THE ROLE OF FOURIER MODES IN EXTENSION THEOREMS OF HARTOGS-CHIRKA TYPE

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ABSTRACT. We generalize Chirka's theorem on the extension of functions holomorphic in a neighbourhood of $\varGamma(F) \cup (\partial D \times D)$ – where D is the open unit disc and $\varGamma(F)$ is the graph of a continuous D-valued function F – to the bidisc. We extend holomorphic functions by applying the Kontinuitätssatz to certain continuous families of analytic annuli, which is a procedure suited to configurations not covered by Chirka's theorem.

1. Introduction and statement of results

This article is motivated by the paper [3] by Chirka, in which the following theorem is proved (in what follows, D will represent the open unit disc in $\mathbb{C}$ with centre at the origin, and given a function F defined in some region in $\mathbb{C}$, $\varGamma(F)$ will denote the graph of F over its domain) :

Theorem (Chirka). *Let $F \in \mathcal{C}(\overline{D}; \mathbb{C})$ and assume that $\sup_{\overline{D}} |F| < 1$. Let Ω be a connected neighbourhood of $\varGamma(F) \cup (\partial D \times D)$ contained in $\mathbb{C} \times D$. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to the bidisc $D \times D$.*

The requirement that $\sup_{\overline{D}} |F| < 1$ is rather essential to the extension theorem stated above (in contrast, refer to [4] for a version by Chirka & Rosay, in which the condition $\sup_{\overline{D}} |F| < 1$ is relaxed, but in which only the functions holomorphic in the union of a neighbourhood of $\varGamma(F)$ with $\{z \in \mathbb{C} \ : \ |z| > 1\} \times D$ – i.e. holomorphic in a *large* domain – extend holomorphically). A pertinent counterexample, when $\sup_{\overline{D}} |F| > 1$, to the sort of holomorphic extension described in Chirka's theorem – i.e. extension from *small* neighbourhoods of $\varGamma(F) \cup (\partial D \times D)$ – is the case when $\varGamma(F)$ is a Wermer disc. We will discuss this example in §4 below.

The strategy of Chirka – inspired by the methods in [7] – is to construct a continuously varying family of functions $\{F_t\}_{t \in [0,1]} \subset \{G \in \mathcal{C}(\mathbb{C}) \mid \lim_{|z| \to \infty} G(z) = 0\}$ such that $F_1 = \widetilde{F}$ and $F_0 \equiv 0$, and such that $\varGamma(F_t)$ is complex-analytic in a neighbourhood of any $(z, F_t(z)) \notin \Omega \cup (\{\ |z| > 1\} \times D)$. Here, $\widetilde{F}$ is any smooth extension of the F provided by the theorem, that satisfies $\widetilde{F}|_{|z| \geq 2} \equiv 0$. Next, one extends $f \in \mathcal{O}(\Omega)$ to $\Omega \cup (\{\ |z| > 1\} \times D)$ via Laurent decomposition. One can now show that the latter can be analytically continued, owing to the Kontinuitätssatz, via $\{F_t\}_{t \in [0,1]}$ to a neighbourhood of the classical Hartogs configuration $\varGamma(F_0) \cup (\partial D \times D)$. The condition $\sup_{\overline{D}} |F| < 1$ is crucial in ensuring that the extension of $f \in \mathcal{O}(\Omega)$ by Laurent decomposition is single-valued. The strategy described fails in $\mathbb{C}^n$, $n > 2$, and Chirka's theorem does not extend to higher dimensions as shown by Rosay's counterexample in [8].

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The results in this paper are motivated by the two-fold aim of :

a) Showing that functions holomorphic in *small* neighbourhoods of a Hartogs-Chirka type configuration $\Gamma(F) \cup (\partial D \times D)$, with $\sup_{\overline{D}} |F| \gg 1$, extend holomorphically to $D \times D$ (in a manner that will be made precise in Theorem 1.1), given that F satisfies suitable restrictions.
b) Extending Chirka's theorem to higher dimensions, and to a reasonably wide class of Hartogs-Chirka type configurations $\Gamma(F) \cup (\partial D \times D^m)$, $m \geq 2$ (in particular, to configurations in which F is not *merely* real-analytic or $\mathcal{C}^\infty$).

Neither of the above seems to be achievable using Chirka's strategy. In this article, we discuss an alternative strategy for invoking the Kontinuitätssatz, and use it to demonstrate new Hartogs-Chirka type extension phenomena.

The first of the above aims is met by the following theorem. But we first present the following notation : if Ω is a domain in $\mathbb{C}^n$, then $(\widetilde{\Omega}, \pi^\Omega)$ will denote the envelope of holomorphy of Ω.

Theorem 1.1. *Let $F \in \mathcal{C}(\overline{D}; \mathbb{C})$ and assume that $\sup_{\partial D} |F| < 1$. Let $A_j(r)$ represent the j^{th} Fourier coefficient of $F(re^{i\bullet})$, $r > 0$, $j \in \mathbb{Z}$. Assume that F satisfies the condition*

$$\sum_{n \in \mathbb{Z}} \frac{|A_n(r)|}{r^n} \ < \ 1 \ \forall r \in (0, 1]. \tag{1.1}$$

Let Ω_1 be a neighbourhood of $\Gamma(F) \cup (\partial D \times D)$ and let Ω_2 be any connected open set satisfying $\partial D \times D \subset \overline{\Omega_2} \subset \Omega_1 \cap (\{ |z| \geq 1\} \times D)$. If $f \in \mathcal{O}(\Omega_1)$, then $f|_{\Omega_2}$ has a holomorphic extension to $D \times D$.

Note that when $\sup_{\overline{D}} |F| > 1$, $\pi^{\Omega_1}(\widetilde{\Omega_1}) \supsetneq D \times D$ in general. For this reason, the usual arguments justifying that f has a single-valued extension to the bidisc fail. This is the reason behind the particular form of the conclusion of Theorem 1.1. Observe that while the condition (1.1) admits F such that the negative Fourier modes of $F(re^{i\bullet})$ are large, it imposes a severe restriction on the sizes of the positive Fourier modes of $F(re^{i\bullet})$ as $r \to 0^+$. One would like to investigate if such severe restrictions on the positive Fourier modes are necessary. This is a valid concern because if we assume that the function F has *only* positive Fourier modes, the condition (1.1) becomes unnecessary. The relevant theorem in this case is

Theorem 1.2. *Let $F \in \mathcal{C}(\overline{D}; \mathbb{C})$ and assume that $\sup_{\partial D} |F| < 1$. Let $A_j(r)$ represent the j^{th} Fourier coefficient of $F(re^{i\bullet})$, $r > 0$, $j \in \mathbb{Z}$. Assume that $A_j \equiv 0 \ \forall j < 0$. Let Ω_1 be a neighbourhood of $\Gamma(F) \cup (\partial D \times D)$ and let Ω_2 be any connected open set satisfying $\partial D \times D \subset \overline{\Omega_2} \subset \Omega_1 \cap (\{ |z| \geq 1\} \times D)$. If $f \in \mathcal{O}(\Omega_1)$, then $f|_{\Omega_2}$ has a holomorphic extension to $D \times D$.*

In a somewhat different direction, we may consider a continuous mapping $F := (F_1, \ldots, F_m) : \overline{D} \to D^m$, $m \geq 2$, and consider the Hartogs-Chirka type configuration $\Gamma(F) \cup (\partial D \times D^m)$. We know that, in general, Chirka's result is not true for such higher-dimensional configurations – see [8]. In contrast, it has been shown by Bharali [2] that Chirka's result does generalize to a certain class of Hartogs-Chirka type configurations. However, the class of real-analytic maps $(F_1, \ldots, F_m)$ studied in [2] is rather restrictive. We show in this paper that that if we impose a condition analogous to condition (1.1) above, we can demonstrate analytic continuation for a considerably less restrictive set of configurations. We make this precise in the following

Theorem 1.3. *Let $F = (F_1, \dots, F_m) \in \mathcal{C}(\overline{D}; \mathbb{C}^m)$. Assume that $F(e^{i\theta}) \in D^m$ $\forall\theta \in [0, 2\pi)$ and let $A_{jk}(r)$ represent the k^{th} Fourier coefficient of $F_j(re^{i\bullet})$, $r > 0$, $k \in \mathbb{Z}$, $j = 1, \dots, m$. Assume that each F_j satisfies the condition*

$$\sum_{n\in\mathbb{Z}} \frac{|A_{jn}(r)|}{r^n} \; < \; 1 \; \forall r \in (0, 1]. \tag{1.2}$$

Let Ω_1 be a neighbourhood of $\Gamma(F) \cup (\partial D \times D^m)$ and let Ω_2 be any connected open set satisfying $\partial D \times D^m \subset \overline{\Omega_2} \subset \Omega_1 \cap (\{\, |z| \geq 1\} \times D^m)$. If $f \in \mathcal{O}(\Omega_1)$, then $f|_{\Omega_2}$ has a holomorphic extension to $D \times D^m$.

