\qquad - \left\{\ \left|\frac{\partial f}{\partial z}(\xi_*)\right| + \left|\frac{\partial f}{\partial \overline{z}}(\xi_*)\right|\ \right\}|z-\zeta|^2 - O(\ |\xi_*|^{k/\varDelta}|z-\zeta|^2) \\ &\geq \ |\xi_*|^{(k/\varDelta)-1}|z-\zeta|^2 - |\xi_*|^{(k/\varDelta)-1}B|z-\zeta|^2 - O(\ |\xi_*|^{k/\varDelta}|z-\zeta|^2) \quad (\xi_* \neq 0) \end{aligned} \tag{4.8}$$

The final inequality above follows from (4.5) and from the definition of B. Note that by the condition (1.6), $B < 1$. Thus, we can find a constant $\varepsilon_1 > 0$ so small that

$$\begin{aligned} &0 < |\zeta| \leq \varepsilon_1,\ z \in \overline{D}(0;\varepsilon_1) \setminus F_0^{-1}\{F_0(\zeta)\} \text{ and } \xi_* \neq 0 \\ &\qquad \implies |\xi_*|^{(k/\varDelta)-1}|z-\zeta|^2 - |\xi_*|^{(k/\varDelta)-1}B|z-\zeta|^2 - O(\ |\xi_*|^{k/\varDelta}|z-\zeta|^2) \ > \ 0. \end{aligned}$$

It is pertinent to note here that *even though* ξ_* itself may vary somewhat unpredictably as (z,ζ) is varied, the manner in which ξ_* enters the estimate (4.8) makes it possible to choose $\varepsilon_1 > 0$ uniformly in the above estimate. Combining this estimate with (4.6) and (4.8), we get

$$\begin{aligned} &0 < |\zeta| \leq \varepsilon_1,\ z \in \overline{D}(0;\varepsilon_1) \setminus F_0^{-1}\{F_0(\zeta)\} \text{ and } \xi_* \neq 0 \\ &\qquad \implies \psi'_{\zeta,z}(t_*) \ = \ \text{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ > \ 0. \end{aligned} \tag{4.9}$$

From (4.7) and (4.9), we conclude that

Under the conditions of Case 1, there exists a constant $\varepsilon_1 \in (0,\delta]$ such that

$$\begin{aligned} \text{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ \in \ \{\xi \in \mathbb{C} : \text{Re}(\xi) \geq 0,\ \xi \neq 0\},\ &z \in \overline{D}(0;\varepsilon_1) \setminus F_0^{-1}\{F_0(\zeta)\}, \\ &\forall \zeta \in \overline{D}(0;\varepsilon_1) \setminus E. \end{aligned} \tag{4.10}$$

Case 2: *$M = k$ and the line joining z to ζ contains the origin in its interior.*

The analysis of this case is an obvious variation of the method used in Step 1; hence we shall be brief. We define two functions ψ_z and $\psi_\zeta : [0,1] \to \mathbb{R}$ by

$$\begin{aligned}\psi_z(t) &:= \mathrm{Re}[\ (z-\zeta)F_0(tz)\ ],\\ \psi_\zeta(t) &:= \mathrm{Re}[\ (z-\zeta)F_0((1-t)\zeta)\ ].\end{aligned}$$

Both functions are continuous on $[0,1]$ and differentiable in $(0,1)$. Thus, by the Mean Value Theorem, there exist $\tau_1,\ \tau_2 \in (0,1)$ such that

$$\psi_z(1)-\psi_z(0) \ = \ \psi_z'(\tau_1), \qquad \psi_\zeta(1)-\psi_\zeta(0) \ = \ \psi_\zeta'(\tau_2).$$

Note that

$$\mathrm{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ = \ [\psi_z(1)-\psi_z(0)] \ + \ [\psi_\zeta(1)-\psi_\zeta(0)] \ = \ \psi_z'(\tau_1)+\psi_\zeta'(\tau_2). \tag{4.11}$$

