THE DYNAMICS OF HOLOMORPHIC CORRESPONDENCES OF $\mathbb{P}^1$: INVARIANT MEASURES AND THE NORMALITY SET

GAUTAM BHARALI AND SHRIHARI SRIDHARAN

Abstract. This paper is motivated by Brolin's theorem. The phenomenon we wish to demonstrate is as follows: if F is a holomorphic correspondence on $\mathbb{P}^1$, then (under certain conditions) F admits a measure μ_F such that, for any point z drawn from a "large" open subset of $\mathbb{P}^1$, μ_F is the weak*-limit of the normalised sums of point masses carried by the pre-images of z under the iterates of F. Let ${}^{\dagger}F$ denote the transpose of F. Under the condition $d_{top}(F) > d_{top}({}^{\dagger}F)$, where d_{top} denotes the topological degree, the above phenomenon was established by Dinh and Sibony. We show that the support of this μ_F is disjoint from the normality set of F. There are many interesting correspondences on $\mathbb{P}^1$ for which $d_{top}(F) \leq d_{top}({}^{\dagger}F)$. Examples are the correspondences introduced by Bullett and collaborators. When $d_{top}(F) \leq d_{top}({}^{\dagger}F)$, equidistribution *cannot* be expected to the full extent of Brolin's theorem. However, we prove that when F admits a repeller, equidistribution in the above sense holds true.

1. Introduction

The dynamics studied in this paper owes its origin to a work of Bullett [2] and to a series of articles motivated by [2] — most notably [6, 5, 7, 4]. The object of study in [2] is the dynamical system that arises on iterating a certain relation on $\mathbb{C}$. This relation is the zero set of a polynomial $g \in \mathbb{C}[z_1, z_2]$ of a certain form such that:

- $g(\cdot, z_2)$ and $g(z_1, \cdot)$ are generically quadratic; and
- if V_g denotes the biprojective completion of $\{g = 0\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, then no irreducible component of V_g is of the form $\{a\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{a\}$, where $a \in \mathbb{P}^1$.

In [7], this set-up was extended to polynomials $g \in \mathbb{C}[z_1, z_2]$ of arbitrary degree that induce relations $V_g \subset \mathbb{P}^1 \times \mathbb{P}^1$ with similar properties. Since relations can be composed, it would be interesting to know whether the iterated composition of such a relation exhibits equidistribution properties in analogy to Brolin's Theorem [1, Theorem 16.1].

The reader will be aware of recent results by Dinh and Sibony [11] that, it would seem, should immediately solve the above problem. *However, key assumptions in the theorems of* [11] *fail to hold for many interesting correspondences on* $\mathbb{P}^1$. We shall discuss what this assertion means in the remainder of this section.

On the dynamics of multivalued maps between complex manifolds: results of perhaps the broadest scope are established in [11]. We borrow from [11] the following definition.

Definition 1.1. Let X_1 and X_2 be two compact complex manifolds of dimension k. We say that Γ is a *holomorphic k-chain* in $X_1 \times X_2$ if Γ is a formal linear combination of the

1991 *Mathematics Subject Classification.* Primary 37F05, 37F10; Secondary 30G30.
Key words and phrases. Holomorphic correspondence, invariant measure, normality set.
The first author is supported in part by a UGC Centre for Advanced Study grant.

form

$$\mathit{\Gamma} = \sum_{j=1}^{N} m_j \Gamma_j, \tag{1.1}$$

where the m_j's are positive integers and $\Gamma_1, \dots, \Gamma_N$ are distinct irreducible complex subvarieties of $X_1 \times X_2$ of pure dimension k. Let π_i denote the projection onto X_i, $i = 1, 2$. We say that the holomorphic k-chain $\mathit{\Gamma}$ *determines a meromorphic correspondence of* X_1 *onto* X_2 if, for each Γ_j in (1.1), $\pi_1|_{\Gamma_j}$ and $\pi_2|_{\Gamma_j}$ are surjective. $\mathit{\Gamma}$ determines a set-valued map, which we denote by $F_{\mathit{\Gamma}}$, as follows:

$$X_1 \ni x \longmapsto \bigcup_{j=1}^{N} \pi_2\left(\pi_1^{-1}\{x\} \cap \Gamma_j\right).$$

We call $F_{\mathit{\Gamma}}$ a *holomorphic correspondence* if $F_{\mathit{\Gamma}}(x)$ is a finite set for every $x \in X_1$.

Remark 1.2. It is helpful to encode holomorphic correspondences as holomorphic chains. Circumstances arise where, in the notation of (1.1), $m_j \geq 2$. For instance: even if we start with a holomorphic correspondence on $\mathbb{P}^1$ determined by an *irreducible* variety $V \subset \mathbb{P}^1 \times \mathbb{P}^1$, composing V with itself (see Section 2) can result in a variety that is not irreducible and some of whose irreducible components occur with multiplicity ≥ 2.

Suppose (X, ω) is a compact Kähler manifold of dimension k (ω denoting the normalised Kähler form) and F is a meromorphic correspondence of X onto itself. One of the results in [11] says, roughly, that if $d_{k-1}(F) < d_k(F)$, where $d_{k-1}(F)$ and $d_k(F)$ are the dynamical degrees of F of order $(k-1)$ and k respectively (see [11, §3.5] for a definition of dynamical degree), then there exists a probability measure μ_F satisfying $F^*(\mu_F) = d_k(F)\mu_F$, such that

$$\frac{1}{d_k(F)^n}(F^n)^*(\omega^k) \xrightarrow{\text{weak}^*} \mu_F \text{ as measures, as } n \to \infty. \tag{1.2}$$

When $\dim_{\mathbb{C}}(X) = 1$, the assumption $d_{k-1}(F) < d_k(F)$ translates into the assumption that the (generic) number of pre-images under F is *strictly larger* than the number of images under F, both counted according to multiplicity. None of the techniques in the current literature are of help in studying correspondences F for which $d_{k-1}(F) \geq d_k(F)$, even when $(X, \omega) = (\mathbb{P}^k, \omega_{FS})$ (in this paper ω_{FS} will denote the Fubini–Study form).

Why should one be interested in the dynamics of a correspondence $F : X \to X$ for which $d_{k-1}(F) \geq d_k(F)$? The work of Bullett and collaborators suggest several reasons in the case $(X, \omega) = (\mathbb{P}^1, \omega_{FS})$. Thus, *we shall focus on correspondences on* $\mathbb{P}^1$ (although parts of our results hold true for Riemann surfaces). A mating of two monic polynomials on $\mathbb{C}$ is a construction by Douady [12] that, given two monic polynomials $f, g \in \mathbb{C}[z]$ of the same degree, produces a continuous branched covering $(f \amalg g)$ of a topological sphere to itself whose dynamics emulates that of f or of g on separate hemispheres. For certain natural choices of pairs (f, g), one can determine in principle — see [14, Theorem 2.1] — when $(f \amalg g)$ is semiconjugate to a rational map on $\mathbb{P}^1$. In a series of papers [6, 3, 5, 4], Bullett and collaborators extend this idea to matings between polynomial maps and certain discrete subgroups of the Möbius group or certain Hecke groups. The holomorphic objects whose dynamics turn out to be conjugate to that of matings in this new sense are holomorphic correspondences on $\mathbb{P}^1$. Such correspondences are interesting because they expose further the parallels between the dynamics of Kleinian groups and of rational maps. It would be interesting to devise an ergodic theory for such matings. *In all* *known constructions where a holomorphic correspondence F of $\mathbb{P}^1$ models the dynamics of a mating of some polynomial with some group, $d_0(F) = d_1(F)$.* In this context, to produce an invariant measure — and, especially, to give an explicit prescription for it — would require that the techniques in [11] be supplemented by other ideas.

We now give an informal description of our work (rigorous statements are given in Section 3). Since we mentioned Brolin's theorem, we ought to mention that an analogue of Brolin's theorem follows from (1.2) and certain other results in [11] when $d_0(F) < d_1(F)$. To be more precise: there exists a polar set $\mathcal{E} \subsetneq \mathbb{P}^1$ such that

$$\frac{1}{d_1(F)^n}(F^n)^*(\delta_x) \xrightarrow{\text{weak}^*} \mu_F \text{ as measures, as } n \to \infty, \ \forall x \in \mathbb{P}^1 \setminus \mathcal{E}, \tag{1.3}$$

where μ_F is as in (1.2) with $(X, \omega) = (\mathbb{P}^1, \omega_{FS})$. This means that we have extremely precise information about the measure μ_F. Our first theorem (Theorem 3.2) uses this information to show that the support of μ_F is disjoint from the normality set of F, where "normality set" is the analogue of the Fatou set in the context of correspondences.

When F (a holomorphic correspondence on $\mathbb{P}^1$) satisfies $d_0(F) \geq d_1(F)$, there is no reason to expect (1.3). Indeed, consider these examples: $F_1(z) := 1/z$, in which case $d_0(F_1) = d_1(F_1) = 1$; or the holomorphic correspondence F_2 determined by the $\mathbb{P}^1 \times \mathbb{P}^1$−completion of the zero set of the rational function $g(z_1, z_2) = z_2^2 - (1/z_1^2)$, in which case $d_0(F_2) = d_1(F_2) = 2$. When $d_0(F) \geq d_1(F)$, we draw upon certain ideas of McGehee [16]. We show that if F admits a repeller $\mathcal{R} \subset \mathbb{P}^1$ — in the sense of McGehee, which extends the concept of a repeller known for maps — having certain properties, then there exists a neighbourhood $U(F, \mathcal{R}) \supset \mathcal{R}$ and a probability measure μ_F satisfying $F^*(\mu_F) = d_1(F)\mu_F$, such that

$$\frac{1}{d_1(F)^n}(F^n)^*(\delta_x) \xrightarrow{\text{weak}^*} \mu_F \text{ as measures, as } n \to \infty, \ \forall x \in U(F, \mathcal{R}). \tag{1.4}$$

A rigorous statement of this is given by Theorem 3.5. The condition that F admit a repeller is very natural, and was motivated by the various examples constructed by Bullett *et al.* We take up one class of these examples in Section 7 and show that the conditions stated in Theorem 3.5 hold true for this class. Observe that (1.4) differs from (1.3) in that it does not state that $\mathbb{P}^1 \setminus U(F, \mathcal{R})$ is polar (or even nowhere dense), but this is the best one can expect (see Remark 3.6 below).

The measure μ_F in (1.3) and (1.4) is not, in general, invariant under F in the measure-theoretic sense. It is merely F^*-invariant; see Section 3 for details. What is interesting to find is the phenomenon of equidistribution, which arises in so many situations (see, e.g., work of Clozel, Oh and Ullmo [8], which involves correspondences in a different context). Yet, for a holomorphic correspondence F on $\mathbb{P}^1$ with $d_1(F) \leq d_0(F)$, we can show that there exists a measure that *is* invariant under F in the usual sense; see Corollary 3.7.

Acknowledgements. A part of this work was carried out during a visit by Shrihari Sridharan to the Indian Institute of Science in 2012. He thanks the Department of Mathematics, Indian Institute of Science, for its support and hospitality. Gautam Bharali thanks Shaun Bullett for his very helpful answers to several questions pertaining to the example in Section 7.2. The authors thank the anonymous referee of an earlier version of this article for useful suggestions for improving the exposition herein.

2. Fundamental definitions

In this section, we isolate certain essential definitions that are somewhat long. Readers who are familiar with the rule for composing holomorphic correspondences can proceed to Section 2.2, for a definition of the normality set of a holomorphic correspondence.

2.1. **The composition of two holomorphic correspondences.** Let X be a complex manifold of dimension k. For any holomorphic k-chain $\mathit{\Gamma}$ on X, we define the *support* of $\mathit{\Gamma}$, assuming the representation (1.1), by $|\mathit{\Gamma}| := \cup_{j=1}^{N}\Gamma_j$. Consider two holomorphic correspondences, determined by the k-chains

$$\mathit{\Gamma}^1 = \sum_{j=1}^{N_1} m_{1,j}\Gamma_{1,j}, \qquad \mathit{\Gamma}^2 = \sum_{j=1}^{N_2} m_{2,j}\Gamma_{2,j},$$

in $X \times X$. The k-chains $\mathit{\Gamma}_1$, $\mathit{\Gamma}_2$ have the alternative representations

$$\mathit{\Gamma}^1 = {\sum}'_{1\leq j\leq L_1}\Gamma^{\bullet}_{1,j}, \qquad \mathit{\Gamma}^2 = {\sum}'_{1\leq j\leq L_2}\Gamma^{\bullet}_{2,j}, \tag{2.1}$$

where the primed sums indicate that the irreducible subvarieties $\Gamma^{\bullet}_{s,j}$, $j = 1,\dots,L_s$, are *not necessarily distinct* and are repeated according to the coefficients $m_{s,j}$. Before we give the definition of $\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1$, observe that we may view $\mathit{\Gamma}^1$ and $\mathit{\Gamma}^2$ as *relations* $|\mathit{\Gamma}^1|$ and $|\mathit{\Gamma}^2|$ on X. The composition-rule for relations is classical. Denoting the composition of $|\mathit{\Gamma}^1|$ and $|\mathit{\Gamma}^2|$ in the classical sense by $|\mathit{\Gamma}^2| \star |\mathit{\Gamma}^1|$, we recall:

$$|\mathit{\Gamma}^2| \star |\mathit{\Gamma}^1| := \{(x,z) \in X \times X : \exists y \in X \text{ s.t. } (x,y) \in |\mathit{\Gamma}^1|,\ (y,z) \in |\mathit{\Gamma}^2|\}. \tag{2.2}$$

This is the view we take in Section 6, where we need to make use of McGehee's results from [16] on the dynamics of closed relations on compact spaces.