We note that if, in Theorem 1.3, F were to satisfy the restriction $F(\zeta) \in D^m$ $\forall\zeta \in \overline{D}$, then all functions $f \in \mathcal{O}(\Omega)$ – where Ω is a connected neighbourhood of $\Gamma(F) \cup (\partial D \times D^m)$ contained in $\mathbb{C} \times D^m$ – would extend to D^m, which is just Chirka's extension phenomenon in a restricted, higher-dimensional setting.

The approach used in the first and the third theorem is to construct a continuous family of analytic annuli which are attached to $\Gamma(H)$ – where H is an appropriately selected perturbation of F – along their inner boundaries, and to $\partial D \times D^m$ (with $m = 1$ in Theorem 1.1 and $m \geq 2$ in Theorem 1.3) along their outer boundaries. Once this family is constructed, analytic continuation is achieved by invoking the Kontinuitätssatz. The proof of Theorem 1.2 uses a similar idea, but involves continuous families of analytic discs. These proofs may be found in §3. The technical construction of the aforementioned families of annuli/discs is carried out in the next section.

In the final section of this paper, we discuss a few examples. Firstly, we show that one can construct Hartogs-Chirka type configurations $\Gamma(F) \cup (\partial D \times D)$ such that $\sup_{\overline{D}} |F|$ is as large as we want and such that functions holomorphic in small neighbourhoods of this configuration extend. Next, we discuss a configuration involving Wermer's disc (see Example 4.2 for a definition) – for which the extension phenomenon occuring in the previous example fails. And lastly, we show how Rosay's counterexample to a higher-dimensional analogue of Chirka's theorem fails to satisfy the hypotheses of Theorem 1.3.

2. Preliminary lemmas

We need a few preliminary lemmas before we can prove our main theorems. In what follows, $\mathrm{Ann}(a; r, R)$ will denote the open annulus with centre at $a \in \mathbb{C}$ and having inner and outer radii r and R respectively, while $D(a; R)$ will denote the open disc of radius R with centre at a. The symbol $\mathcal{C}^\infty(\overline{D}; \mathbb{C}^m)$, $m = 1, 2, \dots$, will denote the class of infinitely differentiable functions on the unit disc, all of whose derivatives extend to continuous functions on $\overline{D}$.

The reader will notice that in the following lemma the hypothesis $G \in \mathcal{C}^\infty(\overline{D}; \mathbb{C})$ is much stronger than is required for the conclusion of Lemma 2.1. The only place where we use this hypothesis is in showing the existence of a certain limit towards the end of the proof. However, stating the strongest versions of Lemmas 2.1 and 2.3 – which are of relatively minimal utility in themselves – merely results in statements that are overly technical. For this reason, the G occuring in Lemmas 2.1-2.4 shall be assumed to be $\mathcal{C}^\infty$.

Lemma 2.1. *Let $G(re^{i\theta}) = \sum_{n=-N}^{N} b_n(r)e^{in\theta}$ and assume that $G \in \mathcal{C}^\infty(\overline{D};\mathbb{C})$. Assume further that*

$$\sum_{n=-N}^{N} \frac{|b_n(r)|}{r^n} \ < \ 1 \ \forall r \in (0,1]. \tag{2.1}$$

Then the holomorphic function

$$\mathcal{A}_r(\zeta) \ = \sum_{n=-N}^{N} b_n(r)\left(\frac{\zeta}{r}\right)^n, \quad \zeta \in \mathrm{Ann}(0;r,1),$$

which belongs to $\mathcal{O}[\mathrm{Ann}(0;r,1)] \cap \mathcal{C}[\overline{\mathrm{Ann}}(0;r,1)]$, satisfies $|\mathcal{A}_r(e^{i\theta})| < 1$. Fix $\nu \in \mathbb{N}$ and let $K \Subset \mathrm{Ann}(0;1/\nu,1)$ be a compact subset. The function $(0,1/\nu] \times K \ni (r,\zeta) \mapsto \mathcal{A}_r(\zeta)$ extends to a continuous function on $[0,1/\nu] \times K$.

Proof. To prove the first part of this lemma, note that

$$|\mathcal{A}_r(e^{i\theta})| \ \leq \sum_{n=-N}^{N} |b_n(r)| \left|\frac{e^{i\theta}}{r}\right|^n \ = \sum_{n=-N}^{N} \frac{|b_n(r)|}{r^n} \ < \ 1. \tag{2.2}$$

We fix $\nu \in \mathbb{N}$ and then fix a compact set $K \Subset \mathrm{Ann}(0;1/\nu,1)$. It is obvious that $(0,\nu] \times K \ni (r,\zeta) \mapsto b_{-n}(r)(r/\zeta)^n$ extends to a continuous function on $[0,1/\nu] \times K$, which simply vanishes when $r = 0$, for each $n = 1,2,\dots,N$. Now consider the function $(r,\zeta) \mapsto b_n(r)(\zeta/r)^n$, $n = 1,2,\dots,N$. Note that (2.1) $\implies$ $|b_n(r)| < r^n \ \forall n = 1,2,\dots,N$. This implies, since G is assumed to be smooth, that each of the latter functions extends continuously to a function $\varphi_n \in \mathcal{C}([0,1/\nu] \times K)$, which is defined as

$$\varphi_n(r,\zeta) \ := \begin{cases} b_n(r)(\zeta/r)^n, & \text{if } (r,\zeta) \in (0,1/\nu] \times K, \\ \\ \dfrac{1}{n!} \left.\dfrac{d^n b_n}{dr^n}\right|_{r=0} \zeta^n, & \text{if } (r,\zeta) \in \{0\} \times K. \end{cases}$$

Since $\mathcal{A}_r$ is a finite sum of the functions $b_n(r)(r/\zeta)^n$, the last two observations establish the second part of this lemma. $\square$

Lemma 2.2. *Let G be as in Lemma 2.1, but assume additionally that $\sup_{\partial D}|G| < 1$. Let Ω_1 be a neighbourhood of $\Gamma(G) \cup (\partial D \times D)$ such as that described in Theorem 1.1. Then*

a) *$\{\mathcal{A}_r\}_{r\in(0,1)}$ is a continuous family in the sense that for a fixed $\zeta_0 \in D \setminus \{0\}$, $r \mapsto \mathcal{A}_r(\zeta_0)$ is continuous in the interval $(0,|\zeta_0|\ )$.*
b) *$\lim_{r\to 0^+} \mathcal{A}_r(\zeta_0)$ exists for each $\zeta_0 \in D \setminus \{0\}$, and there exists a $\psi \in \mathcal{O}(D)$ such that $\psi(\zeta) = \lim_{r\to 0^+} \mathcal{A}_r(\zeta)$ on $D \setminus \{0\}$.*
c) *Define*

$$\mathfrak{K} \ := \ \Gamma(\psi) \cup \left[\ \cup_{0<r<1}\{(\zeta,\mathcal{A}_r(\zeta)) \in \mathbb{C}^2 \mid r < |\zeta| < 1\}\ \right] \setminus \Omega_1.$$

$\mathfrak{K}$ is compact.

Proof. Part (a) and the first half of part (b) are obvious conclusions of Lemma 2.1. Thus, we may define

$$\psi(\zeta) \ := \lim_{|\zeta|>r\to 0^+} \mathcal{A}_r(\zeta) \quad \forall \zeta \in D \setminus \{0\}.$$

Fix a $\nu \in \mathbb{N}$. Lemma 2.1(b) tells us that

$$(\mathcal{A}_r|_{\mathrm{Ann}(0;1/\nu,1)})(\zeta) \ \longrightarrow \psi(\zeta) \text{ uniformly on each compact } K \Subset \mathrm{Ann}(0;1/\nu,1) \text{ as } r \searrow 0\ .$$

We conclude from this statement that

$$\psi|_{\mathrm{Ann}(0;1/\nu,1)} \in \mathcal{O}[\mathrm{Ann}(0;1/\nu,1)] \quad \forall \nu = 2,3,4,\dots \tag{2.3}$$

Before proceeding any further, we comment that the functions $\mathcal{A}_r$ are so constructed that $\Gamma(\mathcal{A}_r)$, $0 < r < 1$, are analytic annuli that are attached to $\Gamma(G)$ along their inner boundaries and – in view of the inequality (2.2) – to $\partial D \times D$ along their outer boundaries. Therefore,

$$|\mathcal{A}_r(\zeta)| \leq \max\left\{\sup_{|\xi|=r} |\mathcal{A}_r(\xi)|,\ 1\right\} \leq \max\left\{\sup_{\overline{D}} |G|,\ 1\right\} \tag{2.4}$$

$\forall \zeta \in \mathrm{Ann}(0;r,1)$ *and for each* $r \in (0,1)$.