The next observation is vital to our estimates of $\psi_z'(\tau_1)$ and $\psi_\zeta'(\tau_2)$. Since z and ζ lie on a line through the origin, and on *opposite sides of the origin*,

$$(z-\zeta)\overline{z} \ = \ |z||z-\zeta|, \qquad -\overline{\zeta}(z-\zeta) \ = \ |\zeta||z-\zeta|. \tag{4.12}$$

Now, define $\xi_1 := \tau_1 z$, and $\xi_2 := \tau_2\zeta$. We emulate the calculations leading up to (4.8) above, in conjunction with the relations in (4.12), to get

$$\begin{aligned}\psi_z'(\tau_1) \ &= \ \mathrm{Re}\left[z(z-\zeta)\frac{\partial F_0}{\partial z}(\xi_1)+\overline{z}(z-\zeta)\frac{\partial F_0}{\partial \overline{z}}(\xi_1)\right]\\ &\geq \ |z||z-\zeta| - \left\{\ \left|\frac{\partial f}{\partial z}(\xi_1)\right| + \left|\frac{\partial f}{\partial \overline{z}}(\xi_1)\right|\ \right\}|z||z-\zeta| - O(\ |\xi_1|^{k/\varDelta}|z||z-\zeta|\ )\\ &\geq \ |z||z-\zeta| - B|z||z-\zeta| - O(\ |\xi_1||z||z-\zeta|\ ) \end{aligned} \tag{4.13}$$

An analogous calculation gives

$$\psi_\zeta'(\tau_2) \ \geq \ |\zeta||z-\zeta| - B|\zeta||z-\zeta| - O(\ |\xi_2||\zeta||z-\zeta|\ ). \tag{4.14}$$

Arguing exactly as in Case 1, we can infer from (4.11),, (4.13) and (4.14) that:

Under the conditions of Case 2, there exists a constant $\varepsilon_2 \in (0,\delta]$ such that

$$\begin{aligned}&\mathrm{Re}[\ (z-\zeta)\{F_0(z)-F_0(\zeta)\}\ ] \ \in \ \{\xi\in\mathbb{C} : \mathrm{Re}(\xi)>0\},\ z\in\overline{D}(0;\varepsilon_2)\setminus F_0^{-1}\{F_0(\zeta)\},\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\zeta\in\overline{D}(0;\varepsilon_2)\setminus E.\end{aligned} \tag{4.15}$$

Note that (4.10) and (4.15) verify condition (1.1) of Theorem 1.1 for F_0 with $E=\{0\}$ and $\overline{D}=\overline{D}(0;\min(\varepsilon_1,\varepsilon_2))$. It remains to examine the cardinality of $F_0^{-1}\{F_0(\zeta)\}$ for $\zeta\neq 0$. Notice that by the very same considerations as in the calculation (4.5), we obtain

$$\begin{aligned}
&\left|\frac{\partial F_0}{\partial \overline{z}}(\zeta)\right| - \left|\frac{\partial F_0}{\partial z}(\zeta)\right| \\
&\quad \geq \frac{1}{2}\left(\frac{k}{\varDelta}+1\right)|\zeta|^{(k/\varDelta)-1} - \frac{1}{2}\left(\frac{k}{\varDelta}-1\right)|\zeta|^{(k/\varDelta)-1} \\
&\qquad\qquad - \left\{ \left|\frac{\partial f}{\partial z}(\zeta)\right| + \left|\frac{\partial f}{\partial \overline{z}}(\zeta)\right| \right\} - O(\,|\zeta|^{k/\varDelta}) \\
&\quad \geq |\zeta|^{(k/\varDelta)-1} - |\zeta|^{(k/\varDelta)-1}\left\{\frac{k}{\varDelta}\,|\tau_M(\zeta)| \;+\; \frac{|\zeta|}{\varDelta}\|\nabla\tau_M(\zeta)\|\,(1-|\tau_M(\zeta)|\,)^{-1}\right\} - O(\,|\zeta|^{k/\varDelta}) \\
&\quad \geq |\zeta|^{(k/\varDelta)-1} - B|\zeta|^{(k/\varDelta)-1} - O(\,|\zeta|^{k/\varDelta})
\end{aligned}$$