The object $\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1$ is, in essence, just the composition of two relations together with data that allows one to count forward and backward images of points "according to multiplicity". To begin, we define the k-chain $\Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j}$ by the conditions:

$$|\Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j}| := \{(x,z) \in X \times X : \exists y \in X \text{ s.t. } (x,y) \in \Gamma^{\bullet}_{1,j},\ (y,z) \in \Gamma^{\bullet}_{2,l}\}, \tag{2.3}$$

$$\Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j} \equiv \sum_{1\leq s\leq N(j,l)} \nu_{s,jl} Y_{s,jl},$$

where the $Y_{s,jl}$'s are the distinct irreducible components of the subvariety on the right-hand side of (2.3), and $\nu_{s,jl} \in \mathbb{Z}_+$ is the generic number y's as (x,z) varies through $Y_{s,jl}$ for which the membership conditions on the right-hand side of (2.3) are satisfied. Finally, we define the k-chain

$$\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1 := \sum_{j=1}^{L_1}\sum_{l=1}^{L_2} \Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j}. \tag{2.4}$$

If $\mathit{\Gamma}^1$ and $\mathit{\Gamma}^2$ determine holomorphic correspondences on X, then so does $\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1$. It requires some amount of intersection theory to show this, but see Section 4 for an *elementary* proof when $X = \mathbb{P}^1$. The n-fold iterate of $\mathit{\Gamma}$ will be denoted by $\mathit{\Gamma}^{\circ n}$.

We adopt a notational simplification. Given a k-chain $\mathit{\Gamma}$ that determines a holomorphic correspondence and there is no scope for confusion, we shall denote $F_{\mathit{\Gamma}}$ by F. We conclude with a simple observation: for holomorphic correspondences

$$|\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1| = |\mathit{\Gamma}^2| \star |\mathit{\Gamma}^1|. \tag{2.5}$$

2.2. **The normality set of a holomorphic correspondence on a Riemann surface.** Let F be a holomorphic correspondence on a compact Riemann surface X. The essential concept of the normality set of F is not difficult. But we will need formalism that enables good book-keeping. We will use the representation (2.1) for a holomorphic correspondence Γ. The set of integers $\{m, m+1, \dots, n\}$ will be denoted by $[m\,..\,n]$.

Given $N \in \mathbb{Z}_+$, we say that $(z_0, \dots, z_N;\, \alpha_1, \dots, \alpha_N) \in X^{N+1} \times [1\,..\,L]^N$ is *a path of an iteration of F starting at z_0, of length N*, or simply an *N-path starting at z_0*, if

$$(z_{j-1}, z_j) \in \Gamma^\bullet_{\alpha_j}, \;\; j = 1, \dots, N.$$

Next, given any two irreducible subvarieties $\Gamma^\bullet_j$ and $\Gamma^\bullet_k$ in the decomposition of Γ in the sense of (2.1), we define

$$\Gamma^\bullet_{(j,k)} \;:=\; \{(x, y, z) \in X^3 : (x, y) \in \Gamma^\bullet_j,\; (y, z) \in \Gamma^\bullet_k\}.$$

This construction can be extended to any multi-index $\boldsymbol{\alpha} \in [1\,..\,L]^j$:

$$\Gamma^\bullet_{\boldsymbol{\alpha}} \;:=\; \{(x_0, \dots, x_j) \in X^{j+1} : (x_{i-1}, x_i) \in \Gamma^\bullet_{\alpha_i},\; 1 \leq i \leq j\}. \tag{2.6}$$

In all discussions on the normality set of F, we shall work with only those N-paths $(z_0, \dots, z_N;\, \alpha_1, \dots, \alpha_N)$ that satisfy

- $(*)$ For each $j = 1, \dots, N$, $\Gamma^\bullet_{(\alpha_1,\dots,\alpha_j)} \cap B_j$ is an irreducible complex-analytic subvariety of B_j for every sufficiently small open ball $B_j \ni (z_0, \dots, z_j)$.

An N-path will be called an *admissible N-path* if it satisfies $(*)$. Fix $z_0 \in X$ and set

$$\mathscr{P}_N(z_0) \;:=\; \text{the set of all paths of iterations of } F\text{, of length } N\text{, starting at } z_0,$$

$N \in \mathbb{Z}_+$. We will denote an element of $\mathscr{P}_N(z_0)$ by $\boldsymbol{\mathcal{Z}}$. Observe that if $\boldsymbol{\mathcal{Z}}$ is an admissible N-path, $N \geq 2$, then there is a *unique* irreducible component of $\Gamma^\bullet_{(\alpha_1,\dots,\alpha_j)}$ to which $(z_0, \dots, z_j)$ belongs, $j = 2, \dots, N$. Hence, if $\boldsymbol{\mathcal{Z}}$ is an admissible N-path, let us write

$$\Gamma^\bullet_{\boldsymbol{\mathcal{Z}},j} \;:=\; \begin{cases} \Gamma^\bullet_{\alpha_1}, & \text{if } j = 1, \\ \text{the unique irreducible component of } \Gamma^\bullet_{(\alpha_1,\dots,\alpha_j)} \text{ containing } (z_0, \dots, z_j), & \text{if } j \geq 2. \end{cases}$$

Let $\nu_{(\boldsymbol{\mathcal{Z}},j)} : Y_{\boldsymbol{\mathcal{Z}},j} \longrightarrow \Gamma^\bullet_{\boldsymbol{\mathcal{Z}},j}$, where $Y_{\boldsymbol{\mathcal{Z}},j}$ is a compact Riemann surface, denote the desingularization of $\Gamma^\bullet_{\boldsymbol{\mathcal{Z}},j}$. We now have the essential notations needed to define the normality set. The definitions that follow are strongly influenced by the notion introduced by Bullett and Penrose [7]. Yet, what we call a "branch of an iteration" will look vastly different from its namesake in [7]. This is because, for our purposes, we will need to label (resp., track) all the maps involved by (resp., along) the various paths that comprise $\mathscr{P}_N(z_0)$. I.e., the difference is (largely) in formalism. The one alteration that we make to the Bullett–Penrose definition is in working with only admissible paths: this allows us to parametrise holomorphic branches of the iteration of F along $\boldsymbol{\mathcal{Z}}$ — which we shall presently define — simply by $\boldsymbol{\mathcal{Z}}$ itself (see Remark 5.3 as well).

Let proj_k and π^j_k denote the following projections:

$$\mathsf{proj}_k : X^{k+1} \longrightarrow X^k, \;\; \mathsf{proj}_k : (x_0, \dots, x_k) \longmapsto (x_0, \dots, x_{k-1}),$$
$$\pi^j_k : X^{k+1} \longrightarrow X, \;\; \pi^j_k : (x_0, \dots, x_k) \longmapsto x_j, \;\; 0 \leq j \leq k,$$

where $k \in \mathbb{Z}_+$. The idea of a holomorphic branch of an iteration of F along $\boldsymbol{\mathcal{Z}}$, $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_N(z_0)$ and admissible, is to assign to a suitable open neighbourhood $U \ni z_0$ a finite sequence

$(U(\boldsymbol{\mathcal{Z}},j))_{1\leq j\leq N}$ of $\Gamma^{\bullet}_{\boldsymbol{\mathcal{Z}},j}$-open sets. In the case $j=1$, the natural choice of $U(\boldsymbol{\mathcal{Z}},1)$ is the irreducible component of $\Gamma^{\bullet}_{\boldsymbol{\mathcal{Z}},1}\cap(U\times X)$ containing (z_0,z_1). Proceed inductively and set:

$$\begin{aligned} U(\boldsymbol{\mathcal{Z}},j) \ &= \ \text{the irreducible component of } \Omega(\boldsymbol{\mathcal{Z}},j)\cap\Gamma^{\bullet}_{\boldsymbol{\mathcal{Z}},j} \text{ containing } (z_0,\dots,z_j), \text{ and} \\ \Omega(\boldsymbol{\mathcal{Z}},j) \ &:= \ \begin{cases} U\times X, & \text{if } j=1, \\ \Big[U\times\prod_{k=1}^{j-1}\pi^k_k(U(\boldsymbol{\mathcal{Z}},k))\Big]\times X, & \text{if } j\geq 2. \end{cases} \end{aligned}$$

The phrase "irreducible component" above signifies that $\Omega(\boldsymbol{\mathcal{Z}},j)\cap\Gamma^{\bullet}_{\boldsymbol{\mathcal{Z}},j}$ be viewed as a complex-analytic subvariety of $\Omega(\boldsymbol{\mathcal{Z}},j)$. Now, if each $U(\boldsymbol{\mathcal{Z}},j)$ were obtainable as the graph of some holomorphic map $\Psi_j : U\longrightarrow\mathbb{C}^j$, then the sequence $(\Psi_j)_{1\leq j\leq N}$ would be a natural definition of a *holomorphic* branch along $\boldsymbol{\mathcal{Z}}$; see Remark 2.1. However, if some $U(\boldsymbol{\mathcal{Z}},j)$ has a singularity at $(z_0,\dots,z_j)$, or does not project injectively under π^0_j, then the preceding idea does not work. Therefore, we modify this idea by substituting the aforementioned $(\Psi_j)_{1\leq j\leq N}$ by a sequence of natural parametrisations of the $U(\boldsymbol{\mathcal{Z}},j)$'s.

To this end, fix a $z_0\in X$ and an *admissible* path $\boldsymbol{\mathcal{Z}}\in\mathscr{P}_N(z_0)$. Each $U(\boldsymbol{\mathcal{Z}},j)$ is parametrised via $\nu_{(\boldsymbol{\mathcal{Z}},j)}$ by an open patch in the Riemann surface $Y_{\boldsymbol{\mathcal{Z}},j}$. We say that *there exists a holomorphic branch of an iteration of F on U along $\boldsymbol{\mathcal{Z}}$*, which we denote by $(\psi_{(\boldsymbol{\mathcal{Z}},1)},\dots,\psi_{(\boldsymbol{\mathcal{Z}},N)};U)$, if there exist a connected neighbourhood U of z_0 and holomorphic mappings $\psi_{(\boldsymbol{\mathcal{Z}},j)}:D\longrightarrow Y_{(\boldsymbol{\mathcal{Z}},j)}$ of a planar domain D containing 0 that, for each $j=1,\dots,N$, satisfy three conditions:

1) $\nu_{(\boldsymbol{\mathcal{Z}},j)}\circ\psi_{(\boldsymbol{\mathcal{Z}},j)}(0)=(z_0,\dots,z_j)$.
2) (Compatibility condition) $\mathsf{proj}_j\circ\nu_{(\boldsymbol{\mathcal{Z}},j)}\circ\psi_{(\boldsymbol{\mathcal{Z}},j)}=\nu_{(\boldsymbol{\mathcal{Z}},j-1)}\circ\psi_{(\boldsymbol{\mathcal{Z}},j-1)}$, $j\neq 1$.

A part of our third condition will encode the requirement that $U(\boldsymbol{\mathcal{Z}},1),\dots,U(\boldsymbol{\mathcal{Z}},N)$ contain no singularities at which they are locally reducible (i.e., as analytic germs). (Loosely speaking, this ensures that any point $z_*\neq z_0$ sufficiently close to z_0 will have a holomorphic branch of an iteration of F along *some* admissible path in $\mathscr{P}_N(z_*)$ that is "sufficiently close" to $(\psi_{(\boldsymbol{\mathcal{Z}},1)},\dots,\psi_{(\boldsymbol{\mathcal{Z}},N)})$ — see Remark 2.3.) More precisely:

3) $\psi_{(\boldsymbol{\mathcal{Z}},j)}$ is a finite-sheeted (perhaps branched) covering map onto $\nu^{-1}_{(\boldsymbol{\mathcal{Z}},j)}(U(\boldsymbol{\mathcal{Z}},j))$, and $\nu_{(\boldsymbol{\mathcal{Z}},j)}$ maps the latter set **homeomorphically onto** $U(\boldsymbol{\mathcal{Z}},j)$.

Remark 2.1. Note that if F is a non-constant rational *map* on $\mathbb{P}^1$, then the maps

$$\psi_{(\boldsymbol{\mathcal{Z}},j)} \ := \ \big(\mathsf{id}_D+z_0,\, F(\cdot+z_0)|_D\,,\dots,\, F^{j-1}\circ F(\cdot+z_0)\big|_D\big)\,, \quad j=1,2,3,\dots,$$

where D is a small disc around 0, satisfy all the conditions above (taking each $Y_{(\boldsymbol{\mathcal{Z}},j)}$ to be the graph of the appropriate $(\mathbb{P}^1)^j$-valued map).

Having defined holomorphic branches, we can give the following definition.

Definition 2.2. Let X be a compact Riemann surface and let F be a holomorphic correspondence on X. A point $z_0\in X$ is said to belong to the *normality set of F*, denoted by $\mathcal{N}(F)$, if there exists a connected neighbourhood U of z_0 and a **single** planar domain D containing 0, which depends on z_0, such that

1) For each $n\in\mathbb{Z}_+$ and each $\boldsymbol{\mathcal{Z}}\in\mathscr{P}_n(z_0)$, there exists a holomorphic branch $(\psi_{(\boldsymbol{\mathcal{Z}},1)},\dots,\psi_{(\boldsymbol{\mathcal{Z}},n)};U)$ of an iteration of F along $\boldsymbol{\mathcal{Z}}$ with $\mathsf{Dom}(\psi_{(\boldsymbol{\mathcal{Z}},j)})=D$ for every $(\boldsymbol{\mathcal{Z}},\,j)$.
2) The family

$$\mathscr{F}(z_0) \ := \ \big\{\pi^n_n\circ\nu_{(\boldsymbol{\mathcal{Z}},n)}\circ\psi_{(\boldsymbol{\mathcal{Z}},n)} : n\in\mathbb{Z}_+,\ \boldsymbol{\mathcal{Z}}\in\mathscr{P}_n(z_0),\ \&\ (\psi_{(\boldsymbol{\mathcal{Z}},1)},\dots,\psi_{(\boldsymbol{\mathcal{Z}},n)};U)$$

is a holomorphic branch of an iteration of F along $\boldsymbol{\mathcal{Z}}\}$

is a normal family on D.