By (2.3), ψ is already holomorphic on $D \setminus \{0\}$. The bounds above imply, since $\psi(\zeta)$ is the limit of the $\mathcal{A}_r(\zeta)$'s, provided $\zeta \neq 0$, that $|\psi(\zeta)| \leq \sup_{\xi \neq 0} |G(\xi)|$ in a punctured neighbourhood of the origin. Thus, ψ extends to a holomorphic function on D. This establishes (b).

Notice that by the estimates (2.4) and part (b) of this lemma, $\mathfrak{K}$ is a bounded set. Therefore, it suffices to show that $\mathfrak{K}$ is closed. Now consider a point $(z,w) \notin \Omega_1$ with the property that there exist sequences $\{r(\nu)\}_{\nu \in \mathbb{N}} \subset (0,1)$ and $\{\zeta_\nu\}_{\nu \in \mathbb{N}} \subset D$ such that $r(\nu) \to 0$ as $\nu \to \infty$ and $(\zeta_\nu, \mathcal{A}_{r(\nu)}(\zeta_\nu)) \longrightarrow (z,w)$ as $\nu \to \infty$. It is easy to see that to prove (c), it suffices to show that all such points $(z,w) \in \mathfrak{K}$. Notice that, by construction, there is a $\delta(\Omega_1) > 0$ depending only on Ω_1 such that $(\zeta, \mathcal{A}_r(\zeta)) \in \Omega_1\ \forall r, |\zeta| < \delta(\Omega_1)$. Thus, as $(z,w) \notin \Omega_1$, $z \neq 0$. Now, given that the $\mathcal{A}_r$'s converge uniformly on compact subsets lying away from 0, there exists $\kappa_1 \in \mathbb{N}$ such that

$$|\mathcal{A}_{r(\nu)}(\zeta) - \psi(\zeta)| < \varepsilon/2 \quad \forall \nu \geq \kappa_1,\ \forall \zeta \in \overline{D}(z;|z|/2).$$

Let $\kappa_2 \in \mathbb{N}$ be such that

$$\zeta_\nu \in D(z;|z|/2) \qquad \textit{and} \qquad |w - \mathcal{A}_{r(\nu)}(\zeta_\nu)| < \varepsilon/2 \quad \forall \nu \geq \kappa_2.$$

The above inequalities imply that

$$|w - \psi(\zeta_\nu)| \leq |w - \mathcal{A}_{r(\nu)}(\zeta_\nu)| + |\mathcal{A}_{r(\nu)}(\zeta_\nu) - \psi(\zeta_\nu)| < \varepsilon \quad \forall \nu \geq \max(\kappa_1, \kappa_2).$$

This tells us that $w = \lim_{\nu\to\infty} \psi(\zeta_\nu)$, whence $(z,w) \in \Gamma(\psi) \setminus \Omega_1$. This establishes (c), and concludes our proof. ☐

Lemma 2.3. *Let $G(re^{i\theta}) = \sum_{n=0}^N b_n(r)e^{in\theta}$ – i.e. we assume that $G(re^{i\boldsymbol{\cdot}})$ has no negative Fourier modes. Assume further that $G \in \mathcal{C}^\infty(\overline{D};\mathbb{C})$. Then the holomorphic function*

$$\mathcal{D}_r(\zeta) = \sum_{n=0}^N b_n(r)\left(\frac{\zeta}{r}\right)^n, \quad \zeta \in D(0;r),$$

which belongs to $\mathcal{O}[D(0;r)] \cap \mathcal{C}[\overline{D}(0;r)]$, satisfies $\mathcal{D}_r(re^{i\theta}) = G(re^{i\theta})\ \forall \theta \in [0,2\pi)$. Fix $\nu \in \mathbb{N}$ and let $K \Subset D(0;1-1/\nu)$ be a compact subset. The function $(r,\zeta) \mapsto \mathcal{D}_r(\zeta)$ is a continuous function on $[1-1/\nu,1] \times K$.

The above lemma is a triviality; we merely state it as an element that will be needed in the proof of our next result.

Lemma 2.4. *Let G be as in Lemma 2.3, but assume additionally that $\sup_{\partial D} |G| < 1$. Let Ω_1 be a neighbourhood of $\Gamma(G) \cup (\partial D \times D)$ such as that described in Theorem 1.2. Then*

a) $\{\mathcal{D}_r\}_{r\in(0,1)}$ *is a continuous family in the sense that for a fixed* $\zeta_0 \in D$, $r \mapsto \mathcal{D}_r(\zeta_0)$ *is continuous in the interval* $(\,|\zeta_0|, 1)$.

b) $\lim_{r\to 1^-} \mathcal{D}_r(\zeta)$ *exists for each* $\zeta \in D$, *and this limit defines a holomorphic function* $\psi \in \mathcal{O}(D)$.

c) *Define*

$$\mathfrak{K} \;:=\; \Gamma(\psi) \cup \left[\, \cup_{0<r<1}\{(\zeta, \mathcal{D}_r(\zeta)) \in \mathbb{C}^2 \mid |\zeta| < r\} \,\right] \setminus \Omega_1.$$

$\mathfrak{K}$ *is compact.*

Proof. Part (a) and the first half of part (b) are direct consequences of Lemma 2.3. The inference that

$$\psi(\zeta) \;:=\; \lim_{|\zeta|<r\to 1^-} \mathcal{D}_r(\zeta) \quad \forall \zeta \in D$$

is holomorphic follows from Lemma 2.3. The uniform-convergence argument is exactly analogous to the argument used in proving Lemma 2.2. We therefore omit the details. We remark that

$$\psi(\zeta) \;=\; \sum_{n=0}^{N} b_n(1)\zeta^n.$$

The functions $\mathcal{D}_r$ are so constructed that $\Gamma(\mathcal{D}_r)$, $0 < r < 1$, are analytic discs that are attached to $\Gamma(G)$ along their boundaries. Therefore,

$$|\mathcal{D}_r(\zeta)| \;\leq\; \sup_{|\xi|=r} |\mathcal{D}_r(\xi)| \;\leq\; \sup_{\overline{D}} |G| \quad \forall \zeta \in D(0; r) \text{ and for each } r \geq 1 - 1/\nu.$$

Thus, $\mathfrak{K}$ is a bounded set, and we argue that $\mathfrak{K}$ is closed exactly as we did in Lemma 2.2(c). $\square$

The following lemma is key to the proofs of Theorems 1.1-1.3. Before proving it, we explicitly state the following simple

Fact : *Due to the continuity of the functions F and G occuring in the statements of the various theorems and lemmas above, the associated Fourier coefficients $A_n(r)$ and $b_n(r)$ satisfy* $A_n(0) = b_n(0) = 0\ \forall n \neq 0$.

This fact is used implicitly at several places in the next lemma.

Lemma 2.5. *Let* $F \in \mathcal{C}(\overline{D}; \mathbb{C})$ *and let* $A_j(r)$ *represent the* j^{th} *Fourier coefficient of* $F(re^{i\boldsymbol{\cdot}})$, $r > 0$, $j \in \mathbb{Z}$. *Assume that :*

1) $\sup_{\partial D} |F| < 1$, *and*
2) F *satisfies the condition*

$$\sum_{n\in\mathbb{Z}} \frac{|A_n(r)|}{r^n} \;<\; 1 \quad \forall r \in (0, 1]. \tag{2.5}$$

Given $\varepsilon > 0$ *there exists a function* $G \in \mathcal{C}^\infty(\overline{D}; \mathbb{C})$ *of the form*

$$G(re^{i\theta}) \;=\; \sum_{n=-N}^{N} B_n(r)e^{in\theta},$$

where N is some large positive integer and $B_n \in \mathcal{C}^\infty([0, 1]; \mathbb{C})$, *such that*

- $|F(\zeta) - G(\zeta)| < \varepsilon\ \forall \zeta \in \overline{D}$,
- *G has the property (1) and satisfies the analogue of (2) above (with $B_n(r)$ replacing $A_n(r)$ in* (2.5) *above).*

Furthermore, if property (2) is replaced by

2*) *F has no negative Fourier modes,*

then G can be constructed so that it has property (1) and $B_{-j} \equiv 0$ *for* $j = 1, 2, \ldots, N$.