Once again, as $B < 1$, we can find a constant ε_3 that is so small that

$$\left|\frac{\partial F_0}{\partial \overline{z}}(\zeta)\right| - \left|\frac{\partial F_0}{\partial z}(\zeta)\right| \;\geq\; (1-B)|\zeta|^{(k/\varDelta)-1} - O(\,|\zeta|^{k/\varDelta}) \;>\; 0 \;\forall\zeta : 0 < |\zeta| \leq \varepsilon_3. \tag{4.16}$$

We can view F_0 as a mapping of the disc $\overline{D(0;\delta)}$ in $\mathbb{R}^2$ into $\mathbb{R}^2$. Therefore, we can define the real Jacobian of this mapping (except at $\zeta = 0$ when $M = k$), which we denote by $\mathrm{Jac}_{\mathbb{R}}(F_0)$. The inequality (4.16) tells us that

$$\mathrm{Jac}_{\mathbb{R}}(F_0)(\zeta) \;=\; \left|\frac{\partial F_0}{\partial z}(\zeta)\right|^2 - \left|\frac{\partial F_0}{\partial \overline{z}}(\zeta)\right|^2 \;<\; 0 \quad \forall\zeta : 0 < |\zeta| \leq \varepsilon_3.$$

By the Inverse Function Theorem, we conclude from the above statement that for each $\zeta \in D(0,\varepsilon_3)\setminus\{0\}$, $F_0^{-1}\{F_0(\zeta)\} \cap D(0;\varepsilon_3)$ is a discrete set. Thus, if we define

$$\varepsilon \;:=\; \min\{\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3/2\},$$

then, both the hypotheses of Theorem 1.1 are satisfied for F_0 with $E = \{0\}$ and $\overline{D} = \overline{D(0;\varepsilon)}$. Hence, $[z, F_0]_{\overline{D(0;\varepsilon)}} = \mathcal{C}(\overline{D(0;\varepsilon)})$. Now, note that

$$\mathcal{P}(\mathfrak{S}_j(\varepsilon)) = [z, C_*\omega_j F_0]_{\overline{D(0;\varepsilon)}} = [z, F_0]_{\overline{D(0;\varepsilon)}} = \mathcal{C}(\overline{D(0;\varepsilon)}) = \mathcal{C}(\mathfrak{S}_j(\varepsilon)), \quad j = 1, \dots, \varDelta.$$

Hence, the second step of our proof is accomplished.

The following step, as we shall see, completes the proof of Theorem 1.2.

Step III: *We apply Kallin's Lemma to the conclusions of Step II to prove the desired result.* Consider the polynomial $p(z,w) = zw/C_*$. For any $(z,w) \in \mathfrak{S}_1(\varepsilon)$,

$$\begin{aligned}
\mathrm{Re}\{p(z,w)\} \;&=\; |z|^{(k/\varDelta)+1} + \mathrm{Re}\left\{\,|z|^{(k/\varDelta)+1}\sum_{\nu=1}^{\infty}\alpha_\nu\tau_M(z)^\nu + zR(z)\right\} \\
&\geq\; |z|^{(k/\varDelta)+1} - |z|^{(k/\varDelta)+1}\sum_{\nu=1}^{\infty}|\alpha_\nu||\tau_M(z)|^\nu - O(\,|z|^{(k/\varDelta)+2}) \\
&\geq\; |z|^{(k/\varDelta)+1} - |z|^{(k/\varDelta)+1}\left[1 - (1-|\tau_M(z)|\,)^{1/\varDelta}\right] - O(\,|z|^{(k/\varDelta)+2}) \\
&\geq\; |z|^{(k/\varDelta)+1} - A|z|^{(k/\varDelta)+1} - O(\,|z|^{(k/\varDelta)+2}). \qquad (4.17)
\end{aligned}$$

