ON PEAK-INTERPOLATION MANIFOLDS FOR $A(\Omega)$ FOR CONVEX DOMAINS IN $\mathbb{C}^n$

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Abstract. Let Ω be a bounded, weakly convex domain in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundary. $A(\Omega)$ is the algebra of all functions holomorphic in Ω and continuous upto the boundary. A submanifold $\boldsymbol{M} \subset \partial\Omega$ is said to be complex-tangential if $T_p(\boldsymbol{M})$ lies in the maximal complex subspace of $T_p(\partial\Omega)$ for each $p \in \boldsymbol{M}$. We show that for real-analytic submanifolds $\boldsymbol{M} \subset \partial\Omega$, if $\boldsymbol{M}$ is complex-tangential, then every compact subset of $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$.

1. Statement of Main Result

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $A(\Omega)$ be the algebra of functions holomorphic in Ω and continuous upto the boundary. Recall that a compact subset $K \subset \partial\Omega$ is called a **peak-interpolation set for $\boldsymbol{A(\Omega)}$** if given any $f \in \mathcal{C}(K)$, $f \not\equiv 0$, there exists a function $F \in A(\Omega)$ such that $F|_K = f$ and $|F(\zeta)| < \sup_K |f|$ for every $\zeta \in \overline{\Omega} \setminus K$.

We are interested in determining when a sufficiently smooth submanifold $\boldsymbol{M} \subset \partial\Omega$ is a peak-interpolation set for $A(\Omega)$. When Ω is a strictly pseudoconvex domain having $\mathcal{C}^2$ boundary, and $\boldsymbol{M}$ is of class $\mathcal{C}^2$, the situation is very well understood; refer to the works of Henkin & Tumanov [5], Nagel [8], and Rudin [10]. In the strictly pseudoconvex setting, $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$ if and only if $\boldsymbol{M}$ is **complex-tangential**, i.e. $T_p(\boldsymbol{M}) \subset H_p(\partial\Omega)$ $\forall p \in \boldsymbol{M}$. Here, and in what follows, for any submanifold $\mathcal{M} \subseteq \partial\Omega$, $T_p(\mathcal{M})$ will denote the real tangent space to $\mathcal{M}$ at the point $p \in \mathcal{M}$, while $H_p(\partial\Omega)$ will denote the maximal complex subspace of $T_p(\partial\Omega)$.

Very little is known, however, when Ω is a weakly pseudoconvex of finite type (There are several notions of type for domains in $\mathbb{C}^n$, $n \geq 3$. We shall not define them at this juncture; the interested reader may refer to [2], [3], [4], [7].). In view of a result by Nagel & Rudin [9], it is still necessary for $\boldsymbol{M}$ to be complex-tangential. However, showing even that any smooth, (topologically) closed complex-tangential arc in $\partial\Omega$ is a peak-interpolation set for $A(\Omega)$, for a general smoothly bounded weakly pseudoconvex domain of finite type, is a difficult problem. This is because doing so would necessarily imply that every point in $\partial\Omega$ is a peak point for $A(\Omega)$. Whether or not this is true for general pseudoconvex domains of finite type is an extremely difficult open question in the theory of functions in several complex variables. In this paper we show that when Ω is a *convex* domain and $\partial\Omega$ and $\boldsymbol{M}$ are real-analytic, it suffices for $\boldsymbol{M}$ to be complex-tangential for it to be a peak-interpolation set for $A(\Omega)$.

Our main result is as follows :

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Theorem 1.1. *Let Ω be a bounded (weakly) convex domain in $\mathbb{C}^n$, $n \geq 2$, having real-analytic boundary, and let $\boldsymbol{M}$ be a real-analytic submanifold of $\partial\Omega$. If $\boldsymbol{M}$ is complex-tangential, then $\boldsymbol{M}$ (and thus, every compact subset of $\boldsymbol{M}$) is a peak-interpolation set for $A(\Omega)$.*

2. Some notation and introductory remarks

In what follows, the notation $\langle\ ,\ \rangle$ will denote the usual real inner product on $\mathbb{R}^d$. Furthermore, given vectors $v, w \in \mathbb{R}^d$ and a real $d \times d$ matrix $M = [m_{jk}]$, the notation $\langle v \mid M \mid w\rangle$ will be defined as

$$\langle v \mid M \mid w\rangle := \sum_{j,k=1}^{d} m_{jk} v_j w_k.$$

In what follows, $B_d(a;r)$ will denote the open ball in $\mathbb{R}^d$ centered at $a \in \mathbb{R}^d$ and having radius r, while $\mathbb{B}_d(a;r)$ will denote the closure of $B_d(a;r)$.