Proof. Define

$$\begin{aligned} S_m(\theta, r) &:= \sum_{j=-m}^{m} A_j(r)e^{ij\theta} \\ \sigma_n(\theta, r) &:= \frac{S_0(r,\theta) + \cdots + S_n(\theta, r)}{n+1} \end{aligned}$$

Let us first assume that F has properties (1) and (2). Let $\eta > 0$ be so small that

$$\eta \ < \ 1 - \sup_{\partial D} |F|, \tag{2.6}$$

$$\eta + \sum_{n\in\mathbb{Z}} |A_n(r)|/r^n \ < \ 1 \quad \forall r \in (0; 1],$$

and define $\delta := \min(\varepsilon, \eta)$. There exists a natural number $N > 0$ such that

$$|F(re^{i\theta}) - \sigma_N(\theta, r)| \ < \delta/2 \quad \forall (\theta, r) \in [0, 2\pi) \times [0, 1]. \tag{2.7}$$

This above is a consequence of Fejér's theorem. For a *fixed* $r \in [0, 1]$, (2.7) is precisely the statement of Fejér's theorem applied to the periodic function $F(re^{i\bullet})$. However, on examining the proof of Fejér's theorem, one sees that owing to the equicontinuity of the family $\{F(re^{i\bullet})\}_{r\in[0,1]} \subset \mathcal{C}(\mathbb{T})$, the choice of N in (2.7) is uniform in $r \in [0, 1]$.

One sees immediately that if one writes

$$\sigma_N(\theta, r) \ = \ \sum_{j=-N}^{N} a_j(r)e^{ij\theta},$$

then $|a_j(r)| \le |A_j(r)| \ \forall r \in [0, 1]$. For each $j = 1, 2, \ldots, N$, we pick a function $B_{-j}(r)$ which satisfies the following conditions :

i) $B_{-j} \in \mathcal{C}^\infty([0, 1]; \mathbb{C})$,

ii) B_{-j} vanishes to infinite order at $r = 0$, and

iii) $|a_{-j}(r) - B_{-j}(r)| \le \delta/2(2N + 1) \ \forall r \in [0, 1]$,

provided $a_{-j} \not\equiv 0$, $j = 1, 2, \ldots, N$. *If* $a_{-j} \equiv 0$, *we just choose* $B_{-j} \equiv 0$. Note that by our condition on the Fourier coefficients $\{A_n(r)\}_{n\in\mathbb{Z}}$, $|a_j(r)| \le |A_j(r)| \le r^j \ \forall j = 1, 2, \ldots, N, \ r \in [0, 1)$. Let $0 < R_0 < 1$ be a small number such that

$$R_0 \ \le \ \left\{\frac{\delta}{4(2N + 1)}\right\}^{1/j} \quad \forall j = 1, 2, \ldots, N.$$

For each $j = 1, 2, \ldots, N$ we define a function $B_j(r)$ as follows :

$$B_j(r) \ := \ \begin{cases} \alpha_j(r)r^j, & \text{if } r \le R_0, \\ \beta_j(r), & \text{if } r \ge R_0, \end{cases}$$

such that

i)* $B_j \in \mathcal{C}^\infty([0, 1]; \mathbb{C})$,

ii)* α_j vanishes to infinite order at $r = 0$, *iii**) α_j satisfies

$$|\alpha_j(r)| \;\leq\; \sup_{s\leq 1}\frac{|a_j(s)|}{s^j} \quad \forall s\in[0,R_0],$$

*iv**) β_j satisfies

$$|\beta_j(r)-a_j(r)| \;\leq\; \frac{R_0^j\delta}{2(2N+1)} \quad \forall r\in[R_0,1].$$

Finally, define $B_0(r)$ to be any $\mathcal{C}^\infty$ function such that $|B_0(r)-a_0(r)|<\delta/2(2N+1)\ \forall r\in[0,1]$ and such that $B_0-B_0(0)$ vanishes to high order at $r=0$. Now write

$$G(re^{i\theta}) \;=\; \sum_{j=-N}^{N} B_j(r)e^{ij\theta}.$$

We now make some estimates. We first consider the $B_{-j}(r)$'s, $j=1,2,\ldots,N$. Note that the following statements continue to be true trivially if $B_{-j}\equiv 0$ for any $j=1,2,\ldots,N$.

$$\sum_{j=1}^{N}|B_{-j}(r)-a_{-j}(r)| \;\leq\; \sum_{j=1}^{N}\frac{\delta}{2(2N+1)} \;\leq\; \frac{\delta N}{2(2N+1)}, \tag{2.8}$$

$$\begin{aligned}\sum_{j=1}^{N}|B_{-j}(r)|r^j \;&\leq\; \sum_{j=1}^{N} r^j\left\{|a_{-j}(r)|+\frac{\delta}{2(2N+1)}\right\} \\ &\leq\; \sum_{j=1}^{N}|a_{-j}(r)|r^j+\frac{\delta N}{2(2N+1)} \quad \forall r\in[0,1].\end{aligned} \tag{2.9}$$

Next, we consider the $B_j(r)$'s, $j=1,2,\ldots,N$. First, we let $0\leq r\leq R_0$. We use item (*iii**) in the definition of $B_j(r)$ above to get :

$$\begin{aligned}\sum_{j=1}^{N}|B_j(r)-a_j(r)| \;&\leq\; \sum_{j=1}^{N} r^j\left|\alpha_j(r)-\frac{|a_j(r)|}{r^j}\right| \\ &\leq\; \sum_{j=1}^{N} 2R_0^j\,\sup_{s\leq 1}\frac{|a_j(s)|}{s^j} \\ &\leq\; \sum_{j=1}^{N} 2\left\{\frac{\delta}{4(2N+1)}\right\}\,\sup_{s\leq 1}\frac{|A_j(s)|}{s^j} \\ &\leq\; \frac{\delta N}{2(2N+1)}\;,\end{aligned} \tag{2.10}$$

$$\sum_{j=1}^{N}\frac{|B_j(r)|}{r^j} \;\leq\; \sum_{j=1}^{N}\,\sup_{s\leq 1}\frac{|a_j(s)|}{s^j} \quad \forall r\in[0,R_0]. \tag{2.11}$$

And when we consider $R_0 \leq r \leq 1$, we use item (iv^*) in the definition of $B_j(r)$ to get :

$$\sum_{j=1}^{N} |B_j(r) - a_j(r)| \;\leq\; \sum_{j=1}^{N} \frac{R_0^j \delta}{2(2N+1)} \;\leq\; \frac{\delta N}{2(2N+1)}, \tag{2.12}$$

$$\sum_{j=1}^{N} \frac{|B_j(r)|}{r^j} \;\leq\; \sum_{j=1}^{N} \left\{ \frac{|a_j(r)|}{r^j} + \frac{R_0^j \delta}{2(2N+1) R_0^j} \right\} \tag{2.13}$$

$$\leq \sum_{j=1}^{N} \frac{|a_j(r)|}{r^j} + \frac{\delta N}{2(2N+1)} \quad \forall r \in [R_0, 1].$$

Observe that from (2.9) and (2.11), we have

$$\sum_{j=-N}^{N} \frac{|B_j(r)|}{r^j} \;\leq\; \sum_{j=1}^{N} |a_{-j}(r)| r^j + \frac{\delta N}{2(2N+1)} + |a_0(r)| + \frac{\delta}{2(2N+1)} + \sum_{j=1}^{N} \sup_{s \leq 1} \frac{|a_j(s)|}{s^j}$$

$$\leq \sum_{j=-N}^{N} \sup_{s \leq 1} \frac{|a_j(s)|}{s^j} + \frac{\delta(N+1)}{2(2N+1)}$$

$$\leq \sum_{j=-N}^{N} \sup_{s \leq 1} \frac{|A_j(s)|}{s^j} + \frac{\eta}{2} \;<1 \quad \forall r \in [0, R_0]. \tag{2.14}$$

The last inequality follows from the definition of η and the fact that $|a_j(r)| \leq |A_j(r)|\ \forall r \in [0, 1]$. Next, applying (2.9) and (2.13) we get

$$\sum_{j=-N}^{N} \frac{|B_j(r)|}{r^j} \;\leq\; \sum_{j \neq 0} \frac{|a_j(r)|}{r^j} + \frac{2\delta N}{2(2N+1)} + |a_0(r)| + \frac{\delta}{2(2N+1)}$$

$$\leq \sum_{j=-N}^{N} \sup_{s \leq 1} \frac{|a_j(s)|}{s^j} + \frac{\delta}{2}$$

$$\leq \sum_{j=-N}^{N} \sup_{s \leq 1} \frac{|A_j(s)|}{s^j} + \frac{\eta}{2} \;<1 \quad \forall r \in [R_0, 1]. \tag{2.15}$$

From the inequalities (2.6), (2.14) and (2.15), we get

$$\sum_{j=-N}^{N} \frac{|B_j(r)|}{r^j} \;<\; 1 \quad \forall r \in [0, 1],$$

which is to say that G satisfies the analogue of (2), with $B_n(r)$ replacing $A_n(r)$ in the expression (2.5).