Remark 2.3. The set $\mathcal{N}(F)$ is open, although it is not necessarily non-empty. If $z_0 \in \mathcal{N}(F)$ and U is the neighbourhood of z_0 as given by Definition 2.2, then it is routine to show that $U \subset \mathcal{N}(F)$.

3. More definitions and statement of results

We need to present some constructions before we can state our first result.

Given a holomorphic correspondence on X, $\dim_{\mathbb{C}}(X) = k$, determined by a holomorphic k-chain Γ, its *adjoint correspondence* is the meromorphic correspondence determined by the k-chain (assuming the representation (1.1) for Γ)

$$^{\dagger}\Gamma \ := \ \sum_{j=1}^{N} m_j {}^{\dagger}\Gamma_j,$$

where ${}^{\dagger}\Gamma_j := \{(y,x) \in X \times X : (x,y) \in \Gamma_j\}$. In general, ${}^{\dagger}\Gamma$ may not determine a holomorphic correspondence. However, when $\dim_{\mathbb{C}}(X) = 1$, it is easy to see that *any meromorphic correspondence of X is automatically holomorphic*. Thus, if F_{Γ} is a holomorphic correspondence on $\mathbb{P}^1$, then so is $F_{{}^{\dagger}\Gamma}$. In the abbreviated notation introduced in Section 2, we shall henceforth write:

$$F^n \ := \ F_{\Gamma^{\circ n}}, \qquad {}^{\dagger}F \ := \ F_{{}^{\dagger}\Gamma}.$$

Given a holomorphic k-chain Γ on $X \times X$, Γ detemines a current of bidimension (k,k) via the currents of integration given by its constituent subvarieties Γ_j. We denote this current by $[\Gamma]$. If F is determined by Γ, we *formally* define the action of F on currents S on X of bidegree (p,p), $0 \leq p \leq k$, by the prescription:

$$F^*(S) \ := \ (\pi_1)_* \left(\pi_2^*(S) \wedge [\Gamma]\right), \tag{3.1}$$

where, as usual, π_j denotes the projection of $X \times X$ onto the jth factor. This prescription will make sense for those currents for which the pullback by π_2 makes sense and the intersection of this new current with $[\Gamma]$ also makes sense. That this is the case is easy to see when S is a smooth (p,p) form (hence a current of bidegree (p,p) on X). The reader is referred to [11, Section 2.4] for details.

A finite Borel measure μ on X can be viewed as a current of bidegree (k,k). Let us work out $F^*(\mu)$ for a specific example that is central to this paper. Let $x \in X$ and let δ_x be the Dirac mass at x. The prescription (3.1) is interpreted as

$$\begin{aligned}
\langle F^*(\delta_x), \varphi\rangle \equiv_{\text{by duality}} \langle \pi_2^*(\delta_x) \wedge [\Gamma], \pi_1^*\varphi\rangle \ &:= \ \sum_{j=1}^{N} m_j \langle (\left.\pi_2\right|_{\Gamma_j})^*(\delta_x), \pi_1^*\varphi\rangle \\
&= \ \sum_{j=1}^{N} m_j \langle \delta_x, \left(\left.\pi_2\right|_{\Gamma_j}\right)_*(\pi_1^*\varphi)\rangle,
\end{aligned} \tag{3.2}$$

where each summand in the last expression is just the way one defines the pullback of a current under a holomorphic mapping (in this case, $\left.\pi_2\right|_{\Gamma_j}$, $j = 1, \dots, N$) of an analytic space that is submersive on a Zariski open subset. If $\Omega \subset X$ is a Zariski-open subset of X such that $(\pi_2^{-1}(\Omega) \cap \Gamma_j, \Omega, \pi_2)$ is a covering space for each $j = 1, \dots, N$, then, for

$x \in \Omega$, $(\pi_2|_{\Gamma_j})_*(f)(x)$ is just the sum of the values of f on the fibre $\pi_2^{-1}\{x\} \cap \Gamma_j$ for any $f \in \mathcal{C}^\infty(X \times X)$. Thus, when $x \in \Omega$, (3.2) equals the quantity

$$\sum_{j=1}^{N} m_j \sum_{\zeta:(\zeta,x)\in\Gamma_j} \varphi(\zeta) \ =: \ \mathbf{\Lambda}[\varphi](x) \qquad x \in \Omega. \tag{3.3}$$

For any fixed *continuous* function φ, $\mathbf{\Lambda}[\varphi]$ extends continuously to each $x \in X \setminus \Omega$. We shall denote this continuous extension of the left-hand side of (3.3) also as $\mathbf{\Lambda}[\varphi]$. In other words, $F^*(\delta_x)$ can be defined as a measure supported on the set ${}^\dagger F(x)$, and

$$\langle F^*(\delta_x), \varphi\rangle \ = \ \mathbf{\Lambda}[\varphi](x) \ \ \forall x \in X, \ \ \forall \varphi \in \mathcal{C}(X). \tag{3.4}$$

The arguments preceding (3.3) continue to be valid if, in (3.2), δ_x is replaced by μ, a finite, positive Borel measure on X.

The push-forward of a current S by F is defined by the equation $F_*(S) := ({}^\dagger F)^*(S)$ whenever the latter makes sense.

We define two numbers that are essential to the statement of our theorems. With F as above, let $d_1(F)$ denote the generic number of preimages under F of a point in $\mathbb{P}^1$, counted according to multiplicity. More rigorously, this means — Ω being any Zariski-open set of the type described prior to the equation (3.3) — that

$$d_1(F) \ = \ \sum_{j=1}^{N} m_j \mathsf{Card}\{x : (x,y) \in \Gamma_j\}, \quad y \in \Omega, \tag{3.5}$$

which is independent of the choice of $y \in \Omega$. In other words, $d_1(F)$ is the *topological degree of* F, often denoted as $d_{top}(F)$. Define $d_0(F) := d_{top}({}^\dagger F)$.

We will first consider a holomorphic correspondence F of $\mathbb{P}^1$ such that $d_1(F) > d_0(F)$. A very special case of a result of Dinh and Sibony is that there exists a probability measure μ_F such that

$$\frac{1}{d_1(F)^n}(F^n)^*(\omega_{FS}) \xrightarrow{\text{weak*}} \mu_F \ \text{ as measures, as } n \to \infty, \tag{3.6}$$

where ω_{FS} denotes the Fubini–Study form on $\mathbb{P}^1$, treated as a normalised area form. Let us call this measure the *Dinh–Sibony measure associated to* F. Since equidistribution is among the themes of this paper, we should mention that for a generic $z \in \mathbb{P}^1$, μ_F is the asymptotic distribution of the iterated pre-images of z. More precisely:

Fact 3.1. *Let F be a holomorphic correspondence on $\mathbb{P}^1$ such that $d_0(F) < d_1(F)$ and let μ_F be the Dinh–Sibony measure associated to F. There exists a polar set $\mathcal{E} \subsetneq \mathbb{P}^1$ such that for each $z \in \mathbb{P}^1 \setminus \mathcal{E}$*

$$\frac{1}{d_1(F)^n}(F^n)^*(\delta_z) \xrightarrow{\textit{weak*}} \mu_F \ \textit{ as measures, as } n \to \infty.$$

Consequently, $F^*(\mu_F) = d_1(F)\mu_F$.

The above follows by combining (3.6) with another result from [11]. Fact 3.1 will have *no role to play* herein except to set the context for our results. For instance, it shows that $\mathsf{supp}(\mu_F)$ could be viewed — especially in view of Brolin's results — as the analogue of the Julia set in the general context of correspondences. While, for a rational map on $\mathbb{P}^1$, its Julia set is *definitionally* disjoint from its normality set, in the case of correspondences we have:

Theorem 3.2. *Let F be a holomorphic correspondence on $\mathbb{P}^1$ and assume that $d_0(F) < d_1(F)$. Let μ_F be the Dinh–Sibony measure associated to F. Then,* $\mathrm{supp}(\mu_F)$ *is disjoint from the normality set of F.*

The ideas behind the proof are as follows. For $\mathcal{N}(F) \neq \varnothing$, we shall show that one can apply Marty's normality criterion in such a manner as to deduce that the volumes of any compact $K \Subset \mathcal{N}(F)$ given by the measures induced by $(F^n)^*(\omega_{FS})$ are bounded independent of n. The result then follows due to the fact that $d_1(F)^{-n} \to 0$ as $n \to \infty$.

The situation is *very different* when $d_0(F) \geq d_1(F)$. To repeat: we should not expect asymptotic equidistribution of preimages in general, even when $d_0(F), d_1(F) \geq 2$, as the holomorphic correspondence F whose graph is the $\mathbb{P}^1 \times \mathbb{P}^1-$completion of the zero set of the rational function $g(z_1, z_2) = z_2^2 - (1/z_1^2)$ illustrates. We require some dynamically meaningful condition for things to work. It is this need that motivates the next few definitions. Let X be a compact Hausdorff space and let $f \subset X \times X$ be a relation of X to itself such that $\pi_1(f) = X$. For any set $S \subset X$, we write

$$f(S) \ := \ \pi_2\left(\pi_1^{-1}(S) \cap f\right).$$

We define the nth iterated relation by

$$\begin{aligned} f^{(n)} \ &:= \ f \star f^{(n-1)} \ \text{ for } n \geq 2, \\ f^{(1)} \ &:= \ f, \end{aligned} \tag{3.7}$$

where the composition operation $\star$ is as given by (2.2) above. It is useful to have a notion of omega limit sets in the context of iterating a relation analogous to the case of maps. This definition is provided by McGehee in [16, Section 5]. Following McGehee, for a subset $S \subset X$, let us write

$$\mathfrak{K}(S; f) \ := \{K \subset_{\text{closed}} X : K \text{ satisfies } f(K) \subset K \text{ and } f^{(n)}(S) \subset K \text{ for some } n \geq 0\}$$

(with the understanding that $f^{(0)}$ is the diagonal in $X \times X$). The *omega limit set of S under f*, denoted by $\omega(S; f)$, is the set

$$\omega(S; f) \ := \ \bigcap \mathfrak{K}(S; f).$$

We say that a set $\mathcal{A} \subset X$ is an *attractor* for f if $\mathcal{A} \neq X$ and there exists a set U such that $\mathcal{A} \subset U^\circ$ and such that $\omega(U; f) = \mathcal{A}$.

These concepts motivate the following two definitions in the context of holomorphic correspondences.

Definition 3.3. Let F be a holomorphic correspondence on a Riemann surface X given by the holomorphic 1-chain Γ. A set $\mathcal{A} \subset X$ is called an *attractor* for F if it is an attractor for the relation $|\Gamma|$ in the sense of [16] (i.e., as discussed above). A set $\mathcal{R}$ is called a *repeller* for F if it is an attractor for the relation $|{}^\dagger\Gamma|$.

We must note here McGehee calls the relation on X induced by $|{}^\dagger\Gamma|$ the *transpose* of $|\Gamma|$, and our $|{}^\dagger\Gamma|$ is $|\Gamma|^*$ in the notation of [16].

Definition 3.4. Let F be as above and let $\mathcal{R}$ be a repeller for F. We say that $\mathcal{R}$ is a *strong repeller* for F if there exists a point $a_0 \in \mathcal{R}$ and an open set $U \supset \mathcal{R}$ such that for each $w \in U$, there exists a sequence $\{a_n(w)\}_{n \in \mathbb{Z}_+}$ such that

- $a_n(w) \in ({}^\dagger F)^n(w) \ \forall n \in \mathbb{Z}_+$; and
- $a_n(w) \longrightarrow a_0$ as $n \to \infty$.

The term *strong attractor* has an analogous definition.

We call $w \in \mathbb{P}^1$ a *critical value* if there exists an irreducible component Γ_j such that at least one of the irreducible germs of Γ_j at some point in $\pi_2^{-1}\{w\} \cap \Gamma_j$ is either non-smooth or does *not* project injectively under π_2.

We are now in a position to state our next result.

Theorem 3.5. *Let F be a holomorphic correspondence on $\mathbb{P}^1$ for which $d_0(F) \geq d_1(F)$. Assume that F has a strong repeller $\mathcal{R}$ that is disjoint from the set of critical values of F. Then, there exist a probability measure μ_F on $\mathbb{P}^1$ that satisfies $F^*(\mu_F) = d_1(F)\mu_F$ and an open set $U(F, \mathcal{R}) \supset \mathcal{R}$ such that*

$$\frac{1}{d_1(F)^n}(F^n)^*(\delta_z) \xrightarrow{weak^*} \mu_F \;\; \textit{as measures, as } n \to \infty \;\; \forall z \in U(F, \mathcal{R}). \tag{3.8}$$

It may seem to the reader that (3.8) could be stronger, since the theorem does not state that $\mathbb{P}^1 \setminus U(F, \mathcal{R})$ is polar (or even nowhere dense). However, given that $d_0(F) \geq d_1(F)$, this is very much in the nature of things. In this regard, we make the following remark.