Let ρ be a defining function for $\partial\Omega$. Recall that for $p \in \partial\Omega$ and a vector $v \in T_p(\partial\Omega)$, the second fundamental form for $\partial\Omega$ at p is the quadratic form

$$T_p(\partial\Omega) \ni v \mapsto \langle v \mid (\mathfrak{H}\rho)(p) \mid v\rangle,$$

where $\mathfrak{H}\rho$ denotes the real Hessian of ρ. We define $\mathfrak{N}_p \subseteq T_p(\partial\Omega)$ to be the null space of the second fundamental form at p, i.e. $\mathfrak{N}_p = \{v \in T_p(\partial\Omega) : \langle v \mid (\mathfrak{H}\rho)(p) \mid v\rangle = 0\}$

A final piece of notation : if ϕ is a $\mathcal{C}^1$ function defined in some open set in $\mathbb{C}^n$, $\partial_k\phi$ and $\partial_{\bar{k}}\phi$ will denote

$$\partial_k\phi = \frac{\partial\phi}{\partial z_k}, \qquad \partial_{\bar{k}}\phi = \frac{\partial\phi}{\partial\overline{z}_k}.$$

A standard approach to proving that $\boldsymbol{M} \subset \partial\Omega$ is a peak-interpolation set – $\boldsymbol{M}$, $\partial\Omega$ smooth and $\Omega \Subset \mathbb{C}^n$, $n \geq 2$ – which is encountered in the papers [5] and [10], is to use Bishop's theorem [1], which states :

Theorem 2.1 (Bishop)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $K \subset \partial\Omega$ be a compact subset. If K is a totally-null set – i.e. if for every annihilating measure $\mu \perp A(\Omega)$, $|\mu|(K) = 0$ – then K is a peak-interpolation set for $A(\Omega)$.*

In the above theorem, an **annihilating measure** refers to a regular, complex Borel measure on $\overline{\Omega}$ which, viewed as a bounded linear functional on $\mathcal{C}(\overline{\Omega})$, annihilates $A(\Omega)$.

Bishop's theorem implies that it suffices to show that $\boldsymbol{M}$ is a countable union of totally-null sets, which is the approach taken in [10]. The essential difference between the proof of Theorem 1.1 and the earlier results lies in the very particular manner in which we decompose $\boldsymbol{M} \subset \partial\Omega$, in the weakly convex setting, into countably many totally-null subsets. As we shall see, the manner in which we decompose $\boldsymbol{M}$ is necessitated by the fact that there may be submanifolds of $\boldsymbol{M}$ along which the second fundamental form for $\partial\Omega$ is *not* strictly positive – a phenomenon that is absent in the strictly convex setting.

The proof of Theorem 1.1 relies on four main ingredients. We need, for our proof, to show that

(1) If $\Omega \subset \mathbb{R}^N,\ N \geq 2$, is a convex domain having a smooth boundary that contains no line segments, there cannot be a smooth curve $\sigma : I \to \partial\Omega$ of class $\mathcal{C}^1$ with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ on an entire interval. Consequently – as we will show in Section 3 – if $\boldsymbol{M} \subset \partial\Omega$ is a smooth submanifold, $\mathfrak{N}_\zeta \cap T_\zeta(\boldsymbol{M}) = \{0\}$ for each ζ belonging to an open, dense subset of $\boldsymbol{M}$.

(2) If $\Omega \subset \mathbb{R}^N,\ N \geq 2$, is a bounded convex domain with real-analytic boundary, $\boldsymbol{M} \subset \partial\Omega$ is a real-analytic submanifold and $p \in \boldsymbol{M}$, there is a neighbourhood $V \ni p$ and a stratification of $\boldsymbol{M} \cap V$ into finitely many real-analytic submanifolds (not necessarily closed) of $\partial\Omega \cap V$ such that if $\mathcal{M}$ is a stratum of positive dimension, $T_\zeta(\mathcal{M}) \cap \mathfrak{N}_\zeta = \{0\}\ \forall \zeta \in \mathcal{M}$.

(3) For each stratum $\mathcal{M} \subset \partial\Omega$ of the aforementioned local stratification with $\dim_{\mathbb{R}}(\mathcal{M}) \geq 1$, and for each $q \in \mathcal{M}$, there is a small neighbourhood $U \ni q$ such that the compact $\mathcal{M} \cap \overline{U}$ is a totally-null set.