We now exploit the estimates (2.8), (2.10) and (2.12), to get

$$|F(re^{i\theta}) - G(re^{i\theta})| \;\leq\; |F(re^{i\theta}) - \sigma_N(re^{i\theta})| + \sum_{j=-N}^{N} |a_j(r) - B_j(r)|$$

$$< \;\frac{\delta}{2} + 2 \cdot \frac{\delta N}{2(2N+1)} + \frac{\delta}{2(2N+1)} \;= \delta. \tag{2.16}$$

Given the way in which δ is defined, we see that G has the property (1), and $|G(\zeta) - F(\zeta)| < \varepsilon\ \forall \zeta \in \overline{D}$. G is, of course, smooth by construction.

Note further that in the above construction, if F has no negative Fourier modes, neither does G. So, if F had the property (2*) instead of property (2), in addition to choosing $B_{-j} \equiv 0$ we would use the same rule by which we selected $B_0(r)$ in the above argument to define $B_j(r),\ j = 1, 2, \dots, N$. It is easy to verify that this modified construction would yield the second part of this lemma. $\square$

3. Proofs of the theorems

3.1. The proof of Theorem 1.1. Let $\varepsilon > 0$ be so small that $F(D) + \eta \subset \Omega_1$ $\forall \eta \in \mathbb{C}$ such that $|\eta| < 2\varepsilon$. By Lemma 2.5, there exists a function $H \in \mathcal{C}^\infty(\overline{D}; \mathbb{C})$ with $H(re^{i\theta}) = \sum_{n=-N}^{N} B_n(r)e^{in\theta}$ such that

$$|H(\zeta) - F(\zeta)| \ < \ \varepsilon \quad \forall \zeta \in \overline{D}, \qquad \sup_{\partial D} |H| \ < \ 1,$$
$$\sum_{n=-N}^{N} \frac{|B_n(r)|}{r^n} \ < \ 1 \quad \forall r \in (0, 1].$$

Let $\delta > 0$ be so small that

- $\sup_{\partial D} |H| + \delta < 1$;
- $\delta + \sum_{n=-N}^{N} |B_n(r)|/r^n < 1$ $\forall r \in (0; 1]$; and
- $H(D) + \eta \subset \Omega_1$ $\forall \eta \in \mathbb{C}$ such that $|\eta| < \delta$.

Define, for each η such that $|\eta| < \delta$

$$\begin{aligned} H^{(\eta)}(\zeta) \ &:= \ H(\zeta) + \eta, \\ \mathcal{A}_r^{(\eta)}(\zeta) \ &:= \ \sum_{n \neq 0} B_n(r) \left(\frac{\zeta}{r}\right)^n + (B_0(r) + \eta), \quad \zeta \in \mathrm{Ann}(0; r, 1), \end{aligned}$$

We apply Lemma 2.2 to $\{\mathcal{A}_r^{(\eta)}\}_{r \in (0;1)}$ for each η, by taking

$$b_n(r) \ = \ B_n(r) \quad \forall n \neq 0, \qquad b_0(r) \ = \ B_0(r) + \eta$$

in that lemma. We conclude that there is a function $\psi \in \mathcal{O}(D)$ such that

1) For any fixed $\zeta_0 \in D \setminus \{0\}$, $r \mapsto \mathcal{A}_r^{(\eta)}(\zeta_0)$ is continuous in $(0, |\zeta_0|\ )$ for $|\eta| < \delta$.
2) For $\zeta \in D \setminus \{0\}$, $\lim_{r \to 0^+} \mathcal{A}_r^{(\eta)}(\zeta) = \psi(\zeta) + \eta$ for $|\eta| < \delta$.
3) For each $\eta \ :\ |\eta| < \delta$, $\mathfrak{K}^{(\eta)}$ is compact, where we define

$$\mathfrak{K}^{(\eta)} \ := \ \Gamma(\psi + \eta) \cup \Big[\cup_{0<r<1} \{(\zeta, \mathcal{A}_r^{(\eta)}(\zeta)) \in \mathbb{C}^2 \mid r < |\zeta| < 1\} \Big] \setminus \Omega_1.$$

In other words, for each fixed η, the family $\{\Gamma(\mathcal{A}_r^{(\eta)})\}_{r \in (0,1)}$ is a continuous family of analytic annuli attached to $\Gamma(H^{(\eta)}) \cup (\partial D \times D)$, which accumulate onto $\Gamma(\psi + \eta)$ as $r \to 0^+$. The analytic annuli $\Gamma(\mathcal{A}_r^{(\eta)})$ with $r \approx 1$ are contained in Ω_1. We can therefore apply the Kontinuitätssatz to conclude that

$$U(\Delta) \ := \ \{(z, w) \in D \times \mathbb{C} \ :\ (w - \psi(z)) \in \Delta\} \ = \ \bigcup_{\eta \in \Delta} \Gamma(\psi + \eta) \ \subset \ \pi^{\Omega_1}(\widetilde{\Omega_1}), \tag{3.1}$$

where Δ is any disc contained in $D(0; \delta)$

There is a canonical holomorphic imbedding of Ω_1 into $\widetilde{\Omega_1}$. We denote this imbedding by $\mathrm{j} : \Omega_1 \hookrightarrow \widetilde{\Omega_1}$. Corresponding to each $f \in \mathcal{O}(\Omega_1)$ there is a holomorphic function on $\widetilde{\Omega_1}$, which we shall denote by $\mathcal{E}(f)$, such that $\mathcal{E}(f)\circ \mathrm{j} = f$. It is now a standard argument – see, for instance [5] or [3] – to show that there exist holomorphic mappings

$$\widetilde{\mathcal{A}_r}(\cdot\,;\eta) : \mathrm{Ann}(0;r,1) \to \widetilde{\Omega_1}\ \forall r \in (0,1),\ \text{and}\quad \widetilde{\psi}(\cdot\,;\eta) : D \to \widetilde{\Omega_1}$$

such that

a) For each η with $|\eta| < \delta$:

$$\pi^{\Omega_1}\circ \widetilde{\mathcal{A}_r}(\zeta;\eta) = (\zeta, \mathcal{A}_r^{(\eta)}(\zeta))\ \forall \zeta \in \mathrm{Ann}(0;r,1) \text{ when } r \in (0,1),\ \text{and}$$
$$\pi^{\Omega_1}\circ \widetilde{\psi}(\zeta;\eta) = (\zeta, \psi(\zeta) + \eta)\ \forall \zeta \in D.$$

b) $\mathrm{j}(\zeta, \mathcal{A}_r^{(\eta)}(\zeta)) = \widetilde{\mathcal{A}_r}(\zeta;\eta)$ wherever the left-hand side is defined, and $\forall r \in (0,1)$.

c) $\mathrm{j}(\zeta, \psi(\zeta) + \eta) = \widetilde{\psi}(\zeta;\eta)$ wherever the left-hand side is defined.

Notice that – in view of item (a) – for each fixed $\zeta \in D$, $\eta \mapsto \widetilde{\psi}(\zeta;\eta)$ is holomorphic.