Remark 3.6. If F is as in Theorem 3.5, we *cannot conclude, in general, that the set $\mathbb{P}^1 \setminus U(F, \mathcal{R})$ is polar.* The following example constitutes a basic obstacle to $\mathbb{P}^1 \setminus U(F, \mathcal{R})$ being even nowhere dense. Let P be any polynomial whose filled Julia set has non-empty interior. Consider the holomorphic correspondence F determined by

$$\Gamma^P \;:=\; \text{the completion in } \mathbb{P}^1 \times \mathbb{P}^1 \text{ of the zero set of } (z_1 - P(z_2)).$$

Here, $d_0(F) = \deg(P) > 1 = d_1(F)$. Note that $\{\infty\}$ is a strong repeller. However, $U(F, \{\infty\})$ cannot contain any points from the filled Julia set of P.

From the perspective of studying the problem of equidistribution of *inverse* images, Theorem 3.5 is, to the best of our knowledge, the first theorem concerning the equidistribution problem for holomorphic correspondences F such that $d_0(F) \geq d_1(F)$.

We must point out that, in general, given $d_1(F) > d_0(F)$, the Dinh–Sibony measure is *not* an invariant measure. (It is what one calls an F^*-invariant measure.) A Borel measure μ is said to be *invariant under F* if its push-forward by F preserves (compensating for multiplicity if F is not a map) measure of all Borel sets — i.e., if $F_*\mu = d_0(F)\mu$. Thus, under the condition $d_0(F) \geq d_1(F)$, we can actually construct measures that are invariant under F:

Corollary 3.7. *Let F be a holomorphic correspondence on $\mathbb{P}^1$ for which $d_0(F) \geq d_1(F)$.*

i) If $d_0(F) > d_1(F)$, there exists a measure μ_F that is invariant under F.

ii) Suppose $d_0(F) = d_1(F)$. If F has a strong attractor that is disjoint from the critical values of ${}^{\dagger}F$, then there exists a measure μ_F that is invariant under F.

The proof of Theorem 3.5 relies on techniques developed by Lyubich in [15]. Given our hypothesis on the existence of a repeller $\mathcal{R}$, one can show that there exists a compact set B such that $\mathcal{R} \subset B^{\circ}$ and ${}^{\dagger}F(B) \subset B$. This allows us to define a Perron–Frobenius-type operator $\mathbb{A}_B : \mathcal{C}(B; \mathbb{C}) \longrightarrow \mathcal{C}(B; \mathbb{C})$, where

$$\mathbb{A}_B \;:=\; \frac{1}{d_1(F)}\,\mathbf{\Lambda}|_B\,,$$

with $\mathbf{\Lambda}|_B$ being the operator given by (3.3) with B replacing Ω. Our proof relies on showing that the family $\{\mathbb{A}_B^n : n = 1, 2, 3, \dots\}$ satisfies the conditions of the main result in

[15, §2] (which provides us a candidate for μ_F). This goal is achieved, *in part*, by showing that there exists an open neighbourhood W of the B above such that ${}^{\dagger}F(W) \subset W$, and that around each $z \in W$ and for each $n \in \mathbb{Z}_+$, there are $d_1(F)^n$ holomorphic branches (counting according to multiplicity) of n-fold iteration of the correspondence ${}^{\dagger}F$.

The examples in Remark 3.6 have some very particular features. One might ask whether there are plenty of holomorphic correspondences F on $\mathbb{P}^1$ with $d_0(F) \geq d_1(F)$ — *without* such special features as in the correspondences in Remark 3.6 — that satisfy the conditions stated in Theorem 3.5. One might also ask whether any of the correspondences alluded to in Section 1 satisfy the conditions in Theorem 3.5. The reader is referred to Section 7 concerning these questions. In the next section, we shall establish a few technical facts which will be of relevance throughout this paper. The proofs of our theorems will be provided in Sections 5 and 6.

4. Technical propositions

We begin by showing that the composition of two holomorphic correspondences on $\mathbb{P}^1$, under the composition rule (2.4), produces a holomorphic correspondence.

One way to see this is to begin with how one computes $F_2 \circ F_1$ if one is given exact expressions for F_1 and F_2. Let $\mathit{\Gamma}^s$ be the graph of F_s, $s = 1, 2$, and consider the representations given by (2.1). Fix indices j and l such that $1 \leq j \leq L_1$ and $1 \leq l \leq L_2$. It follows that there exist irreducible polynomials $P_1, P_2 \in \mathbb{C}[z, w]$ such that

$$\Gamma^{\bullet}_{1,j} \cap \mathbb{C}^2 = \{(z,w) \in \mathbb{C}^2 : P_1(z,w) = 0\}, \quad \Gamma^{\bullet}_{2,l} \cap \mathbb{C}^2 = \{(z,w) \in \mathbb{C}^2 : P_2(z,w) = 0\};$$

see, for instance, [18, pp. 23-24]. Now, given any polynomial $P \in \mathbb{C}[z, w]$, set

$$\begin{aligned} \operatorname{supp}(P) &:= \{(\alpha,\beta) \in \mathbb{N}^2 : \partial_z^{\alpha}\partial_w^{\beta} P(0,0) \neq 0\}, \\ d_z(P) &:= \max\{\alpha \in \mathbb{N} : (\alpha,\beta) \in \operatorname{supp}(P)\}, \\ d_w(P) &:= \max\{\beta \in \mathbb{N} : (\alpha,\beta) \in \operatorname{supp}(P)\}. \end{aligned}$$

Then, there is a choice of projective coordinates on $\mathbb{P}^1$ such that

$$\begin{aligned} \Gamma^{\bullet}_{1,j} &= \{([z_0 : z_1], [w_0 : w_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 : z_0^{d_z(P_1)} w_0^{d_w(P_1)} P_1(z_1/z_0, w_1/w_0) = 0\}, \\ \Gamma^{\bullet}_{2,l} &= \{([z_0 : z_1], [w_0 : w_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 : z_0^{d_z(P_2)} w_0^{d_w(P_2)} P_2(z_1/z_0, w_1/w_0) = 0\}. \end{aligned}$$

With these notations, we are in a position to state our first proposition.

Proposition 4.1. *Let $\Gamma^{\bullet}_{1,j}$ and $\Gamma^{\bullet}_{2,l}$ be irreducible subvarieties belonging to the holomorphic 1-chains $\mathit{\Gamma}^1$ and $\mathit{\Gamma}^2$ respectively. Let P_1 and P_2 be the defining functions of $\Gamma^{\bullet}_{1,j} \cap \mathbb{C}^2$ and $\Gamma^{\bullet}_{2,l} \cap \mathbb{C}^2$ respectively.*

i) Let $R(z,w) := \mathsf{Res}(P_1(z,\cdot), P_2(\cdot,w))$, where Res denotes the resultant of two univariate polynomials. Let V_R denote the biprojective completion in $\mathbb{P}^1 \times \mathbb{P}^1$ of $\{(z,w) \in \mathbb{C}^2 : R(z,w) = 0\}$. Then $|\Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j}| = V_R$.

ii) V_R has no irreducible components of the form $\{a\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{a\}$, $a \in \mathbb{P}^1$.

Proof. Let us write $V := |\Gamma^{\bullet}_{2,l} \circ \Gamma^{\bullet}_{1,j}|$. Since two polynomials $p, q \in \mathbb{C}[X]$ have a common zero if and only if $\mathsf{Res}(p,q) = 0$,

$$V \cap \mathbb{C}^2 = \{(z,w) \in \mathbb{C}^2 : \mathsf{Res}(P_1(z,\cdot), P_2(\cdot,w)) = 0\}.$$

Hence, as V is the biprojective completion of $V \cap \mathbb{C}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$, (i) follows.

To prove (ii), let us first consider the case when $a \neq [0:1]$. Then, it suffices to show that R has no factors of the form $(z-a)$ or $(w-a)$. We shall show that R has no factors of the form $(z-a)$. An analogous argument will rule out factors of the form $(w-a)$. To this end, assume that there exists an $a \in \mathbb{C}$ such that $(z-a)|R$ in $\mathbb{C}[z,w]$. This implies

$$R(a,w) \;=\; 0 \;\; \forall w \in \mathbb{C}.$$

Thus, for each $w \in \mathbb{C}$, the polynomial $P_2(\boldsymbol{\cdot}, w)$ has a zero in common with $p_1 := P_1(a, \boldsymbol{\cdot}) \in \mathbb{C}[X]$. Note that $p_1 \not\equiv 0$ because, otherwise, $(z-a)|P_1$, which would contradict the fact that $\pi_s|_{\Gamma^\bullet_{1,j}}$ is surjective, $s = 1, 2$. Thus, there exists an uncountable set $S \subset \mathbb{C}$ and a point $b \in p_1^{-1}\{0\}$ such that

$$P_2(b,w) \;=\; 0 \;\; \forall w \in S.$$

But this implies $P_2(b, \boldsymbol{\cdot}) \equiv 0$, i.e. that $(z-b)|P_2$. This is impossible, for exactly the same reason that $(z-a)\nmid P_1$. Hence R has no factors of the form $(z-a)$, $a \in \mathbb{C}$.

Note that, if we write $\mathbb{C}' := \{[z_0 : z_1] \in \mathbb{P}^1 : z_1 \neq 0\}$, then, arguing as in the beginning of this proof,

$$V \cap (\mathbb{C}' \times \mathbb{C}) \;=\; \{(z,w) \in \mathbb{C}^2 : z^{d_z(R)} \mathsf{Res}(P_1(1/z, \boldsymbol{\cdot}), P_2(\boldsymbol{\cdot}, w)) = 0\},$$

where $d_z(R)$ is as defined in the beginning of this section. If we define $\widetilde{R} \in \mathbb{C}[z,w]$ by

$$\widetilde{R}(z,w) \;:=\; z^{d_z(R)} \mathsf{Res}(P_1(1/z, \boldsymbol{\cdot}), P_2(\boldsymbol{\cdot}, w)),$$

then we get $z \nmid \widetilde{R}$ in $\mathbb{C}[z,w]$. Thus, $\{[0:1]\} \times \mathbb{P}^1$ is not an irreducible component of V. By a similar argument, $\mathbb{P}^1 \times \{[0:1]\}$ is not an irreducible component of V either. ☐

It is now easy to see that $\Gamma^2 \circ \Gamma^1$ determines a holomorphic correspondence on $\mathbb{P}^1$. Let us pick $\Gamma^\bullet_{1,j}$ and $\Gamma^\bullet_{2,l}$ as in Proposition 4.1 and let C be an irreducible component of $|\Gamma^\bullet_{2,l} \circ \Gamma^\bullet_{1,j}|$. By the fundamental theorem of algebra, $\pi_s|_C$ would fail to be surjective for some $s \in \{1,2\}$ only if C is of the form $\{a\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{a\}$, $a \in \mathbb{P}^1$. This is impossible by Part (ii) of Proposition 4.1. Hence, we have the following:

Corollary 4.2. *Let Γ^1 and Γ^2 be two holomorphic correspondences on $\mathbb{P}^1$. Then $\Gamma^2 \circ \Gamma^1$ is a holomorphic correspondence on $\mathbb{P}^1$.*

The next lemma will be useful in simplifying expressions of the form $(F^n)^*\delta_z$ or $(F^n)_*\delta_z$. Its proof is entirely routine, so we shall leave the proof as an exercise.

Lemma 4.3. *Let X be a compact complex manifold and let F be a holomorphic correspondence on X. Then ${}^\dagger(F^n) = ({}^\dagger F)^n \;\forall n \in \mathbb{Z}_+$.*

The final result in this section is important because it establishes that the measures $d_1(F)^{-n}(F^n)^*(\delta_z)$ appearing in Theorem 3.5 are probability measures.

Proposition 4.4. *Let F be a holomorphic correspondence on $\mathbb{P}^1$. Then $d_{top}(F^n) = d_{top}(F)^n \;\forall n \in \mathbb{Z}_+$.*

The result above is obvious if F is a map. Essentially the same reasoning applies when F is not a map. In the definition of the topological degree, any point $y \in \mathbb{P}^1$ that satisfies equation (3.5) is generic. Since the correspondence Γ (which determines F) has no irreducible components of the form $\{a\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{a\}$, $a \in \mathbb{P}^1$, all the preimages of each such $y \in \mathbb{P}^1$ — with, perhaps, the exception of finitely many points — is generic in the sense of (3.5). Hence, d_{top} is multiplicative under composition of correspondences.

5. The proof of Theorem 3.2

We begin this section with some remarks on our use of the results of Dinh–Sibony [11]. The result we shall use is the one that leads to (3.6). The precise result is [11, Théorème 5.1], read together with Remarques 5.2. We state a version below *specifically for correspondences on* $\mathbb{P}^1$. We leave it to the reader to verify our transcription of [11, Théorème 5.1] to the present context. It might be helpful for readers who are unfamiliar with [11] to mention a couple of identities needed for this transcription. We shall not define here the notion of *intermediate degrees of* F *of order* s; we just refer to [11, Section 3.1].

Fact 5.1. *Let* (X, ω) *be a compact Kähler manifold of dimension* k *and assume* $\int_X \omega^k = 1$. *Let* $F : X \longrightarrow X$ *be a holomorphic correspondence. Let* $\lambda_s(F)$ *denote the intermediate degree of* F *of order* s, $s = 0, 1, \dots, k$. *Then:*

i) $\lambda_{k-s}({}^\dagger F) = \lambda_s(F)$.

ii) $\lambda_k(F) = d_{top}(F)$.