The central idea in [10] is to show that one can write $\boldsymbol{M} = \cup_{j\in\mathbb{N}} K_j$, where each K_j is compact, in such a manner that each K_j is totally-null. This relies on the ability to construct a family of functions $\{h_\delta\}_{\delta>0} \subset A(\Omega)$ that is uniformly bounded on $\overline{\Omega}$, such that $h_\delta(z) \to 0$ as $\delta \to 0$, for each $z \in \Omega$, and which, in the limit, has a specified behaviour on an $\boldsymbol{M}$-open neighbourhood of K_j. The analogue of this construction, in our context, is the following claim, which is valid in the more general setting of *smoothly* bounded, weakly convex domains. Item (3) above is a consequence of the following claim, which is the last key ingredient in the proof of our main theorem.

(4) Let $\Omega \subset \mathbb{C}^n,\ n \geq 2$, be a bounded, weakly convex domain having a smooth boundary that contains no line segments, and let $\gamma : B_d(0;R) \to \partial\Omega$ be a smooth imbedding whose image is complex-tangential. Also assume that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x$. There exists a $\varrho > 0$ such that if $f \in \mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$, then defining

$$h_\delta(z) = \int_{B_d(0;\varrho)} \frac{\delta^d f(x)/G(x)\ dx}{\Big\{\delta^2 + \sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]\Big\}^d}\ ,\ z \in \overline{\Omega},$$

where G is defined as

$$G(x) = \int_{\mathbb{R}^d} \left\{1 + \frac{1}{2}\sum_{j,k=1}^n \Big(\partial^2_{j\overline{k}}\rho(\gamma(x))\ [d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k + \partial^2_{jk}\rho(\gamma(x))\ [d\gamma(x)v]_j\ [d\gamma(x)v]_k\Big)\right\}^{-d} dv\ ,$$

we have :

(i) $\{h_\delta\}_{\delta>0} \subset A(\Omega)$ and is uniformly bounded on $\overline{\Omega}$,
(ii) $\lim_{\delta\to 0} h_\delta(z) = 0$ if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$,
(iii) $\lim_{\delta\to 0} h_\delta[\gamma(s)] = f(s)\ \forall s \in B_d(0;\varrho)$.

We remark that the object $\gamma[\mathbb{B}_d(0;\varrho)]$ – $\gamma,\ \varrho$ as above – is the prototye for compact sets of the sort described in item (3). Furthermore, we observe that the family of integrals given above is the same as that appearing in [10], although that paper is about a result similar to Theorem 1.1 but which applies only to *strictly* convex domains. Item (4) says that if for every $x \in B_d(0;R)$ the second fundamental form for $\partial\Omega$ is strictly positive on $T_{\gamma(x)}\gamma[B_d(0;R)] \subset T_{\gamma(x)}(\partial\Omega)$, the aforementioned integrals have the estimates similar to those in [10]. We present these estimates in Section 5, Theorem 5.5.

In Section 3, we state and prove propositions relating to item (1) above. The geometric results from Section 3 and real-analyticity are both crucial to the claim made in item (2) above. The existence of a local stratification of $\boldsymbol{M}$ having certain geometric properties is proved in Section 4 below. This local stratification is *essential* to our proof, and the need for it is what necessitates the hypothesis of real-analyticity in Theorem 1.1. We finally complete the proof of Theorem 1.1 in Section 6.

3. Results on Convex Domains

Lemma 3.1. *Let Ω be a convex domain in $\mathbb{R}^N$, $N \geq 2$, having a $\mathcal{C}^2$ boundary and containing no line segments in its boundary. Then, there is no smooth curve $\sigma : I \to \partial\Omega$ of class $\mathcal{C}^1$ with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ $\forall t \in I$ (where I is some interval of the real line).*

Proof. Assume the result is false. Let $\sigma : I \to \partial\Omega$ be a curve with $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ $\forall t \in I$ (I is some interval). Let ρ be a defining function for $\partial\Omega$ with $\|\nabla\rho\| = 1$. For $t \in I$, set

$$n(t) = \nabla\rho(\sigma(t)) \in \mathbb{R}^N,$$
$$H(t) = (\mathfrak{H}\rho)(\sigma(t)) \in \mathbb{R}^{N\times N}.$$