Let V be the connected component of $(\pi^{\Omega_1})^{-1}(U(\ |\eta| < \delta))$ containing $\mathcal{C}_0 := \mathrm{image}(\widetilde{\psi}(\cdot\,;0))$. For each point $q \in \mathcal{C}_0$ there is a neighbourhood $W(q) \Subset V$ of q such that $\pi^{\Omega_1}|_{W(q)} : W(q) \to \mathbb{C}^2$ is a biholomorphism. Let Δ_* be a disc centered at the origin that is so small that

$$\mathrm{image}(\widetilde{\psi}(\cdot\,;\eta)) \ \subset\ \bigcup_{q\in\mathcal{C}_0} W(q) \quad \forall \eta \in \Delta_*.$$

We define $\Omega^* := U(\Delta_*) \cup \omega_2$, ω_2 being a connected open set satisfying

- $\overline{\Omega_2}\ \subset\ \omega_2\ \subset\ \Omega_1$; and
- $\omega_2 \cap U(\Delta_*)$ is connected,

where Ω_2 is as described in Theorem 1.1, and $U(\Delta_*)$ is as defined by (3.1). Our goal is to map Ω^* into $\widetilde{\Omega_1}$ in such a way that this mapping extends j. This mapping will allow us – given any $f \in \mathcal{O}(\Omega_1)$ – to extend $f|_{\omega_2}$ to Ω^*. But Ω^* is a neighbourhood of a classical Hartogs configuration, whence $f|_{\Omega_2}$ would extend to the bidisc. To this end, we define

$$\widetilde{\mathrm{j}}\,(z,w) \ :=\ \begin{cases} \widetilde{\psi}(z; w - \psi(z)), & \text{if } (z,w) \in U(\Delta_*), \\ \\ \mathrm{j}(z,w), & \text{if } (z,w) \in \omega_2. \end{cases}$$

Note that if $(z,w) \in U(\Delta_*) \cap \omega_2$, then, in view of item (c) above

$$\widetilde{\psi}(z; w - \psi(z)) \ =\ \mathrm{j}(z, \psi(z) + \{w - \psi(z)\}) \ =\ \mathrm{j}(z,w) \quad \forall (z,w) \in U(\Delta_*) \cap \omega_2. \tag{3.2}$$

Thus, $\widetilde{\mathrm{j}}$ is well-defined, and extends j. From our foregoing remarks, $\widetilde{\mathrm{j}}$ is holomorphic. Given any $f \in \mathcal{O}(\Omega_1)$, we define $\widetilde{f} \in \mathcal{O}(\Omega_*)$ by $\widetilde{f} := \mathcal{E}(f)\circ \widetilde{\mathrm{j}}$. In view of (3.2), $\widetilde{f}|_{\omega_2} \equiv f|_{\omega_2}$. Notice that Ω_* is a neighbourhood of $\Gamma(\psi) \cup (\partial D \times D)$, which is the classical Hartogs configuration. Thus $\widetilde{f}$ has a holomorphic extension to $D \times D$, whence $f|_{\Omega_2}$ has a holomorphic extension to $D \times D$. This concludes our proof. □

3.2. The proof of Theorem 1.2. Since the proof of this theorem is similar to that of Theorem 1.1, we shall be brief. Let $\varepsilon > 0$ be so small that $F(D) + \eta \subset \Omega_1\ \forall \eta \in \mathbb{C}$ such that $|\eta| < 2\varepsilon$. By Lemma 2.5, there exists a function $H \in \mathcal{C}^\infty(\overline{D};\mathbb{C})$ with $H(re^{i\theta}) = \sum_{n=0}^{N} B_n(r)e^{in\theta}$ such that

$$|H(\zeta) - F(\zeta)| \ <\ \varepsilon \quad \forall \zeta \in \overline{D}, \qquad \sup_{\partial D} |H| \ <\ 1.$$

Recall that the H that has the two properties above can be so chosen that it has no negative Fourier modes. Let $\delta > 0$ be so small that

- $\sup_{\partial D} |H| + \delta < 1$; and
- $H(D) + \eta \subset \Omega_1$ $\forall \eta \in \mathbb{C}$ such that $|\eta| < \delta$.

For each η such that $|\eta| < \delta$, we define

$$\begin{aligned} H^{(\eta)}(\zeta) &:= H(\zeta) + \eta, \\ \mathcal{D}_r^{(\eta)}(\zeta) &:= \sum_{n=1}^{N} B_n(r) \left(\frac{\zeta}{r}\right)^n + (B_0(r) + \eta), \quad \zeta \in D(0;r), \end{aligned}$$

We apply Lemma 2.4 to $\{\mathcal{D}_r^{(\eta)}\}_{r\in(0;1)}$ for each η, by taking

$$b_n(r) \ = \ B_n(r) \quad \forall n = 1, 2, \dots, N, \qquad b_0(r) \ = \ B_0(r) + \eta$$

in that lemma. For each fixed η, the family $\{\mathit{\Gamma}(\mathcal{D}_r^{(\eta)})\}_{r\in(0,1)}$ is a continuous family of analytic discs which are attached to $\mathit{\Gamma}(H^{(\eta)})$ along their boundaries, and which accumulate onto a holomorphic graph $\mathit{\Gamma}(\psi + \eta)$ as $r \to 1^-$. Furthermore, for $|\eta| < \delta$, each

$$\mathfrak{K}^{(\eta)} \ := \ \mathit{\Gamma}(\psi + \eta) \cup \Big[\cup_{0<r<1} \{(\zeta, \mathcal{D}_r^{(\eta)}(\zeta)) \in \mathbb{C}^2 \mid |\zeta| < r\} \Big] \setminus \Omega_1$$

is compact. We may therefore apply the Kontinuitätssatz to conclude that

$$U(\Delta) \ := \ \{(z, w) \in D \times \mathbb{C} \ : \ (w - \psi(z)) \in \Delta\} \ = \ \bigcup_{\eta\in\Delta} \mathit{\Gamma}(\psi + \eta) \ \subset \ \pi^{\Omega_1}(\widetilde{\Omega_1}), \tag{3.3}$$

where Δ is any disc contained in $D(0;\delta)$

Let $\mathrm{j} : \Omega_1 \hookrightarrow \widetilde{\Omega_1}$ be the canonical holomorphic imbedding of Ω_1 into $\widetilde{\Omega_1}$. As before, there exist holomorphic mappings

$$\widetilde{\mathcal{D}_r}(\,\bullet\,;\eta) : D(0;r) \to \widetilde{\Omega_1} \ \forall r \in (0,1), \ \text{and} \quad \widetilde{\psi}(\,\bullet\,;\eta) : D \to \widetilde{\Omega_1}$$

such that

a) For each η with $|\eta| < \delta$:

$$\begin{aligned} &\pi^{\Omega_1} \circ \widetilde{\mathcal{D}_r}(\zeta;\eta) = (\zeta, \mathcal{D}_r^{(\eta)}(\zeta)) \ \forall \zeta \in D(0;r) \text{ when } r \in (0,1), \text{ and} \\ &\pi^{\Omega_1} \circ \widetilde{\psi}(\zeta;\eta) = (\zeta, \psi(\zeta) + \eta) \ \forall \zeta \in D. \end{aligned}$$

b) $\mathrm{j}(\zeta, \mathcal{D}_r^{(\eta)}(\zeta)) = \widetilde{\mathcal{D}_r}(\zeta;\eta)$ wherever the left-hand side is defined, and $\forall r \in (0,1)$.

c) $\mathrm{j}(\zeta, \psi(\zeta) + \eta) = \widetilde{\psi}(\zeta;\eta)$ wherever the left-hand side is defined.

Arguing exactly as in the proof of Theorem 1.1, we can find a disc $\Delta_* \subset D(0;\delta)$, centered at the origin, and an appropriate open set ω_2 satisfying $\overline{\Omega_2} \subset \omega_2 \subset \Omega_1$, such that if we define

$$\widetilde{\mathrm{j}}\,(z, w) \ := \ \begin{cases} \widetilde{\psi}(z; w - \psi(z)), & \text{if } (z, w) \in U(\Delta_*), \\ \\ \mathrm{j}(z, w), & \text{if } (z, w) \in \omega_2, \end{cases}$$

(where $U(\Delta_*)$ is as defined by (3.3) above), then $\widetilde{\mathrm{j}}$ is holomorphic and well defined. Holomorphicity follows from (a) above, while (c) implies that

$$\widetilde{\psi}(z; w - \psi(z)) \ = \ \mathrm{j}(z, \psi(z) + \{w - \psi(z)\}) \ = \ \mathrm{j}(z, w) \quad \forall (z, w) \in U(\Delta_*) \cap \omega_2. \tag{3.4}$$

We define $\Omega^* := U(\Delta_*)\cup\omega_2$. As before, given any $f \in \mathcal{O}(\Omega_1)$, we define $\widetilde{f} \in \mathcal{O}(\Omega_*)$ by $\widetilde{f} := \mathcal{E}(f)\circ\widetilde{\mathrm{j}}$. In view of (3.4), $\widetilde{f}|_{\omega_2} \equiv f|_{\omega_2}$. Since Ω_* is a neighbourhood of $\Gamma(\psi) \cup (\partial D \times D)$, i.e. is a classical Hartogs configuration, $\widetilde{f}$ has a holomorphic extension to $D \times D$, whence $f|_{\Omega_2}$ has a holomorphic extension to $D \times D$. This concludes our proof. $\square$