In what follows ω_{FS} shall denote the Fubini–Study form normalised so that $\int_{\mathbb{P}^1} \omega_{FS} = 1$. The key result needed is:

Result 5.2 (Théorème 5.1 of [11] paraphrased for correspondences of $\mathbb{P}^1$)**.** *Let* F_n, $n = 1, 2, 3, \dots,$ *be holomorphic correspondences of* $\mathbb{P}^1$. *Suppose that the series* $\sum_{n \in \mathbb{Z}_+} (d_0(F_1)/d_1(F_1)) \dots (d_0(F_n)/d_1(F_n))$ *converges. Then, there exists a probability measure* μ *such that*

$$d_1(F_1)^{-1} \dots d_1(F_n)^{-1} (F_n \circ \dots \circ F_1)^*(\omega_{FS}) \xrightarrow{weak^*} \mu \ \text{ as measures, as } n \to \infty.$$

In fact, the convergence statement (3.6) follows from the above result by taking $F_n = F$ for $n = 1, 2, 3, \dots$

The proof of Theorem 3.2. We assume that $\mathcal{N}(F) \neq \varnothing$; there is nothing to prove otherwise. Let us fix a $z_0 \in \mathcal{N}(F)$. Then, any $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$ is admissible, and by Property (2) in sub-section 2.2, assuming that $n \geq 2$, we get

$$\begin{aligned} \pi^0_{j-1} \circ \nu_{(\boldsymbol{\mathcal{Z}}, j-1)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, j-1)} &= \pi^0_{j-1} \circ \mathsf{proj}_j \circ \nu_{(\boldsymbol{\mathcal{Z}}, j)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, j)} \\ &= \pi^0_j \circ \nu_{(\boldsymbol{\mathcal{Z}}, j)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, j)} \quad \forall j = 2, \dots, n. \end{aligned}$$

Iterating this argument, we deduce the following:

For any $n \geq 2$, $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$ (and admissible)

$$\Rightarrow \ \pi^0_j \circ \nu_{(\boldsymbol{\mathcal{Z}}, j)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, j)} \ = \ \pi^0_1 \circ \nu_{(\boldsymbol{\mathcal{Z}}, 1)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, 1)} \quad \forall j = 1, \dots, n. \tag{5.1}$$

Let us now *fix* a disc Δ around $0 \in \mathbb{C}$ such that $\Delta \Subset D$ (where D is as given by Definition 2.2). Define:

$$K := \bigcap_{\boldsymbol{\mathcal{X}} \in \mathscr{P}_1(z_0)} \pi^0_1 \circ \nu_{(\boldsymbol{\mathcal{X}}, 1)} \circ \psi_{(\boldsymbol{\mathcal{X}}, 1)}(\overline{\Delta}).$$

Clearly, there is a region $G \Subset D$, containing 0, such that

$$\left(\nu_{(\boldsymbol{\mathcal{X}}, 1)} \circ \psi_{(\boldsymbol{\mathcal{X}}, 1)}\right)^{-1} \left(\left(\pi^0_1 |_{U(\boldsymbol{\mathcal{X}}, 1)} \right)^{-1} (K) \right) \Subset G \quad \text{for every } \boldsymbol{\mathcal{X}} \in \mathscr{P}_1(z_0),$$

where $U(\boldsymbol{\mathcal{X}}, 1)$ is as described in sub-section 2.2. From the above, from (5.1), and from the fact that any $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$ has some $\boldsymbol{\mathcal{X}} \in \mathscr{P}_1(z_0)$ as an initial 1-path, we deduce:

$$\text{for any } n \geq 1 \ \& \ \boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0), \ \left(\nu_{(\boldsymbol{\mathcal{Z}}, n)} \circ \psi_{(\boldsymbol{\mathcal{Z}}, n)}\right)^{-1} \left(\left(\pi^0_n |_{U(\boldsymbol{\mathcal{Z}}, n)} \right)^{-1} (K) \right) \Subset G. \tag{5.2}$$

We can deduce from Definition 2.2 that $K \subset \mathcal{N}(F)$. As K has non-empty interior, it suffices to show that for any *non-negative* function $\varphi \in \mathcal{C}(\mathbb{P}^1;\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset K$, $\int_{\mathbb{P}^1} \varphi\, d\mu_F = 0$. Hence, let us pick some function $\varphi \in \mathcal{C}(\mathbb{P}^1;\mathbb{R})$ as described. For any path $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$, let us write

$$\Phi_{\boldsymbol{\mathcal{Z}}} \;:=\; \left(\pi_n^0 \times \pi_n^n\right) \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)}.$$

We adopt notation analogous to that developed in the beginning of sub-section 2.2. For any multi-index $\boldsymbol{\alpha} \in [1\,..\,L]^j$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_j)$, $j = 1, 2, 3, \dots$, let us define

$$\Gamma^{\bullet\boldsymbol{\alpha}} \;:=\; \begin{cases} \Gamma^\bullet_{\alpha_1}, & \text{if } j = 1,\\ \Gamma^\bullet_{\alpha_j} \circ \Gamma^{\bullet(\alpha_1,\dots,\alpha_{j-1})}, & \text{if } j \geq 2. \end{cases}$$

If we write $\boldsymbol{\mathcal{Z}}$ as $(z_0, \dots, z_n;\, \boldsymbol{\alpha})$, then $\Phi_{\boldsymbol{\mathcal{Z}}}(D)$ is a $\Gamma^{\bullet\boldsymbol{\alpha}}$-open neighbourhood of the point (z_0, z_n). Furthermore, by our constructions in sub-section 2.2, $\pi_1^{-1}(K) \cap |\Gamma^{\circ n}|$ is covered by the sets $\Phi_{\boldsymbol{\mathcal{Z}}}(D)$ as $\boldsymbol{\mathcal{Z}}$ varies through $\mathscr{P}_n(z_0)$. Hence, by definition:

$$\langle (F^n)^*(\omega_{FS}), \varphi\rangle \;=\; \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \int\limits_{\operatorname{reg}(\Phi_{\boldsymbol{\mathcal{Z}}}(D))} \left(\,\pi_1|_{\Phi_{\boldsymbol{\mathcal{Z}}}(D)}\,\right)^*(\varphi)\left(\,\pi_2|_{\Phi_{\boldsymbol{\mathcal{Z}}}(D)}\,\right)^*(\omega_{FS}). \tag{5.3}$$

It is routine to show that $\pi_n^0 \times \pi_n^n\big|_{U(\boldsymbol{\mathcal{Z}},n)}$ is a branched covering map onto its image. As $\nu_{(\boldsymbol{\mathcal{Z}},n)}$ maps $\psi_{(\boldsymbol{\mathcal{Z}},n)}(D)$ homeomorphically onto $U(\boldsymbol{\mathcal{Z}},n)$, the topological degree of $(\pi_n^0 \times \pi_n^n) \circ \nu_{(\boldsymbol{\mathcal{Z}},n)}\big|_{\psi_{(\boldsymbol{\mathcal{Z}},n)}(D)}$ equals the topological degree of $\pi_n^0 \times \pi_n^n\big|_{U(\boldsymbol{\mathcal{Z}},n)}$. Let us denote this number by $\deg_1(\boldsymbol{\mathcal{Z}})$. By the change-of-variables formula, we get:

$$\langle (F^n)^*(\omega_{FS}), \varphi\rangle \;=\; \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \frac{1}{\deg_1(\boldsymbol{\mathcal{Z}})} \int\limits_{\psi_{(\boldsymbol{\mathcal{Z}},n)}(D)} (\varphi \circ \pi_n^0 \circ \nu_{(\boldsymbol{\mathcal{Z}},n)})(\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)})^*(\omega_{FS}).$$

Since $\Phi_{\boldsymbol{\mathcal{Z}}}(D)$, in general, has singularities, we discuss briefly what is meant above by "change-of-variables formula". Note that:

- The magnitude of the form $(\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)})^*(\omega_{FS})$ stays bounded on punctured neighbourhoods of any singular point of $\Phi_{\boldsymbol{\mathcal{Z}}}(D)$.
- $\operatorname{reg}(\Phi_{\boldsymbol{\mathcal{Z}}}(D))$ after at most finitely many punctures is the image of a Zariski-open subset of $\psi_{(\boldsymbol{\mathcal{Z}},n)}(D)$ under a $\deg_1(\boldsymbol{\mathcal{Z}})$-to-1 covering map.

Given these facts, it is a standard calculation that the right-hand side of (5.3) transforms to the last integral above. For each $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$, let us write

$$\deg_2(\boldsymbol{\mathcal{Z}}) \;:=\; \text{the degree of the map } \psi_{(\boldsymbol{\mathcal{Z}},n)} : D \longrightarrow \nu^{-1}_{(\boldsymbol{\mathcal{Z}},n)}(U(\boldsymbol{\mathcal{Z}},n)).$$

By the change-of-variables formula for branched coverings of finite degree, we get:

$$\begin{aligned}
&\langle (F^n)^*(\omega_{FS}), \varphi\rangle\\
&\quad= \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \frac{1}{\deg_2(\boldsymbol{\mathcal{Z}})\deg_1(\boldsymbol{\mathcal{Z}})} \int\limits_D (\varphi \circ \pi_n^0 \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)})(\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)})^*(\omega_{FS})\\
&\quad= \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \frac{1}{\deg_2(\boldsymbol{\mathcal{Z}})\deg_1(\boldsymbol{\mathcal{Z}})} \int\limits_{\overline{G}} (\varphi \circ \pi_n^0 \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)})(\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)})^*(\omega_{FS}).
\end{aligned} \tag{5.4}$$

The last expression follows from (5.2) and the fact that $\operatorname{supp}(\varphi) \subset K$.

Endow $\mathbb{P}^1$ with homogeneous coordinates. Given the form of the argument made below, we can assume without loss of generality that $\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)}(D)$ does not contain *both* $[0:1]$ and $[1:0]$. Write

$$\pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)} \;=\; [X_{(\boldsymbol{\mathcal{Z}},n)} : Y_{(\boldsymbol{\mathcal{Z}},n)}],$$

where $X_{(\boldsymbol{\mathcal{Z}},n)}, Y_{(\boldsymbol{\mathcal{Z}},n)} \in \mathcal{O}(D)$ and have no common zeros in D, and define

$$q_{(\boldsymbol{\mathcal{Z}},n)} \;:=\; \begin{cases} X_{(\boldsymbol{\mathcal{Z}},n)}/Y_{(\boldsymbol{\mathcal{Z}},n)}, & \text{if } [0:1] \in \pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)}(D), \\ Y_{(\boldsymbol{\mathcal{Z}},n)}/X_{(\boldsymbol{\mathcal{Z}},n)}, & \text{if } [1:0] \in \pi_n^n \circ \nu_{(\boldsymbol{\mathcal{Z}},n)} \circ \psi_{(\boldsymbol{\mathcal{Z}},n)}(D). \end{cases}$$

From the expression for the Fubini–Study metric in local coordinates and from (5.4), we have the estimate (where C is a constant ≥ 1 that is independent of n):

$$d_1(F)^{-n}|\langle (F^n)^*(\omega_{FS}), \varphi\rangle| \;\leq\; \frac{C}{d_1(F)^n} \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \int_{\overline{G}} \|\varphi\|_\infty \left(\frac{|q'_{(\boldsymbol{\mathcal{Z}},n)}(\zeta)|}{1+|q_{(\boldsymbol{\mathcal{Z}},n)}(\zeta)|^2} \right)^2 dA(\zeta).$$

Since, by hypothesis, $\mathscr{F}(z_0)$ is a normal family, it follows by Marty's normality criterion — see, for instance, Conway [9, Chapter VII/§3] — that the family

$$\left\{ \frac{|q'_{(\boldsymbol{\mathcal{Z}},n)}|}{1+|q_{(\boldsymbol{\mathcal{Z}},n)}|^2} : n \in \mathbb{Z}_+ \text{ and } \boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0) \right\}$$

is locally uniformly bounded. As $G \Subset D$, there exists an $M > 0$ such that

$$\frac{|q'_{(\boldsymbol{\mathcal{Z}},n)}(\zeta)|}{1+|q_{(\boldsymbol{\mathcal{Z}},n)}(\zeta)|^2} \;\leq\; M \;\; \forall \zeta \in \overline{G},\; n \in \mathbb{Z}_+ \text{ and } \forall \boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0).$$

Given $n \in \mathbb{Z}_+$, the number of summands in (5.4) is $d_{top}({}^\dagger(F^n)) = d_0(F)^n$. The equality is a consequence of Proposition 4.4 and Lemma 4.3. From it and the last two estimates, it follows that

$$d_1(F)^{-n}|\langle (F^n)^*(\omega_{FS}), \varphi\rangle| \;\leq\; C \left[\frac{d_0(F)}{d_1(F)} \right]^n M^2 \mathsf{Area}(\overline{G}) \;\longrightarrow\; 0 \text{ as } n \to \infty, \tag{5.5}$$

since, by hypothesis, $d_0(F) < d_1(F)$.