We compute that

$$n'(t) = H(t)\sigma'(t). \tag{3.1}$$

Now, notice that as $\sigma'(t) \in \mathfrak{N}_{\sigma(t)}$ by assumption, and as $H(t)$ is a symmetric matrix and is positive semi-definite on $T_{\sigma(t)}(\partial\Omega)$ $\forall t \in I$, we have

$$\langle H(t)\sigma'(t), v\rangle = \langle\sigma'(t),\ H(t)v\rangle = 0 \quad \forall v \in T_{\sigma(t)}(\partial\Omega),\ \forall t \in I.$$

The last equality follows from the fact that, since $H(t)$ is positive semi-definite on $T_{\sigma(t)}(\partial\Omega)$, for any $v \in T_{\sigma(t)}(\partial\Omega)$ we have

$$0 \leq \langle\sigma'(t) + \lambda v\ ,\ H(t)(\sigma'(t) + \lambda v)\rangle \ = \ \lambda^2\langle v,\ H(t)v\rangle + 2\lambda\langle\sigma'(t),\ H(t)v\rangle \quad \forall\lambda \in \mathbb{R},$$

which forces $\langle\sigma'(t),\ H(t)v\rangle$ to vanish. Thus, by (3.1), $n'(t)$ is orthogonal to $T_{\sigma(t)}(\partial\Omega)$, $\forall t \in I$.

Next, observe that

$$\langle n(t), n(t)\rangle = 1$$
$$\Rightarrow\ 2\langle n'(t), n(t)\rangle = 0, \quad \text{[by differentiating the above equation]}$$

whence $n'(t)$ is orthogonal to the outward unit normal at $\sigma(t)$ for each $t \in I$. We infer, thus, that $n'(t) = 0$ $\forall t \in I$. Thus n is constant on I.

Write $c = n(t)$, and define a function

$$G(s,t) = \langle\sigma(s) - \sigma(t), c\rangle,\ s,t \in I.$$

Clearly

$$\frac{\partial G}{\partial s} = \frac{\partial G}{\partial t} = 0, \quad \text{[since } \sigma'(\cdot) \perp n(\cdot)\text{]}$$

whence $G \equiv$ const. Since $G(s,s) = 0$, $G \equiv 0$. Thus,

$$\sigma(s) - \sigma(t) \in T_{\sigma(t)}(\partial\Omega). \tag{3.2}$$

By the convexity of Ω, the line segment joining $\sigma(s)$ and $\sigma(t)$ must lie in $\overline{\Omega}$. In view of (3.2), this means that the line segment joining $\sigma(s)$ to $\sigma(t)$ lies in $\partial\Omega$. This is a contradiction, whence the initial assumption if false. $\square$

Lemma 3.2. *Let Ω be a convex domain in $\mathbb{R}^N$, $N \geq 2$, having a $\mathcal{C}^2$ boundary and containing no line segments in its boundary, and let $\boldsymbol{M}$ be a submanifold of $\partial\Omega$ of class $\mathcal{C}^2$. Then, the set $\{p \in \boldsymbol{M} \mid T_p(\boldsymbol{M}) \cap \mathfrak{N}_p = \{0\}\ \}$ is open and dense in $\boldsymbol{M}$.*

Proof. Let $\dim_{\mathbb{R}}(\boldsymbol{M}) = d > 0$. Define

$$\mathfrak{S} = \{p \in \boldsymbol{M} \mid T_p(\boldsymbol{M}) \cap \mathfrak{N}_p \supsetneq \{0\}\ \}.$$

Let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be a non-singular parametrization of $\boldsymbol{M}$ near $p \in \boldsymbol{M}$ of class $\mathcal{C}^2$. We will show that $\mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]$ cannot contain an open subset of $\boldsymbol{M}$. Define

$$H(s) = d\gamma(s)^T (\mathfrak{H}\rho)(\gamma(s))\ d\gamma(s),$$
$$\mathcal{N}_s = \{v \in \mathbb{R}^d : H(s)v = 0\},$$

Assume that $\mathrm{int}[\mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]] \neq \emptyset$. Without loss of generality, we may assume that there exists an $\varepsilon_* \in (0,\varepsilon]$ such that $\gamma[B_d(0;\varepsilon_*)] \subset \mathfrak{S} \cap \gamma[B_d(0;\varepsilon)]$.

$$B_d(0;\varepsilon_*) \ = \ \coprod_{j=1}^{d} \{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) = j\}.$$