3.3. The proof of Theorem 1.3. The proof of Theorem 1.3 proceeds along the same lines as the proof of the first theorem. The essential difference is that we find an $\varepsilon > 0$ such that $F(D)+\eta \subset \Omega$ $\forall \eta \in D(0; 2\varepsilon)^m$, and then apply Lemma 2.5 to the pairs $(F_1, \varepsilon), \dots, (F_m, \varepsilon)$ to obtain a map $G = (G_1, \dots, G_m)$ all of whose components obey the conclusions of Lemma 2.5. Let us write $G_j(re^{i\theta}) = \sum_{n=-N(j)}^{N(j)} B_n(r)e^{in\theta}$, $j = 1, 2, \dots, m$. We apply Lemma 2.1 by defining

$$b_n(r) \ := \ B_{jn}(r), \quad j \neq 0, \qquad b_0(r) \ := \ B_{j0}(r) + \eta,$$

for each $j = 1, \dots, m$, and obtain

$$\mathcal{A}_r^{(\eta)} := (\mathcal{A}_{1,r}^{(\eta)}, \dots, \mathcal{A}_{m,r}^{(\eta)}) : \overline{\mathrm{Ann}}(0; r, 1) \to D^m,$$
$$\mathcal{A}_r^{(\eta)} \in \mathcal{O}[\mathrm{Ann}(0; r, 1)] \cap \mathcal{C}[\overline{\mathrm{Ann}}(0; r, 1); D^m] \text{ for every } \eta \in D(0; \delta)^m \text{ and } \forall r \in (0; 1),$$

where $\delta > 0$ is chosen to be so small that :

- $\sup_{\partial D} |H_j| + \delta < 1$ for $j = 1, \dots, m$;
- $\delta + \sum_{n=-N}^{N} |B_{jn}(r)|/r^n < 1$ $\forall r \in (0; 1]$ and $j = 1, \dots, m$; and
- $H(D) + \eta \subset \Omega_1$ $\forall \eta \in D(0; \delta)^m$.

As before, there exists a D^m-valued function $\psi := (\psi_1, \dots, \psi_m) \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D}; D^m)$ such that for each $\eta \in D(0; \delta)^m$

$$\lim_{r\to 0^+} \mathcal{A}_r^{(\eta)}(\zeta_0) \ = \ \psi(\zeta_0) + \eta \text{ for each fixed } \zeta_0 \in D \setminus \{0\}.$$

Defining

$$H^{(\eta)} \ := \ (H_1(\zeta) + \eta_1, \dots, H_n(\zeta) + \eta_m) \quad \forall \eta = (\eta_1, \dots, \eta_m) \in D(0; \delta)^m,$$
$$\mathfrak{K}^{(\eta)} \ := \ \Gamma(\psi + \eta) \cup \Big[\cup_{0<r<1} \Gamma(\mathcal{A}_r^{(\eta)}) \Big] \setminus \Omega_1,$$

we see that properties (1)–(3) in the proof of Theorem 1.1 hold for $\{\mathcal{A}_r^{(\eta)}\}_{r\in(0,1)}$ and ψ in our new context – the only difference being that the relevant functions are vector-valued, and η varies in a polydisc $D(0; \delta)^m$. Therefore, $\{\Gamma(\mathcal{A}_r^{(\eta)})\}_{r\in(0,1)}$ is a continuous family of analytic annuli attached to $\Gamma(H^{(\eta)}) \cup (\partial D \times D^m)$, which accumulate onto $\Gamma(\psi+\eta)$ as $r \to 0^+$. The analytic annuli $\Gamma(\mathcal{A}_r^{(\eta)})$ with $r \approx 1$ are contained in Ω_1. As before, the Kontinuitätssatz tells us that

$$U(\mathbf{P}) \ := \ \bigcup_{\eta\in\mathbf{P}} \Gamma(\psi + \eta) \ \subset \ \pi^{\Omega_1}(\widetilde{\Omega}_1), \tag{3.5}$$

where $\mathbf{P}$ is any polydisc contained in $D(0; \delta)^m$.

Arguing exactly as before, we can find a sufficiently small polydisc $\mathbf{P}_* \subset D(0; \delta)^m$ centered at the origin, an appropriately chosen domain ω_2 such that $\overline{\Omega_2} \subset \omega_2 \subset \Omega_1$, and a mapping $\widetilde{\mathrm{j}} : U(\mathbf{P}_*) \cup \omega_2$ (here, $U(\mathbf{P}_*)$ is as defined in (3.5) above) such that

- $\widetilde{\mathrm{j}} \in \mathcal{O}(U(\mathbf{P}_*) \cup \omega_2)$, and
- $\widetilde{\mathrm{j}}|_{\omega_2} \equiv \mathrm{j}|_{\omega_2}$.

We use this mapping $\widetilde{\mathrm{j}}$ exactly as in the previous two theorems to complete this proof. $\square$

4. Examples

We begin by showing that one can construct a Hartogs-type configuration $\Gamma(F)\cup(\partial D\times D)$ such that $\sup_{\overline{D}}|F|$ is as large as we want and such that functions holomorphic in any small neighbourhood of this configuration extend to $D\times D$ in the manner described in Theorem 1.1. A related *counterexample* to this sort of a phenomenon is the case when $\Gamma(F)$ is a Wermer disc (see Example 4.2 below for a definition). In that case, *there is no analytic continuation*, provided $\sup_{\overline{D}}|F|$ is sufficiently large. We explain under the heading Example 4.2 why this does not contradict Theorem 1.1

Theorem 1.3 is not true if F occuring therein is replaced by an arbitrary smooth, D^m-valued function. This is the content of Rosay's counterexample in [8]. Under Example 4.3 below, we discuss how Rosay's counterexample fails to meet the hypotheses of Theorem 1.3.

We begin with our first example.

Example 4.1. *An example showing that given any $N\in\mathbb{N}$, there is a $F\in\mathcal{C}^\infty(\overline{D};\mathbb{C})$ such that $\sup_{\overline{D}}|F|>N$, $\sup_{\partial D}|F|<1$, and such that small connected neighbourhoods of $\Gamma(F)\cup(\partial D\times D)$ exhibit the analytic-continuation phenomenon described in Theorem 1.1.*

We are given $N\in\mathbb{N}$. Let $\chi\in\mathcal{C}^\infty(\,[0,\infty);[0,1))$ be a smooth cut-off on $[0,\infty)$ such that

$$\begin{aligned}\chi|_{[1/(N+1),\infty)} &\equiv \frac{N+(1/2)}{N+1},\\ \chi &\equiv 0 \text{ in a relatively open neighbourhood of } 0.\end{aligned}$$

We define

$$F(re^{i\theta}) := \frac{\chi(r)}{r}e^{-i\theta}.$$

Clearly

$$\begin{aligned}|F(e^{i\theta}/(N+1))| &= N+(1/2) > N,\\ |F(\zeta)| &= \frac{N+(1/2)}{N+1} < 1 \quad \forall\zeta\in\partial D,\\ r|A_{-1}(r)| &= \chi(r) \leq \frac{N+(1/2)}{N+1} < 1 \quad \forall r\in(0,1].\end{aligned}$$

These conditions imply that for any connected neighbourhood $\Omega_1\supset\Gamma(F)\cup(\partial D\times D)$, any connected open set Ω_2 satisfying $(\partial D\times D)\subset\overline{\Omega_2}\subset\Omega_1\cap(\{\,|z|\geq 1\}\times D)$, and for any $f\in\mathcal{O}(\Omega_1)$, $f|_{\Omega_2}$ extends holomorphically to $D\times D$.

Wermer presents an example of a function $g\in\mathcal{C}^\infty(\overline{D})$ [6] with the property that $\Gamma(g)$ is totally real, but $g|_{\partial D}\equiv 0$. This allows us to define a function $F=Mg$ – where $M>0$ is sufficiently large – such that the configuration $\Gamma(F)\cup(\partial D\times D)$ resists analytic continuation of the type described in Theorem 1.1. We now explain how the relevant F fails to satisfy the hypothesis of Theorem 1.1.