In view of Result 5.2, taking $F_n = F$ for $n = 1, 2, 3, \dots$, we have

$$\int_{\mathbb{P}^1} \varphi \, d\mu_F \;=\; \lim_{n\to\infty} \frac{1}{d_1(F)^n} \langle (F^n)^*(\omega_{FS}), \varphi\rangle \;=0.$$

In view of our remarks earlier, the theorem follows. □

Remark 5.3. The reader will observe that to keep a count of multiplicities in the above proof, it is essential that the integrands that appear be labelled by $\boldsymbol{\mathcal{Z}}$ in $\mathscr{P}_n(z_0)$ and by n itself: i.e., the length of the forward iteration of F under consideration. If we do **not** make the restriction $(*)$ about the paths to be considered in Definition 2.2, then it can happen that, for a path $(z_0, \dots, z_N; \boldsymbol{\alpha})$, there exists some j, $1 \leq j \leq N$, for which the set $U(\boldsymbol{\mathcal{Z}}, j)$, as defined in Section 2 is *not* irreducible when viewed as a *germ* of an analytic variety at $(z_0, \dots, z_j)$. Clearly, the label $(\boldsymbol{\mathcal{Z}}, j)$ wouldn't then suffice to index the various analytic irreducible components into which $U(\boldsymbol{\mathcal{Z}}, j)$ splits. In short, the assumption of admissibility is made so that the basic motivation for the normality set is not obscured by too much book-keeping paraphernalia.

6. The proof of Theorem 3.5 and Corollary 3.7

As the discussion in Section 3 preceding the statement of Theorem 3.5 suggests, its proof relies on several notions introduced in [16]. We therefore begin this section with a definition and a couple of results from [16].

Definition 6.1. Let X be a compact Hausdorff space and let $f \subset X \times X$ be a relation of X to itself such that $\pi_1(f) = X$. A set $B \subset X$ is called an *attractor block* for f if $f(\overline{B}) \subset B^\circ$.

We recall that, given a relation f and a set $S \subset X$, $f(S)$ is as defined in Section 3.

Result 6.2 (McGehee, Theorem 7.2 of [16])**.** *Let X be a compact Hausdorff space and let $f \subset X \times X$ be a relation of X to itself such that $\pi_1(f) = X$. Assume f is a closed set. If B is an attractor block for f, then B is a neighbourhood of $\omega(B; f)$.*

Result 6.3 (McGehee, Theorem 7.3 of [16])**.** *Let X be a compact Hausdorff space and let $f \subset X \times X$ be a relation of X to itself such that $\pi_1(f) = X$. Assume f is a closed set. If $\mathcal{A}$ is an attractor for f and V is a neighbourhood of $\mathcal{A}$, then there exists a closed attractor block B for f such that $B \subset V$ and $\omega(B; f) = \mathcal{A}$.*

We clarify that, given two subsets A and B of some topological space, B is called a neighbourhood of A here (as in [16]) if $A \subset B^\circ$.

Before we can give the proof of Theorem 3.5, we need one more concept. For this purpose, we shall adapt some of the notations developed in Section 2.2. Here, F will denote a holomorphic correspondence on $\mathbb{P}^1$. Firstly: given $N \in \mathbb{Z}_+$, we say that $(w_0, w_{-1} \dots, w_{-N}; \alpha_1, \dots, \alpha_N) \in (\mathbb{P}^1)^{N+1} \times [1\,..\,L]^N$ (see (2.1) for the meaning of L) is *a path of a backward iteration of F starting at w_0, of length N*, if

$$(w_{-j}, w_{-j+1}) \in \Gamma^\bullet_{\alpha_j}, \quad j = 1, \dots, N.$$

In analogy with the notation in Section 2.2, we set:

$$\mathscr{P}_{-N}(w_0) \ := \ \text{the set of all paths of } \textit{backward} \text{ iterations of } F\text{, of length } N\text{, starting at } w_0.$$

Next, we say that a point $w \in \mathbb{P}^1$ is a *regular value* of F if it is not a critical value (recall that we have defined this in Section 3). We can now make the following definition:

Definition 6.4. Let F be a holomorphic correspondence on $\mathbb{P}^1$, let $N \in \mathbb{Z}_+$, and let $w_0 \in \mathbb{P}^1$. Let $\boldsymbol{\mathcal{W}} := (w_0, w_{-1}, \dots, w_{-N}; \alpha_1, \dots, \alpha_N) \in \mathscr{P}_{-N}(w_0)$. We call the list $({}^\dagger F_{(\boldsymbol{\mathcal{W}},1)}, \dots, {}^\dagger F_{(\boldsymbol{\mathcal{W}},N)})$ a *regular branch of a backward iteration of F along $\boldsymbol{\mathcal{W}}$* if:

1) $w_0, w_{-1}, \dots, w_{-N+1}$ are regular values.
2) For each $j = 1, \dots, N$, ${}^\dagger F_{(\boldsymbol{\mathcal{W}},j)}$ is a holomorphic function defined by

$${}^\dagger F_{(\boldsymbol{\mathcal{W}},j)}(\zeta) \ := \ \pi_1 \circ \big(\pi_2|_{\mathcal{D}(\boldsymbol{\mathcal{W}},j)} \big)^{-1}(\zeta) \ \ \forall \zeta \in \pi_2(\mathcal{D}(\boldsymbol{\mathcal{W}},j)),$$

where $\mathcal{D}(\boldsymbol{\mathcal{W}},j)$ is a *local* irreducible component of $\Gamma^\bullet_{\alpha_j}$ at the point (w_{-j}, w_{-j+1}) such that: (a) $\mathcal{D}(\boldsymbol{\mathcal{W}},j)$ is smooth; (b) π_2 restricted to $\mathcal{D}(\boldsymbol{\mathcal{W}},j)$ is injective; and (c) $\pi_2(\mathcal{D}(\boldsymbol{\mathcal{W}},j)) \supset \pi_1(\mathcal{D}(\boldsymbol{\mathcal{W}},j-1))$ when $j \geq 2$.

The above is a paraphrasing — for the setting in which we are interested — of the notion of a "regular inverse branch of F of order N" introduced by Dinh in [10].

The following is the key proposition needed to prove Theorem 3.5.

Proposition 6.5. *Let F be a holomorphic correspondence of $\mathbb{P}^1$ having all the properties stated in Theorem 3.5 and let $\mathcal{R}$ be a strong repeller that is disjoint from the set of critical values of F. Then, there exists a closed set $B \subset \mathbb{P}^1$ such that $B^\circ \supset \mathcal{R}$ and such that:*

i) *The operator $\mathbb{A}_B := d_1(F)^{-1}\, \mathbf{\Lambda}|_B$, where $\mathbf{\Lambda}|_B$ is as defined in* (3.3) *with B replacing Ω, maps $\mathcal{C}(B;\mathbb{C})$ into itself.*

ii) *There exists a probability measure $\mu_B \in \mathcal{C}(\mathbb{P}^1;\mathbb{R})^*$ that satisfies $\mu_B \circ \mathbb{A}_B = \mu_B$ and such that*

$$\lim_{n\to\infty} \sup_B \left| \mathbb{A}^n[\varphi] - \int_B \varphi\, d\mu_B \right| \ = \ 0 \quad \forall \varphi \in \mathcal{C}(B;\mathbb{C}). \tag{6.1}$$

Proof. Let $a_0 \in \mathcal{R}$ and let U be an open set containing $\mathcal{R}$ such that:

- For each $w \in U$, there is a sequence $\{a_n(w)\}_{n\in\mathbb{Z}_+}$ such that $a_n(w) \in {}^\dagger F^n(w)$ for each n, and $a_n(w) \longrightarrow a_0$ as $n \to \infty$.
- U contains no critical values of F.

By Result 6.2 and Result 6.3, we can find an open neighbourhood W of $\mathcal{R}$ such that $\overline{W} \subset U$ and $\overline{W}$ is a closed attractor block for the relation $|{}^\dagger\Gamma|$.

Repeating the last argument once more, we can find a closed attractor block, B, for $|{}^\dagger\Gamma|$ such that

$$\mathcal{R} \subset B^\circ \subset B \subset W \Subset U.$$

By the above chain of inclusions and by the definition of the term "attractor block", it follows that the operator $\mathbb{A}_B$ maps $\mathcal{C}(B;\mathbb{C})$ into itself.

Claim 1. *For each fixed $\varphi \in \mathcal{C}(B;\mathbb{C})$, $\{\mathbb{A}_B^n[\varphi]\}_{n\in\mathbb{Z}_+}$ is an equicontinuous family.*

It is easy to see that $\mathcal{R}$ is a closed *proper* subset. We can thus make a useful observation:

(∗∗) We can choose W so that $\mathbb{P}^1 \setminus \overline{W}$ is non-empty. Hence, we can choose coordinates in such a way that we may view W as lying in $\mathbb{C}$, and that $W \Subset \mathbb{C}$. We shall work with respect to these coordinates in the remainder of this proof.

Let us pick a point w_0 in B (which, by construction, is a regular value) and let $D(w_0)$ be a small disc centered at w_0 such that $\overline{D(w_0)} \subset W$. Let us fix an $N > 1$ and consider a path $\boldsymbol{\mathcal{W}} \in \mathscr{P}_{-N}(w_0)$. Recall that, by construction:

$${}^\dagger F(\overline{W}) \ \subset \ W. \tag{6.2}$$

We can infer from (6.2) that there exists a regular branch $({}^\dagger F_{(\boldsymbol{\mathcal{W}},1)}, \dots, {}^\dagger F_{(\boldsymbol{\mathcal{W}},N)})$ of a backward iteration of F along $\boldsymbol{\mathcal{W}}$. To see why, first note that, as w_0 is a regular value and $\overline{D(w_0)} \subset W$, we get:

(a_1) There is an open neighbourhood U_0 of w_0 containing only regular values.

(b_1) Writing

$$\widetilde{U}_1 \ := \ \text{the connected component of } \pi_2^{-1}(U_0) \cap \Gamma^\bullet_{\alpha_1} \text{ containing } (w_{-1}, w_0),$$

and defining $\mathcal{D}(\boldsymbol{\mathcal{W}},1)$:= any one of the irreducible components of the (local) complex-analytic variety $\widetilde{U}_1$, the function

$${}^\dagger F_{(\boldsymbol{\mathcal{W}},1)} \ := \ \pi_1 \circ \left(\pi_2|_{\mathcal{D}(\boldsymbol{\mathcal{W}},1)} \right)^{-1} \text{ is holomorphic on } U_0.$$

(c_1) The open set $U_1 := {}^\dagger F_{(\boldsymbol{\mathcal{W}},1)}(U_0) \subset W$ and hence contains only regular values.

The assertion (c_1) follows from the fact that $U_1 \subset {}^{\dagger}F(\overline{W}) \subset W$ and that the latter contains no critical values.

Let us now, for some $k \in \mathbb{Z}_+$, $k \leq N-1$, *assume* the truth of the statements (a_k), (b_k) and (c_k), which are obtained by replacing all the subscripts 0 and 1 in (a_1), (b_1) and (c_1) (*except* the subscript in π_1) by $k-1$ and k, respectively. Now, (a_{k+1}) follows from (c_k). Defining $\mathcal{D}(\boldsymbol{\mathcal{W}}, k+1)$ in exact analogy to $\mathcal{D}(\boldsymbol{\mathcal{W}}, 1)$, and writing

$$ {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, k+1)} \ := \ \pi_1 \circ \left(\left.\pi_2\right|_{\mathcal{D}(\boldsymbol{\mathcal{W}},\, k+1)} \right)^{-1}, $$

the holomorphicity of $(\left.\pi_2\right|_{\mathcal{D}(\boldsymbol{\mathcal{W}},\, k+1)})^{-1}$ follows from (a_{k+1}). Thus (b_{k+1}) holds true. We get (c_{k+1}) by appealing once again to (6.2) and using the fact that $U_k \subset W$. By induction, therefore, a regular branch of a backward iteration of F along $\boldsymbol{\mathcal{W}}$ exists.

Furthermore, we can conclude that:

$$ {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, N)} \circ \dots \circ {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, 1)}(D(w_0)) \subset W \Subset \mathbb{C} \quad \text{(see $(**)$ above)}. $$

Recall that $\boldsymbol{\mathcal{W}}$ was arbitrarily chosen from $\mathscr{P}_{-N}(w_0)$ and that the arguments in the last two paragraphs hold true for any choice of $\mathcal{D}(\boldsymbol{\mathcal{W}}, j)$, $1 \leq j \leq N$, and for any $N \in \mathbb{Z}_+$. Thus, by Montel's theorem, we infer the following important fact: the family

$$ \begin{aligned} {}^{\dagger}\mathscr{F}(w_0) \ := \ \Big\{ & {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, N)} \circ \dots \circ {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, 1)} \in \mathcal{O}(D(w_0)) : \boldsymbol{\mathcal{W}} \in \mathscr{P}_{-N}(w_0) \textit{ for some } N \in \mathbb{Z}_+ \\ & \textit{\& } ({}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, N)}, \dots, {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, 1)}) \textit{ is a regular branch of a backward iteration of } F \Big\} \\ & \text{is a normal family.} \end{aligned} \tag{6.3} $$

Pick a $\varphi \in \mathcal{C}(B; \mathbb{C})$ and let $\varepsilon > 0$. As B is compact, there exists a number $\delta(\varepsilon) > 0$ such that:

$$ |z_1 - z_2| < \delta(\varepsilon) \ \Rightarrow \ |\varphi(z_1) - \varphi(z_2)| < \varepsilon \quad \forall z_1, z_2 \in B. \tag{6.4} $$

We pick a $\zeta \in B$. By taking $w_0 = \zeta$ in the discussion in the previous paragraph, we infer from the normality of the family ${}^{\dagger}\mathscr{F}(\zeta)$ that we can find a sufficiently small number $r(\varepsilon, \zeta) > 0$ such that:

$$ |\xi - \zeta| < r(\varepsilon, \zeta) \text{ and } \xi \in B \ \Rightarrow \ |\psi(\xi) - \psi(\zeta)| < \delta(\varepsilon) \quad \forall \psi \in {}^{\dagger}\mathscr{F}(\zeta). \tag{6.5} $$