Now, the function $s \mapsto \dim_{\mathbb{R}}(\mathcal{N}_s)$ is upper semi-continuous. Thus, for each $j = 1, \dots, d$,

$$\begin{aligned}&\{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) = j\}\\ &\qquad = \ \{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) \geq j\} \ \setminus \ \{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) \geq (j+1)\},\end{aligned}$$

where both the sets on the right-hand side are closed in $B_d(0;\varepsilon_*)$. But, since the union of the sets $\{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) = j\}$, $j = 1, \dots, d$, is all of $B_d(0;\varepsilon_*)$, there must exist a $k : 1 \leq k \leq d$ such that $\{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) = k\}$ has a non-empty interior. Thus, there is an open ball $B_d(s_0;\delta) \subset \{s \in B_d(0;\varepsilon_*) \mid \dim_{\mathbb{R}}(\mathcal{N}_s) = k\}$, and it is a standard fact that the subspaces $\mathcal{N}_s$ vary in such a manner that they form a k-dimensional vector bundle over $B_d(s_0;\delta)$. Let F be any non-vanishing section of this bundle. Or equivalently, we can find a vector field $F = (F_1, \dots, F_d) : B_d(s_0;\delta) \to \mathbb{R}^d \setminus \{0\}$ of class $\mathcal{C}^2$ (shrinking $\delta > 0$ if necessary) *that takes values in $\mathcal{N}_s$* for each $s \in B_d(s_0;\delta)$. Then

$$F_1(s)\frac{\partial\gamma}{\partial s_1}(s) + \cdots + F_d(s)\frac{\partial\gamma}{\partial s_d}(s) \in \mathfrak{N}_{\gamma(s)} \quad \forall s \in B(s_0;\delta). \tag{3.3}$$

Let $\sigma : (-a,a) \to B_d(s_0;\delta)$ be the integral curve to F through s_0, i.e.

$$\sigma(0) = \ s_0,$$
$$\sigma'(t) = \ F(\sigma(t)) \quad \forall t \in (-a,a).$$

Then

$$(\gamma \circ \sigma)'(t) = F_1(\sigma(t))\frac{\partial\gamma}{\partial s_1}(\sigma(t)) + \cdots + F_d(\sigma(t))\frac{\partial\gamma}{\partial s_d}(\sigma(t)) \quad \forall t \in (-a,a). \tag{3.4}$$

From (3.3) and (3.4),

$$(\gamma \circ \sigma)'(t) \in \mathfrak{N}_{\gamma\circ\sigma(t)} \quad \forall t \in (-a,a)$$

which is impossible, by Lemma 3.1. Thus, $\mathfrak{S}$ does not contain any open subsets of $\boldsymbol{M}$.

In particular $\boldsymbol{M} \setminus \mathfrak{S} \neq \emptyset$. Consider any point $p \in \boldsymbol{M} \setminus \mathfrak{S}$, and let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be as before. Define $G : B_d(0;\varepsilon) \times S^{d-1} \to \mathbb{R}$ by

$$G(s,v) \ := \ \langle d\gamma(s)v \mid (\mathfrak{H}\rho)(\gamma(s)) \mid d\gamma(s)v\rangle.$$

$G^{-1}[\mathbb{R} \setminus \{0\}]$ is an open set and $G^{-1}[\mathbb{R} \setminus \{0\}] \supset \{0\} \times S^{d-1}$, since $p \in \boldsymbol{M} \setminus \mathfrak{S}$. From this, we infer that there is an $\boldsymbol{M}$-open neighbourhood of p contained in $\boldsymbol{M} \setminus \mathfrak{S}$. This last fact completes the proof. □

4. A Stratification Theorem

In this section, we shall state precisely, and prove, the informally stated fact in item (2) in Section 2. A key fact that we will use to do so is the structure theorem for real-analytic subvarieties of $\mathbb{R}^N$, $N \geq 2$. This theorem is due to Łojasiewicz [6], which we now state.

Theorem 4.1 (Łojasiewicz). *Let F be a non-constant real-analytic function defined in a neighbourhood of $0 \in \mathbb{R}^N$, and assume that $V(F) = F^{-1}\{0\} \ni 0$. Then, there is a small neighbourhood $U \ni 0$ such that $V(F) \cap U$ has the decomposition*

$$V(F) \cap U = \cup_{j=0}^{N-1} S_j,$$

where each S_j is a finite, disjoint union of (not necessarily closed) j-dimensional real-analytic submanifolds contained in U, such that each connected component of S_j is a closed real-analytic submanifold of $U \setminus \left(\cup_{k=0}^{j-1} S_k\right)$, $j = 1, ..., (N-1)$.