Similarly, for any $(z,w)\in\mathfrak{S}_1(\varepsilon)$, we estimate

$$\begin{aligned}|\mathrm{Im}\{p(z,w)\}| \ &\leq\ |z|^{(k/\Delta)+1}\left\{\sum_{\nu=1}^{\infty}|\alpha_\nu||\tau_M(z)|^\nu+|R(z)|\right\}.\\ &\leq\ A|z|^{(k/\Delta)+1}+O(\,|z|^{(k/\Delta)+2}).\end{aligned}\tag{4.18}$$

Recall that $A:=\sup_{|\zeta|=1}|\tau_M(\zeta)|$. Let us fix a constant C such that

$$A\ <\ C\ <\frac{\tan(\pi/\Delta)}{1+\tan(\pi/\Delta)}.\tag{4.19}$$

Examining the expressions (4.17) and (4.18), we see that we can, lowering the value of $\varepsilon>0$ if necessary, arrange for

$$\begin{aligned}|\mathrm{Im}\{p(z,w)\}|\ &\leq\ C|z|^{(k/\Delta)+1},\\ \mathrm{Re}\{p(z,w)\}\ &\geq\ (1-C)|z|^{(k/\Delta)+1},\quad \forall(z,w)\in\mathfrak{S}_1(\varepsilon).\end{aligned}$$

Note that lowering the value of $\varepsilon>0$ *does not* alter the conclusion of Step II above. In view of the last inequalities:

$$p(\mathfrak{S}_1(\varepsilon))\ \subsetneq\ \left\{x+iy\in\mathbb{C}:|y|\leq\frac{C}{1-C}x,\ x\geq 0\right\}.$$

The above expression says that $p(\mathfrak{S}_1(\varepsilon))$ is a proper subset of the closed sector W_1 that is centered on the positive x-axis, and has an vertex-angle of

$$2\arctan\left(\frac{C}{1-C}\right)\ <\ 2\arctan\{\tan(\pi/\Delta)\}\ =\ 2\pi/\Delta.$$

The above inequality is a consequence of the condition (4.19) on C. Note that by construction, when $j\neq 1$, $p(\mathfrak{S}_j(\varepsilon))$ is a proper subset of the closed sector W_j, which is simply a copy of W_1 rotated by $(2\pi(j-1)/\Delta)$, $j=2,\dots,\Delta$. We have shown so far that:

- For each $\mathfrak{S}_j(\varepsilon)$, $\mathcal{P}(\mathfrak{S}_j(\varepsilon))=\mathcal{C}(\mathfrak{S}_j(\varepsilon))$, $j=1,\dots,\Delta$;
- $p(\mathfrak{S}_j(\varepsilon))\subsetneq W_j,\ j=1,\dots,\Delta$;
- $W_\mu\cap W_\nu=\{0\}\ \forall\mu\neq\nu$, because the vertex-angle of each W_j is less than $2\pi/\Delta$; and
- $p^{-1}\{0\}\cap\left\{\ \bigcup_{j=1}^{\Delta}\mathfrak{S}_j(\varepsilon)\ \right\}=\{(0,0)\}$.

The above facts allow us to apply Kallin's Lemma repeatedly to show that

$$\mathcal{P}\left(\cup_{j=1}^{\Delta}\mathfrak{S}_j(\varepsilon)\right)\ =\ \mathcal{C}\left(\cup_{j=1}^{\Delta}\mathfrak{S}_j(\varepsilon)\right).\tag{4.20}$$

Now let $\psi\in\mathcal{C}(\{(z,w):|z|\leq\varepsilon\}\cap\mathcal{S})$. Define $\widehat{\psi}:=\psi\circ\Psi:\Psi^{-1}(\{(z,w):|z|\leq\varepsilon\}\cap\mathcal{S})\to\mathbb{C}$. As $\Psi^{-1}(\{(z,w):|z|\leq\varepsilon\}\cap\mathcal{S})=\bigcup_{j=1}^{k}\mathfrak{S}_j(\varepsilon)$, $\widehat{\psi}\in\mathcal{C}\left(\bigcup_{j=1}^{\Delta}\mathfrak{S}_j(\varepsilon)\right)$. We can paraphrase (4.20) in the following way: for each $\epsilon>0$, there exists a polynomial g_ϵ such that

$$|\widehat{\psi}(z,e^{2\pi i(j-1)/\Delta}w)-g_\epsilon(z,e^{2\pi i(j-1)/\Delta}w)|\ <\ \epsilon\quad\forall(z,w)\in\mathfrak{S}_1(\varepsilon),\quad j=1,\dots,\Delta.\tag{4.21}$$