Now, for each $\zeta \in B$ write:

$$ \mathscr{IB}(N, \zeta) \ := \ \{(\psi, \boldsymbol{\mathcal{W}}) : \psi \text{ is some } {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, N)} \circ \dots \circ {}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, 1)} \text{ in } {}^{\dagger}\mathscr{F}(\zeta), \ \boldsymbol{\mathcal{W}} \in \mathscr{P}_{-N}(\zeta)\}. $$

If $(\zeta_0, \zeta_{-1}, \dots, \zeta_{-N}; \boldsymbol{\alpha}) =: \boldsymbol{\mathcal{W}}$ is a path of backward iteration (with ζ_0 being the above ζ), basic intersection theory tells us that the local intersection multiplicity of $\Gamma^{\bullet}_{\alpha_j}$ with $\mathbb{P}^1 \times \{\zeta_{-j+1}\}$ at $(\zeta_{-j}, \zeta_{-j+1})$ equals the number of distinct branches ${}^{\dagger}F_{(\boldsymbol{\mathcal{W}}, j)}$ one can construct according to the above inductive prescription (this number is greater than 1 if $\Gamma^{\bullet}_{\alpha_j}$ has a normal-crossing singularity at $(\zeta_{-j}, \zeta_{-j+1})$). From this, and from the iterative construction of the ${}^{\dagger}F_{(\boldsymbol{\mathcal{W}},\, N)}$'s above, it follows that:

$$ \mathsf{Card}[\,\mathscr{IB}(N, \zeta)\,] \ = \ d_1(F)^N \quad \forall \zeta \in B. \tag{6.6} $$

From (6.4), (6.5) and (6.6), we get:

$$ |\mathbb{A}^n_B[\varphi](\xi) - \mathbb{A}^n_B[\varphi](\zeta)| \ = \ \left| \sum_{(\psi, \boldsymbol{\mathcal{W}}) \in \mathscr{IB}(n, \zeta)} d_1(F)^{-n} (\varphi \circ \psi(\zeta) - \varphi \circ \psi(\xi)) \right| $$

$$\leq d_1(F)^{-n} \sum_{(\psi,\boldsymbol{\mathcal{W}})\in\mathscr{IB}(n,\zeta)} |\varphi\circ\psi(\zeta)-\varphi\circ\psi(\xi)| \ < \ \varepsilon$$
$$\forall\xi\in B \text{ such that } |\xi-\zeta|<r(\varepsilon,\xi) \text{ and } \forall n\in\mathbb{Z}_+.$$

The above holds true for each $\zeta\in B$. This establishes Claim 1.

In what follows, the term *unitary spectrum* of an operator on a complex Banach space will mean the set of all eigenvalues of the operator of modulus 1, which we will denote by $\mathsf{Spec}_{\mathsf{U}}$. Observe that $\mathsf{Spec}_{\mathsf{U}}(\mathbb{A}_B)\ni 1$.

Claim 2: $\mathsf{Spec}_{\mathsf{U}}(\mathbb{A}_B)=\{1\}$*, and the eigenspace associated with* 1 *is* $\mathbb{C}$.
The ingredients for proving the above claim are largely those of [15, §4]. However, to make clear the role that the properties of $\mathcal{R}$ play, we shall rework some of the details of Lyubich's argument. Let us fix a $\lambda\in\mathsf{Spec}_{\mathsf{U}}(\mathbb{A}_B)$ and let $\varphi_\lambda\in\mathcal{C}(B;\mathbb{C})$ be an associated eigenfunction. Let $\zeta_*\in B$ be such that $|\varphi_\lambda(\zeta_*)|=\max_B|\varphi_\lambda|$. By definition

$$d_1(F)^{-1} \sum_{(\psi,\boldsymbol{\mathcal{W}})\in\mathscr{IB}(1,\zeta_*)} \varphi_\lambda\circ\psi(\zeta_*) \ = \ \lambda\varphi_\lambda(\zeta_*). \tag{6.7}$$

Since $|\varphi_\lambda(\zeta_*)|=\max_B|\varphi_\lambda|$, and given (6.6), the above equality would fail unless $|\varphi_\lambda\circ\psi(\zeta_*)|=|\varphi_\lambda(\zeta_*)|$ for each ψ occurring in (6.7). It is now obvious from this fact and from (6.7) that $\varphi_\lambda(x)=\lambda\varphi_\lambda(\zeta_*)$ $\forall x\in{}^{\dagger}F(\zeta_*)$. Iterating, we get

$$\varphi_\lambda(x) \ = \ \lambda^n\varphi_\lambda(\zeta_*) \quad \forall x\in{}^{\dagger}F^n(\zeta_*), \ \ n=1,2,3,\dots \tag{6.8}$$

Since, by construction, $B\subset U$, there exists an $x_n\in{}^{\dagger}F^n(\zeta_*)$, $n=1,2,3,\dots$, such that $x_n\longrightarrow a_0$. Therefore, owing to (6.8), the sequence $\{\lambda^n\varphi_\lambda(\zeta_*)\}_{n\in\mathbb{Z}_+}$ is a convergent sequence. As $\varphi_\lambda\not\equiv 0$ (by definition), this implies that $\lambda=1$.

Observe that, having determined that $\lambda=1$, (6.8) also gives

$$\varphi_\lambda(\zeta_*) \ = \ \varphi_\lambda(a_0). \tag{6.9}$$

Note that $\overline{\mathbb{A}[\varphi]}=\mathbb{A}[\overline{\varphi}]$ $\forall\varphi\in\mathcal{C}(B;\mathbb{C})$. Hence, $\mathsf{Re}\varphi_\lambda$ and $\mathsf{Im}\varphi_\lambda$ are also eigenvectors of $\mathbb{A}$ associated to $\lambda=1$. Thus, we have the following analogue of (6.7):

$$d_1(F)^{-1} \sum_{(\psi,\boldsymbol{\mathcal{W}})\in\mathscr{IB}(1,z^\bullet)} \mathsf{Re}\varphi_\lambda\circ\psi(z^\bullet) \ = \ \lambda\mathsf{Re}\varphi_\lambda(z^\bullet),$$

where $z^\bullet\in B$ stands for either a point of global maximum or a point of global minimum of $\mathsf{Re}\varphi_\lambda$. Using the above as a starting point instead of (6.7) and repeating, with appropriate modifications, the argument that begins with (6.7) and ends at (6.9), we get:

$$\min_B(\mathsf{Re}\varphi_\lambda) \ = \ \mathsf{Re}\varphi_\lambda(a_0) \ = \ \max_B(\mathsf{Re}\varphi_\lambda). \tag{6.10}$$

Similarly, we deduce that:

$$\min_B(\mathsf{Im}\varphi_\lambda) \ = \ \mathsf{Im}\varphi_\lambda(a_0) \ = \ \max_B(\mathsf{Im}\varphi_\lambda).$$

Combining the above with (6.10), we conclude that, for any eigenvector φ_λ associated to $\lambda=1$, $\varphi_\lambda\equiv$ *constant*. This establishes Claim 2.

To complete this proof, we need the following:

Result 6.6 (Lyubich, [15])**.** *Let $\mathscr{B}$ be a complex Banach space. Let $A:\mathscr{B}\longrightarrow\mathscr{B}$ be a linear operator such that $\{A^n(v)\}_{n\in\mathbb{Z}_+}$ is a relatively-compact subset of $\mathscr{B}$ for each $v\in\mathscr{B}$. Assume that $\mathsf{Spec}_{\mathsf{U}}(A)=\{1\}$ and that 1 is a simple eigenvalue. Let $h\neq 0$ be an invariant* *vector of A. Then, there exists a linear functional μ that satisfies $\mu \circ A = \mu$ and $\mu(h) = 1$, and such that*

$$A^n(v) \longrightarrow \mu(v)h \quad \text{as } n \to \infty$$

for each $v \in \mathscr{B}$.

Take $\mathscr{B} = \mathcal{C}(B; \mathbb{C})$ in the above theorem. Note that $|\mathbb{A}_B^n[\varphi]|$ is bounded by $\max_B |\varphi|$ for $n = 1, 2, 3, \ldots$ Thus, in view of Claims 1 and 2, $\mathbb{A}_B$ satisfies all the hypotheses of Result 6.6. Hence (recall that the function that is identically 1 on B is an eigenvector of $\mathbb{A}_B$) there is a regular complex Borel measure μ_B on B such that $\int_B 1\, d\mu_B = 1$, and

$$\lim_{n\to\infty} \sup_B \left| \mathbb{A}^n[\varphi] - \int_B \varphi\, d\mu_B \right| \;=\; 0 \quad \forall \varphi \in \mathcal{C}(B; \mathbb{C}).$$

It is clear from the above equation that μ_B is a positive measure. Hence it is a probability measure on B. $\square$

The proof of Theorem 3.5. Let B be any closed set having the properties listed in the conclusion of Proposition 6.5. Let μ_B be the probability measure associated to this B. We claim that μ_F is given by defining:

$$\int_{\mathbb{P}^1} \varphi\, d\mu_F \;:=\; \int_B \varphi|_B\, d\mu_B \quad \forall \varphi \in \mathcal{C}(\mathbb{P}^1; \mathbb{C}).$$

We must show that μ_F does not depend on the choice of B. The proof of this is exactly as given in [15, Theorem 1]. We fix a point $z \in \mathcal{R}$. So, $z \in B$ for any choice of B. Thus,

$$\begin{aligned}
\int_{\mathbb{P}^1} \varphi\, d\mu_F \;&=\; \lim_{n\to\infty} \mathbb{A}_B^n[\varphi](z) && \text{(by Proposition 6.5)} \\
&=\; \lim_{n\to\infty} d_1(F)^{-n} \boldsymbol{\Lambda}^n[\varphi](z), && (\text{since } {}^{\dagger}F^n(z) \subset {}^{\dagger}F^n(\mathcal{R}) \subset \mathcal{R})
\end{aligned}$$

where $\boldsymbol{\Lambda}$ is as described in the passage following (3.3). The last line is independent of B. Hence the claim.

By the above calculation, we also get $\int_{\mathbb{P}^1} 1\, d\mu_F = 1$. Thus, μ_F is a probability measure.

Let U be the open set described at the beginning of the proof of Proposition 6.5. We now define:

$$\begin{aligned}
\mathfrak{B}(\mathcal{R}) \;:=\; \Big\{ & B \subset U : B \text{ is closed, } B^\circ \supset \mathcal{R},\ {}^{\dagger}F(B) \subset B \text{ and there exists a closed} \\
& \text{attractor block } B_* \text{ for } |{}^{\dagger}\Gamma| \text{ s.t. } B \subset B_*^\circ \subset B_* \subset U \Big\}.
\end{aligned}$$

We see from the proof of Proposition 6.5 that, owing to our hypotheses, $\mathfrak{B}$ is non-empty. Hence

$$U(F, \mathcal{R}) \;:= \bigcup\nolimits_{B \in \mathfrak{B}(\mathcal{R})} B^\circ$$

is a non-empty open set that contains $\mathcal{R}$. Let $z \in U(F, \mathcal{R})$. There exists a $B \in \mathfrak{B}$ such that $z \in B^\circ$. A close look at the essential features of its proof reveals that this B has all the properties listed in the conclusion of Proposition 6.5. Consider any $\varphi \in \mathcal{C}(\mathbb{P}^1; \mathbb{C})$. We now apply Lemma 4.3 to get

$$\begin{aligned}
d_1(F)^{-n} \langle (F^n)^*(\delta_z), \varphi \rangle \;&= d_1(F)^{-n} \boldsymbol{\Lambda}^n[\varphi](z) && \text{(from (3.4) and Lemma 4.3)} \\
&= \mathbb{A}_B^n[\varphi](z) && (\text{since } {}^{\dagger}F^n(z) \subset {}^{\dagger}F^n(B^\circ) \subset B)
\end{aligned}$$

$$\longrightarrow \int_{\mathbb{P}^1} \varphi \, d\mu_F \quad \text{as } n \to \infty, \tag{6.11}$$

and this holds true *for any* $z \in U(F, \mathcal{R})$. The last line follows from our observations above on μ_F. Now note that, by construction, ${}^{\dagger}F(z) \subset U(F, \mathcal{R})$ for each $z \in U(F, \mathcal{R})$. Therefore, in view of equation (3.3), it follows from (6.11) that $F^*(\mu_F) = d_1(F)\mu_F$. □

We are now in a position to provide:

The proof of Corollary 3.7. Recall that, by definition, for any Borel measure μ, $F_*(\mu) := ({}^{\dagger}F)^*(\mu)$. Thus, the proof of Corollary 3.7 involves, in each case, applying one of the results above to ${}^{\dagger}F$.

The proof (i) follows from Fact 3.1 applied to ${}^{\dagger}F$.

In view of Definitions 3.3 and 3.4 and the hypothesis of part (ii), ${}^{\dagger}F$ satisfies the conditions of Theorem 3.5 (i.e., with ${}^{\dagger}F$ replacing F). Thus, (ii) follows from Theorem 3.5. □

7. Examples

The purpose of this section is to provide concrete examples that illustrate some of our comments in Sections 1 and 3. We begin by showing that it is easy to construct holomorphic correspondences on $\mathbb{P}^1$ that satisfy all the conditions stated in Theorem 3.5, but, unlike the examples discussed in Remark 3.6, have "large" repellers. Next, we shall discuss one of the classes of holomorphic correspondences studied by Bullett and collaborators. For each correspondence F in this class, $d_0(F) = d_1(F) = 2$, and we shall show that Theorem 3.5 and Corollary 3.7(ii) are applicable to these examples.