We remark that although the above theorem describes the local structure of the zero-set of a single real-analytic function, it, in fact, describes the local structure of a variety near the origin. This is because, given finitely many real-analytic functions $f_1, ..., f_M$ that vanish at the origin, their set of common zeros is exactly the zero-set of the real-analytic function $F := |f_1|^2 + \cdots + |f_M|^2$.

The following theorem is a precise statement of item (2) in Section 2.

Theorem 4.2. *Let Ω be a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, having real-analytic boundary, and let $\boldsymbol{M} \subset \partial\Omega$ be a d-dimensional real-analytic submanifold. Let $p \in \boldsymbol{M}$. There is a neighbourhood $V \ni p$ such that*

$$\boldsymbol{M} \cap V = \cup_{j=0}^{d} M_j, \tag{4.1}$$

where

(i) *Each M_j is a disjoint union of finitely many (not necesarily closed) j-dimensional real-analytic submanifolds contained in V.*
(ii) *Each connected component of M_j is a closed, real-analytic submanifold of $V \setminus \left(\cup_{k=0}^{j-1} M_k\right)$, $j = 1, ..., d$.*
(iii) *For any $j \neq 0$, if $M_{j,\alpha}$ is a connected component of M_j, $\mathfrak{N}_\zeta \cap T_\zeta(M_{j,\alpha}) = \{0\}$ $\forall \zeta \in M_{j,\alpha}$.*

Proof. Fix $p \in \boldsymbol{M}$. Let $\gamma : (B_d(0;\varepsilon), 0) \to (\boldsymbol{M}, p)$ be a real-analytic parametrization of $\boldsymbol{M}$ near p such that $\mathrm{rank}_{\mathbb{R}}[d\gamma(x)]$ is maximal $\forall x$. Consider the real-analytic function $\mathfrak{F} : B_d(0;\varepsilon) \to \mathbb{R}$ defined by

$$\mathfrak{F}(x) = \det\left[d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x))\ d\gamma(x)\right].$$

The matrix in the above expression is simply the pull-back of the Hessian $\mathfrak{H}\rho$ by γ. Since Ω is a bounded domain with real-analytic boundary, $\partial\Omega$ contains no line segments. This is easy to see. Assume that $\partial\Omega$ contains a line segment. Without loss of generality, we may assume that $0 \in \mathbb{R}^N$ and ξ^0 are its end-points. Furthermore, we may assume that, in a neighbourhood U of the origin, $\partial\Omega \cap U = \{x| \ x_N = R(x_1, \dots, x_{N-1})\}$, where R is a real-analytic convex function. Then the function $t \mapsto R(\xi^0 t)$ is a real-analytic function that is identically zero on an interval that has $t = 0$ as an end-point. But by (real-)analytic continuation, $t \mapsto R(\xi^0 t)$ must vanish on a full neighbourhood of $t = 0$, which is a contradiction. Therefore, Ω satisfies the hypotheses of Lemma 3.2, whence $\mathfrak{F} \not\equiv 0$. Without loss of generality, we may assume that if $\mathfrak{F}^{-1}\{0\} \neq \emptyset$, then $\mathfrak{F}^{-1}\{0\} \ni 0$. By Łojasiewicz's theorem [6], there is a neighbourhood $U \ni 0$, $U \subseteq B_d(0;\varepsilon)$, such that

$$\mathfrak{F}^{-1}\{0\} \cap U = \cup_{j=0}^{d-1} \mathcal{S}_j, \tag{4.2}$$

where each $\mathcal{S}_j$ is a disjoint union of finitely many j-dimensional real-analytic submanifolds, and each connected component of $\mathcal{S}_j$ is a closed submanifold of $U \setminus \left(\cup_{k=0}^{j-1} \mathcal{S}_k\right)$, $j = 1, ..., d-1$. Write $\mathcal{S}_d = U \setminus \left(\cup_{j=0}^{d-1} \mathcal{S}_j\right)$.