Example 4.2. *Wermer's disc.*

The graph of the function

$$g(z,\overline{z}) \;=\; \overline{z}(1-|z|^4)+i\overline{z}(1-|z|^2)$$

is a totally-real surface in $\mathbb{C}^2$. This follows from an easy computation; see details in [6, Example 6.1]. Therefore, the domain

$$\mathcal{D}_\delta \;:=\; \{(z,w)\in\mathbb{C}^2 \;:\; |z|<1+\delta,\ |w-g(z)|<\delta\}$$

is a pseudoconvex domain for all $\delta>0$ sufficiently small. Notice that $\mathcal{D}_\delta \supset \partial D\times D(0,\delta)$. Let $\delta^*>0$ be so small that $D\times\{0\}\not\subseteq \mathcal{D}_{\delta^*}$ and $\mathcal{D}_{\delta^*}$ is pseudoconvex. Then, for each domain $\Omega_1\supset \varGamma(g)\cup(\partial D\times D(0;\delta^*))$ such that $\Omega_1\subset\mathcal{D}_{\delta^*}$, there would exist a function $f\in\mathcal{O}(\Omega_1)$ such that f does *not* extend holomorphically to the bidisc $D\times D(0;\delta^*)$, because $\mathcal{D}_{\delta^*}$ is a domain of holomorphy but does not contain $D\times\{0\}$. We now define

$$F(z) \;:=\; \tfrac{1}{\delta^*}g(z), \quad \widetilde{\mathcal{D}_{\delta^*}} \;:=\; \{(z,w)\mid (z,\delta^* w)\in\mathcal{D}_{\delta^*}\}.$$

By construction, $\widetilde{\mathcal{D}_{\delta^*}}$ is a pseudoconvex domain that contains $\varGamma(F)\cup(\partial D\times D)$ but does not contain $D\times\{0\}$. By our preceding remarks, the Hartogs-Chirka type configuration just constructed does not admit analytic continuation in the manner described in Theorem 1.1

Notice that $F(re^{i\theta})=A_{-1}(r)e^{-i\theta}$, where

$$A_{-1}(r) \;=\; \frac{r}{\delta^*}\{\,(1-r^4)+i(1-r^2)\,\}.$$

We will now show that

$$r|A_{-1}(r)| \;=\; \frac{r^2}{\delta^*}\{(1-r^4)^2+(1-r^2)^2\}^{1/2} \;\gg\; 1 \quad \textit{for some } r\in(0,1]\ ,$$

whence Theorem 1.1 is inapplicable to the above configuration. For this purpose, we will need an upper bound for the quantity δ^* introduced above, and we make the following
Claim : $\delta^* < 0.0061$. To see this, we refer to the Berndtsson-Słodkowski inequality – see [1, Prop.2.3/(b)] – determining when a surface of the form

$$\mathcal{S} \;=\; \{(z,w)\in\Omega\times\mathbb{C}\mid |w-G(z)|=e^{-u(z)}\},$$

(here Ω is a domain in $\mathbb{C}$, G and u are smooth functions, and u is real-valued) is pseudoconvex. The desired inequality is

$$\mathcal{S}\text{ is pseudoconvex} \iff -u_{z\overline{z}} \;\le\; e^{2u}|G_{\overline{z}}|^2-e^u|G_{z\overline{z}}+2u_zG_{\overline{z}}|. \tag{4.1}$$

For $\mathcal{D}_{\delta^*}$ to be pseudoconvex, we require that the surface $\mathcal{S}_{\delta^*}:=\{(z,w)\in D\times\mathbb{C}\mid |w-g(z)|=\delta^*\}$ be pseudoconvex. Applying (4.1) to the surface $\mathcal{S}_{\delta^*}$ yields the following restriction on δ^*.

$$0\;<\delta^*\;\le\;\frac{(1-3|z|^4)^2+(1-2|z|^2)^2}{|z|\sqrt{36|z|^4+4}} \quad \forall |z|\le 1.$$

In other words,

$$0\;<\;\delta^*\;\le\;\min_{r\in[0,1]}\frac{(1-3r^4)^2+(1-2r^2)^2}{r\sqrt{36r^4+4}},$$

and one can use any computational software package to show that the right-hand side of the above inequality is greater than 0.0061. Hence the claim.

One can also compute that $\max_{r\in[0,1]} r^2\{(1-r^4)^2+(1-r^2)^2\}^{1/2} \approx 0.456$. Thus

$$\max_{r\in[0,1]} r|A_{-1}(r)| \ > \ \frac{0.456}{0.0061} \ \gg \ 1,$$

which violates condition (1.1).

Example 4.3. *Rosay's counterexample.*

Rosay shows that one can find an arbitrarily small, strictly pseudoconvex neighbourhood Ω of $\partial D\times D^2$ and a D^2-valued function F such that $\mathit{\Gamma}(F)\cup(\partial D\times D^2)$ is holomorphically convex. Specifically

$$\begin{aligned}\Omega \ := \ \{(z_1,z_2,z_3)\in\mathbb{C}^3 \ : \ \{(\,|z_1|^2-1)^2+s_1|z_2|^2\}^N+\left|\tfrac{z_3}{N}\right|^{2N} \\ +\,\alpha\{(\,|z_1|^2-1)^2+s_1|z_2|^2+|z_3|^2\} \ < \ s^{2N}+\alpha s^2\},\end{aligned}$$

where

- $s>0$ is small, and $s_1=s(1-\delta)<s$ for a fixed, small $\delta>0$, and
- One first chooses N large enough that $s_1^{2N}+1/N^{2N}<s^{2N}$, and then chooses α sufficiently small so as to ensure that $\partial D\times D^2\Subset\Omega$.

Write $F=(F_1,F_2)$. In Rosay's construction

$$F_1(re^{i\theta}) \ := \ \kappa\chi(r)e^{i\theta},$$

where $\chi\in\mathcal{C}^\infty[0,1]$ with $0\le\chi\le1$, such that $\chi\equiv1$ off a small relative neighbourhood of $0\in[0,1]$ and $\chi\equiv0$ on a smaller neighbourhood of 0. The quantity κ will be described presently. The function F_2 is required to be identically zero in an open set contained in $\{re^{i\theta} \ : \ r\in\operatorname{supp}(\chi)\ \}$, and to satisfy $\partial F_1/\partial\bar z_1\neq0$ wherever $F_1\equiv0$. Therefore, F_2 will have negative Fourier modes.

Our interest is in examining F_1. The constant κ is so chosen that

$$\mathit{\Gamma}(F)\cap\overline{\Omega} \ = \ \{(Re^{i\theta},\kappa e^{i\theta},0) \ : \ \theta\in[0,2\pi)\ \},$$

and such that $\mathit{\Gamma}(F)\cap\overline{\Omega}$ is a complex-tangential curve in the surface $\partial\Omega\cap\{z_3=0\}$. Write $z_j:=x_j+iy_j,\ j=1,2$. It is easy to determine what the magnitudes of κ and R (which is close to 1) should be by visualizing $\omega:=\overline{\Omega}\cap(\mathbb{R}^2\times\{0\})$. Then (R,κ) are the coordinates of the point of tangency, in the first quadrant, of the line through the origin that is tangential to $\partial\omega$ (then, the complex span of this line contains the tangent line to the curve $\mathit{\Gamma}(F)\cap\overline{\Omega}$). By construction, the point $(1,s/s_1)\in\partial\omega$ lies *below* the line just described, whence the line $x_1=x_2$ lies below this line, in the first quadrant. Thus, in the notation of Theorem 1.3

$$\kappa/R \ := \ A_{11}(R)/R \ > \ 1,$$

whence, by construction

$$A_{11}(r)/r \ > \ 1 \quad \forall r\in\chi^{-1}\{1\}.$$

This violates the condition (1.2).

References

[1] B. Berndtsson, *Levi-flat surfaces with circular sections*, Several Complex Variables, Math. Notes (John Erik Fornaess ed.), Princeton University Press, Princeton, NJ (1993), pp. 136-159.

[2] G. Bharali, *Some generalizations of Chirka's extension theorem*, Proc. Amer. Math. Soc. **129** (2001), 3665-3669.

[3] E.M. Chirka, *Generalized Hartogs' lemma and non-linear $\overline{\partial}$-equation*, Complex Analysis in Contemporary Mathematics (E.M. Chirka, ed.), FAZIS, Moscow (in Russian) (2002).

[4] E.M. Chirka and J.-P. Rosay, *Remarks on the proof of a generalized Hartogs lemma*, Ann. Pol. Math. **70** (1998), 43-47.

[5] E.M. Chirka and E.L. Stout, *A Kontinuitätssatz*, Topics in Complex Analysis (P. Jacóbczak, W. Pleśniak, eds.), Banach Center Publications, Warsaw, **31** (1995), pp. 143-150.

[6] L. Hörmander and J. Wermer, *Uniform approximation on compact sets in* $\mathbb{C}^n$, Math. Scand. **23** (1968), 5-21.

[7] S.M. Ivashkovich and V.V. Shevchishin, *Deformations of noncompact complex curves, and envelopes of meromorphy of spheres* (in Russian), Mat. Sb. **189** (1998), 23-60.

[8] J.-P. Rosay, *A counterexample related to Hartogs' phenomenon (A question by E. Chirka)*, Michigan Math. J. **45** (1998), 529-535.

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