7.1. **Holomorphic correspondences addressed by Theorem 3.5 having large repellers.** Choose a complex polynomial p with $\deg(p) \geq 2$ such that its Julia set $\mathcal{J}_p \subsetneq \mathbb{P}^1$ and such that no critical values of p lie in $\mathcal{J}_p$. It follows — see, for instance, [15, §4] — that there is a compact set B such that $B^\circ \supset \mathcal{J}_p$ and avoids the critical values of p, and such that $p^{-1}(B) \subset B^\circ$. Next, choose a polynomial Q with $\deg(Q) \geq 2$ and having the following properties:

a) Q has an attractive fixed point, call it ζ_0, in $\mathcal{J}_p$;

b) B lies in the basin of attraction (under the action of Q) of ζ_0.

In view of (b), we can find a positive integer N that is so large that

- $Q^N(B) \subset B^\circ$;
- $\deg(Q)^N \geq \deg(p)$.

Let us write $q := Q^N$.

Next, we define:

$$\begin{aligned} \Gamma_1 &:= \{([z_0 : z_1], [w_0 : w_1]) : w_0^{\deg(q)} z_1 - w_0^{\deg(q)} z_0 \, q(w_1/w_0) = 0\}, \\ \Gamma_2 &:= \{([z_0 : z_1], [w_0 : w_1]) : w_1 z_0^{\deg(p)} - w_0 z_0^{\deg(p)} p(z_1/z_0) = 0\}. \end{aligned}$$

The projective coordinates are so taken that $[0 : 1]$ stands for the point at infinity.

We set $\Gamma := \Gamma_1 + \Gamma_2$ and let F denote the correspondence determined by Γ. Clearly $d_0(F) = \deg(q) + 1 \geq \deg(p) + 1 = d_1(F)$. In this type of construction, we will have $d_0(F) > d_1(F)$ in general. However, apart from satisfying the rather coarse properties (a) and (b) above, p and Q can be chosen with considerable independence from each other.

Thus, this construction will also produce holomorphic correspondences F that satisfy $d_0(F) = d_1(F)$.

By construction, ${}^{\dagger}F(B) \subset B^{\circ}$. In other words, B is an attractor block for the relation $|{}^{\dagger}\Gamma|$. Therefore, it follows from Result 6.2 and Definition 3.3 that F has a repeller $\mathcal{R} \subset B^{\circ}$; this repeller is just $\omega(B; |{}^{\dagger}\Gamma|)$.

It is easy to see that

$$\bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} {}^{\dagger}F^k(B)} \ \subseteq\ \omega(B; |{}^{\dagger}\Gamma|)$$

(the reader may look up [16, Theorem 5.1] for a proof). The above implies that

$$\mathcal{J}_p \subseteq \mathcal{R} \subseteq B^{\circ}.$$

Thus, the correspondence F defined above has a repeller that is disjoint from the set of critical values of F. Owing to (a) and (b) above, for each $w \in B^{\circ}$, there exists a point $a_n(w) \in {}^{\dagger}F^n(w)$ such that $a_n(w) \longrightarrow \zeta_0$. Hence, $\mathcal{R}$ is a strong repeller. Unlike the examples in Remark 3.6, $\mathcal{R}$ is "large" in a certain sense.

7.2. **On the mating between a quadratic map and a Kleinian group.** The ideas developed in the last section are of relevance to the correspondences — alluded to in Section 1 — introduced by Bullett and his collaborators. We shall examine one such class of correspondences. We shall not elaborate here upon what precisely is meant by the mating between a quadratic map on $\mathbb{P}^1$ and a Kleinian group. The idea underlying this concept is simple, but a precise definition requires some exposition. We will just state here (rather loosely) that such a mating provides a holomorphic correspondence F on $\mathbb{P}^1$, and partitions $\mathbb{P}^1$ into an open set and two-component closed set (denoted below by Λ) — both totally invariant under F — such that the action of the iterates of F on $(\mathbb{P}^1 \setminus \Lambda)$ resembles the action of the given Kleinian group on its regular set, and the iterates of distinguished branches of F and ${}^{\dagger}F$ — on the components of Λ, respectively — resemble the dynamics of the given quadratic map on its filled Julia set. We refer the reader to the introduction of [6] or to [5, §3]. The example we present here is that of a holomorphic correspondence on $\mathbb{P}^1$ that realises a mating between certain faithful discrete representations r in $PSL_2(\mathbb{C})$ of

$$G \ :=\ \text{the } \textit{free} \text{ product of } \mathbb{Z}_2 \text{ and } \mathbb{Z}_3,$$

and a quadratic map $q_c : z \longmapsto z^2 + c$. In this discussion, r and c will be such that:

- q_c is hyperbolic and its filled Julia set, $\mathcal{K}(q_c)$, is homeomorphic to a closed disc;
- the regular set of r, $\Omega(r)$, is connected.

The fact that is pertinent to this discussion is that the mating of the above two objects is realisable as a holomorphic correspondence F on $\mathbb{P}^1$. This is the main result of the article [5] by Bullett and Harvey. For such an F, $d_0(F) = d_1(F) = 2$. (In fact, [5, Theorem 1] establishes the latter fact for a much larger class of maps q_c. For simplicity, however, we shall limit ourselves to the assumptions above.)

We shall show that the above example satisfies all the conditions of Theorem 3.5 and briefly indicate how the hypothesis of Corollary 3.7(ii) applies to it as well.

Let Γ denote the holomorphic 1-chain that determines the correspondence provided by [5, Theorem 1]. Let us list some of the features of F that are relevant to the present discourse (we will have to assume here that readers are familiar with [5]):

1) There exists a closed subset $\Lambda \subset \mathbb{P}^1$ that is totally invariant under F and is the disjoint union of two copies Λ_+ and Λ_- of a homeomorph of a closed disc.
2) There exist two open neighbourhoods U^- and V^- of Λ_- such that
   a) $\overline{V^-} \subset U^-$;
   b) $\pi_1^{-1}(U^-) \cap |\Gamma|$ is the union of the graphs of two functions $f_j^+ \in \mathcal{O}(U^-)$, $j = 1, 2$;
   c) $f_1^+(V^-) = U^-$;
   d) There is a quasiconformal homeomorphism of U^- onto an open neighbourhood ω^- of $\mathcal{K}(q_c)$ that carries Λ_- onto $\mathcal{K}(q_c)$, $\partial\Lambda_-$ onto the Julia set of q_c, is conformal in the interior of Λ_-, and conjugates f_1^+ to $q_c|_{\omega^-}$.
   e) $f_2^+(U^-) \cap U^- = \varnothing$.
3) There exist two open neighbourhoods U^+ and V^+, $\overline{V^+} \subset U^+$, of Λ_+ and a pair of functions $f_1^-, f_2^- \in \mathcal{O}(U^+)$ such that

$$\pi_2^{-1}(U^+) \cap |\Gamma| \;=\; \{(f_1^-(\zeta), \zeta) : \zeta \in U^+\} \cup \{(f_2^-(\zeta), \zeta) : \zeta \in U^+\},$$

   and the analogues of the properties $2(c)$–$2(e)$ obtained by swapping the "+" and the "−" superscripts hold true.
4) $(f_2^+)^{-1} = f_2^-$ and $\overline{U^-} \cap \overline{U^+} = \varnothing$.

We must record that property (2) is stated in [5, §3] without some of the features stated above. However, it is evident from Sections 3 and 4 of [5] that (under the assumptions stated at the beginning of this subsection) there is a hybrid equivalence, in the sense of Douady–Hubbard, between $f_1^+|_{\Lambda_-}$ and $q_c|_{\mathcal{K}(q_c)}$. The Julia set of q_c, $\mathcal{J}(q_c)$, is a strong repeller for q_c in the sense of Definition 3.4. Furthermore (see [15, §4], for instance) there exists a neighbourhood basis of $\mathcal{J}(q_c)$, $\mathcal{N}_1 = \{W_\alpha\}$, say, such that

$$W_\alpha \subset \omega^+ \cap \omega^-, \;\text{ and }\; q_c^{-1}(\overline{W}_\alpha) \subset W_\alpha \quad \forall W_\alpha \in \mathcal{N}_1. \tag{7.1}$$

Also note that, from properties $2(a)$–$2(e)$, the generic number of pre-images of any $w \in U^-$ under F equals 2. From this and (7.1), we can deduce that there exists a closed set B such that $\partial\Lambda_- \subset B^\circ \subset B$, ${}^\dagger F(B) \subset B^\circ$ and

$$\partial\Lambda_- \;=\; \bigcap_{n\geq 0} {}^\dagger F^n(B).$$

Since ${}^\dagger F(B) \subset B^\circ$, it is not hard to show (see [16, Theorem 5.4], for instance, for a proof) that the right-hand side above equals $\omega(B; |{}^\dagger\Gamma|)$. Then, by definition, we have:

I) $\partial\Lambda_-$ is a repeller for F.

Furthermore, as $\mathcal{J}(q_c)$ is a strong repeller for the map q_c, invoking property $2(d)$ (along with the fact that the generic number of pre-images of any $w \in U^-$ under F equals 2) gives us:

II) F has the property described in Definition 3.4 with $\mathcal{R} = \Lambda_-$ and for some annular neighbourhood U of $\partial\Lambda_-$, whence $\partial\Lambda_-$ is a strong repeller.

Now, let $Q : U^- \longrightarrow \omega^-$ be the quasiconformal homeomorphism such that $f_1^+ = Q^{-1} \circ q_c \circ Q$. Assume $\partial\Lambda_-$ contains a critical value of f_1^+. By the construction described above, and by property $2(d)$, $\partial\Lambda_-$ is totally invariant under f_1^+. Hence, there exists a point $\zeta_0 \in \partial\Lambda_-$ such that $(f_1^+)'(\zeta_0) = 0$. There is a small connected open neighbourhood G of ζ_0 such that $G \subset U^-$ and $G \setminus \{\zeta_0\}$ contains two distinct pre-images under f_1^+ of each point in U^- belonging to a sufficiently small deleted neighbourhood of $f_1^+(\zeta_0)$. As

$q_c = Q \circ f_1^+ \circ Q^{-1}$ and Q is a homeomorphism, an analogous statement holds true around $Q(\zeta_0) \in \mathcal{J}(q_c)$. By the inverse function theorem, this is impossible because $\mathcal{J}(q_c)$ contains no critical points of q_c. The last statement is a consequence of hyperbolicity — see, for instance [17, Theorem 3.13]. Hence, our assumption above must be false. Finally, by property 2(e), any critical values of f_2^+ lie away from Λ_-. Thus,

III) F has no critical values on $\partial\Lambda_-$.

We see that the properties (I)–(III) above are precisely the conditions of Theorem 3.5 with $\mathcal{R} = \partial\Lambda_-$.

We conclude our discussion of the present example by observing that, in view of property (3) above and an argument analogous to the one that begins with the equation (7.1), with F replacing ${}^\dagger F$ and Λ_+ replacing Λ_-, we can also show that Corollary 3.7(ii) applies to this example.

References

[1] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965) 103-14.
[2] S. Bullett, *Dynamics of quadratic correspondences*, Nonlinearity **1** (1988), no. 1, 27-45.
[3] S. Bullett, *A combination theorem for covering correspondences and an application to mating polynomial maps with Kleinian groups*, Conform. Geom. Dyn. **4** (2000), 75-96.
[4] S. Bullett and M. Freiberger, *Holomorphic correspondences mating Chebyshev-like maps with Hecke groups*, Ergodic Theory Dynam. Systems **25** (2005), no. 4, 1057-1090.
[5] S.R. Bullett and W.J. Harvey, *Mating quadratic maps with Kleinian groups via quasiconformal surgery*, Electron. Res. Announc. Amer. Math. Soc. **6** (2000), 21-30.
[6] S. Bullett, C. Penrose, *Mating quadratic maps with the modular group*, Invent. Math. **115** (1994), no. 3, 483-511.
[7] S. Bullett and C. Penrose, *Regular and limit sets for holomorphic correspondences*, Fund. Math. **167** (2001), no. 2, 111-171.
[8] L. Clozel, H. Oh and E. Ullmo, *Hecke operators and equidistribution of Hecke points*, Invent. Math. **144** (2001), no. 2, 327-351.
[9] J.B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1973.
[10] T.-C. Dinh, *Distribution des préimages et des points périodiques d'une correspondance polynomiale*, Bull. Soc. Math. France **133** (2005), no. 3, 363-394.
[11] T.-C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), no. 1, 221-25.
[12] A. Douady, *Systèmes dynamiques holomorphes*, Séminaire Bourbaki, Vol. 1982/83, 39-63, Astérisque **105-106**, 1983.
[13] F. Kirwan, *Complex Algebraic Curves*, London Mathematical Society Student Texts 23, Cambridge University Press, Cambridge, 1992.
[14] T. Lei, *Matings of quadratic polynomials*, Ergodic Theory Dynam. Systems **12** (1992), no. 3, 589-620.
[15] M. Ju. Ljubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 351-385.
[16] R. McGehee, *Attractors for closed relations on compact Hausdorff spaces*, Indiana Univ. Math. J. **41** (1992), no. 4, 1165-1209.
[17] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math. Studies, no. 135, Princeton University Press, Princeton, NJ, 1994.
[18] I.R. Shafarevich, *Basic Algebraic Geometry 1, Varieties in Projective Space* (2nd ed.; translated from the Russian & with notes by Miles Reid), Springer-Verlag, Berlin, 1994.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: `bharali@math.iisc.ernet.in`

Chennai Mathematical Institute, Chennai 603103, India
E-mail address: `shrihari@cmi.ac.in`