We plan to demonstrate the present result by induction. We make the following inductive hypothesis :

For $m < d - 1$ we have, shrinking U if necessary, a stratification of U

$$U = \cup_{j=0}^{d} S_j, \tag{4.3}$$

where,

- (a) Each S_j is a disjoint union of finitely many (not necesarily closed) j-dimensional real-analytic submanifolds contained in U.
- (b) Each connected component of S_j is a closed, real-analytic submanifold of $U \setminus \left(\cup_{k=0}^{j-1} S_k\right)$, $j = 1, ..., d$.
- $(c)_m$ For each $k = 0, ..., m$ and each connected component $S_{d-k,\alpha}$, of S_{d-k} , $T_{\gamma(x)}[\gamma(S_{d-k,\alpha})] \cap \mathfrak{N}_{\gamma(x)} = \{0\}$ $\forall x \in S_{d-k,\alpha}$.
- $(d)_m$ $\cup_{j=0}^{(d-m-1)} S_j$ is a real-analytic subvariety of U.

Consider the real-analytic subvariety $\widetilde{V}$ of U given by

$$\widetilde{V} = \left(\cup_{j=0}^{(d-m-1)} S_j\right) \cap \{x \in U : \mathrm{rank}_{\mathbb{R}} \left[d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x)) \ d\gamma(x)\right] \leq (d-m-2)\},$$

where the S_j's come from the stratification in (4.3). We consider $S_{d-m-1,\alpha}$: a connected component of S_{d-m-1}. $\mathcal{M}^\alpha_{d-m-1} = \gamma(S_{d-m-1,\alpha})$ is a real-analytic submanifold contained in $\partial\Omega$. By Lemma 3.2, there is an $\mathcal{M}^\alpha_{d-m-1}$-open set $\mathcal{V}$ such that $T_\zeta(\mathcal{M}^\alpha_{d-m-1}) \cap \mathfrak{N}_\zeta = \{0\}$, $\forall \zeta \in \mathcal{V}$. Write $\mathcal{U} = \left(\gamma|_{S_{d-m-1,\alpha}}\right)^{-1}(\mathcal{V})$. $\mathcal{U}$ is open in $S_{d-m-1,\alpha}$, and for any $x \in \mathcal{U}$ and any $v \in [T_x(S_{d-m-1,\alpha}) \setminus \{0\}]$, $d\gamma(x)v \notin \mathfrak{N}_{\gamma(x)}$. In other words, for each $x \in \mathcal{U}$,

$$\ker\left\{d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x)) \ d\gamma(x)|_{T_x(S_{d-m-1,\alpha})}\right\} = \{0\},$$

where we identify the matrices $d\gamma(x)^T (\mathfrak{H}\rho)(\gamma(x)) \ d\gamma(x)$ with linear transformations. So, for each $S_{d-m-1,\alpha}$, the real-analytic subvariety $(\widetilde{V} \cap S_{d-m-1,\alpha}) \subsetneq S_{d-m-1,\alpha}$. From this, we infer that

$\dim_{\mathbb{R}}(\widetilde{V}) < (d-m-1)$. By Łojasiewicz's theorem, shrinking U if necessary, we have

$$\widetilde{V} \cap U = \cup_{j=0}^{(d-m-2)} \widetilde{\mathcal{S}}_j$$

where each connected component of $\widetilde{\mathcal{S}}_j$ is a closed submanifold of $U \setminus \left(\cup_{k=0}^{j-1} \widetilde{\mathcal{S}}_k\right)$, $j = 1, ..., d-m-2$. Now write

$$\widetilde{S}_j = \begin{cases} S_j \cap U, & \text{if } j \geq (d-m) \\ (S_{d-m-1,\alpha} \setminus \widetilde{V}) \cap U, & \text{if } j = (d-m-1) \\ (S_j \cup \widetilde{\mathcal{S}}_j) \cap U, & \text{if } j \leq (d-m-2), \end{cases}$$

shrinking U further if necessary so that

$$U = \cup_{j=0}^{d} \widetilde{S}_j$$

is a stratification of U that satisfies (a) and (b) above with $\widetilde{S}_j$ replacing S_j. By construction, each connected component $\widetilde{S}_{d-m-1,\alpha}$, of $\widetilde{S}_{d-m-1}$, satisfies $(c)_{m+1}$ and $\left(\cup_{j=0}^{(d-m-2)} \widetilde{S}_j\right) = \widetilde{V} \cap U$ satisfies $(d)_{m+1}$.