We define

$$Q_\epsilon(z,w)\ :=\ \frac{1}{\Delta}\sum_{j=1}^{\Delta}g_\epsilon(z,e^{2\pi i(j-1)/\Delta}w).$$

Notice that if $g_\epsilon(z,w) = \sum_{0\leq\mu+\nu\leq N} A_{\mu,\nu}z^\mu w^\nu$, then $Q_\epsilon(z,w)$ has the form

$$\begin{aligned} Q_\epsilon(z,w) &= \sum_{(\mu,\nu):\nu=\Delta j} A_{\mu,\Delta j} z^\mu w^{\Delta j} \\ &\equiv P_\epsilon(z,w^\Delta), \end{aligned}$$

where P_ϵ is itself a polynomial. Let us write $w = |w|e^{i\phi}$, $\phi \in [0,2\pi)$. For $(z,w) \in \{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}$, we compute

$$\begin{aligned} |\psi(z,w) - P_\epsilon(z,w)| &= \left| \frac{1}{\Delta}\sum_{j=1}^{\Delta} \widehat{\psi}(z,|w|^{1/\Delta}e^{i(2\pi(j-1)+\phi)/\Delta}) - Q_\epsilon(z,|w|^{1/\Delta}e^{i\phi/\Delta}) \right| \\ &\leq \sum_{j=1}^{\Delta} \frac{|\widehat{\psi}(z,|w|^{1/\Delta}e^{i(2\pi(j-1)+\phi)/\Delta}) - g_\epsilon(z,|w|^{1/\Delta}e^{i(2\pi(j-1)+\phi)/\Delta})|}{\Delta} \;<\; \Delta\left(\frac{\epsilon}{\Delta}\right). \end{aligned}$$

The last inequality follows from the estimate (4.21). This establishes that $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S})$. Equivalently, we have just established (recall that Φ denotes the right-hand side of (1.4)) that $[z,\Phi]_{\overline{D(0;\varepsilon)}} = \mathcal{C}(\overline{D(0;\varepsilon)})$.

We now only need to show that $\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S}$ is polynomially convex. This follows from general abstract considerations. For this purpose, given a compact $K \Subset \mathbb{C}^n$, we define

$$\begin{aligned} \widehat{K} &:= \text{the polynomially convex hull of } K, \\ \mathcal{P}(K;\mathbb{C}^n) &:= \text{the uniform algebra on } K \text{ generated by the class } \{f|_K : f \in \mathbb{C}[z_1,\dots,z_n]\}, \\ \mathcal{M}[\mathcal{P}(K;\mathbb{C}^n)] &:= \text{the maximal ideal space of the uniform algebra } \mathcal{P}(K;\mathbb{C}^n). \end{aligned}$$

It is well known that $\mathcal{M}[\mathcal{P}(K;\mathbb{C}^n)] = \widehat{K}$; see, for instance, [10, Chap.6/§29]. Thus, in our situation, $\mathcal{M}[\mathcal{P}(\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S};\mathbb{C}^2)] = \widehat{\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}}$. But since $\mathcal{P}(\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}) = \mathcal{C}(\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S})$,

$$\widehat{\{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}} \;=\; \mathcal{M}[\mathcal{C}(\{(z,w) : |z| \leq \varepsilon_0\} \cap \mathcal{S})] \;=\; \{(z,w) : |z| \leq \varepsilon\} \cap \mathcal{S}.$$

This concludes our proof.

$\square$

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