Notice that the stratification in (4.2) establishes the case $m = 0$ for the inductive hypothesis above. By induction, therefore, we can find, shrinking U if necessary, a stratification

$$U = \cup_{j=0}^{d} S_j \tag{4.4}$$

where each connected component $S_{j,\alpha}$, of S_j, $j = 1, ..., d$, is a closed, real-analytic submanifold of $U \setminus \left(\cup_{k=0}^{j-1} S_k\right)$, and for each $j \geq 1$ and each α, $T_\zeta[\gamma(S_{j,\alpha})] \cap \mathfrak{N}_\zeta = \{0\}$, $\forall \zeta \in \gamma(S_{j,\alpha})$. We now find a suitably small neighbourhood, say V, of p so that writing

$$M_j = \gamma(S_j) \cap V, \qquad \boldsymbol{M} \cap V = \cup_{j=0}^{d} M_j$$

[where the S_j's come from (4.4)] gives us the result. $\square$

5. Quantitative Results

In this section, we work with bounded convex domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$, having smooth boundaries containing no line segments. Let $\gamma : (B_d(0; R), 0) \to (\partial\Omega, q)$ be a smooth imbedding whose image is complex-tangential, and for which $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x$. For the remainder of this section, γ and $R > 0$ will have the specific meaning just introduced. In the context of Theorem 1.1, given a point $p \in \boldsymbol{M}$, $\gamma[B_d(0; R)]$ serves as the prototype for an open subset of a stratum of positive dimension in the local stratification (4.1) of $\boldsymbol{M}$ near p.

For Ω as above, ρ a smooth defining function for $\partial\Omega$, $\zeta \in \partial\Omega$ and $z \in \mathbb{C}^n$, we write $G(\zeta, z) = \sum_{j=1}^{n} \partial_j \rho(\zeta)(\zeta_j - z_j)$. For a fixed $\zeta \in \partial\Omega$, the equation $G(\zeta, z) = 0$ defines $H_\zeta(\partial\Omega)$, and the real part of $G(z, \zeta)$ is the perpendicular distance of z from $T_\zeta(\partial\Omega)$. Thus, by the convexity of Ω, if $z \in \overline{\Omega}$, $\mathrm{Re}[G(\zeta, z)] \geq 0$ and $G(\zeta, z) = 0 \iff z = \zeta$. In other words, $\{G(\zeta, \bullet)\}_{\zeta \in \partial\Omega}$ is a smoothly varying family of peak functions for $A(\Omega)$.

We now prove a technical lemma, which we will need later in this section.

Lemma 5.1. *Let Ω, γ and R be as described above. For each $r \in (0, R/2)$, there exists an open set $\mathcal{U}(r) \supset \gamma[B_d(0;2r)]$ such that for each $z \in \overline{\Omega} \cap \mathcal{U}(r)$, there exists a $y_r^z \in B_d(0;2r)$ satisfying*

$$\text{(5.1)}\quad \operatorname{Re}\left\{\sum_{j=1}^{n} \partial_j \rho(\gamma(y_r^z))[\gamma_j(y_r^z) - z_j]\right\} \leq \operatorname{Re}\left\{\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x) - z_j]\right\} \quad \textit{for every}$$
$$\textit{x belonging to a small neighbourhood, } U_z \subseteq B_d(0;R), \textit{ of } y_r^z.$$

Proof. In what follows, we will write $\operatorname{Re}\left\{\sum_{j=1}^{n} \partial_j \rho(\zeta)(\zeta_j - z_j)\right\} = F(\zeta, z)$. For $\zeta \in \gamma[B_d(0;R)]$, we define

$$N_\zeta(\gamma;\varepsilon) := \{z \in N_\zeta(\gamma[B_d(0;R)]) : |z - \zeta| < \varepsilon\},$$

where $N_\zeta(\gamma[B_d(0;R)])$ denotes the normal space of $\gamma[B_d(0;R)]$ in $\mathbb{C}^n$ at ζ. Let $\sigma > 0$ be so small that if z lies in a tube around $\gamma[B_d(0;R)]$,

$$\operatorname{dist}[z, \gamma[B_d(0;R)]] \leq \sigma \;\Rightarrow\; \text{there is a unique } x \in B_d(0;R) \text{ such that } z \in \overline{N_{\gamma(x)}(\gamma;\sigma)}.$$

Also we will assume (shrinking $R > 0$ if necessary) that for each $r \in (0, R/2)$ and each $x \in B_d(0;2r)$, $N_{\gamma(x)}(\gamma;2\sigma) \cap \gamma[\partial\mathbb{B}_d(0;2r)] = \emptyset$. We now fix $r \in (0, R/2)$ for the remainder of this proof. For each $t \in (0, 2r)$, define the function $\mathfrak{F}_t : \overline{\Omega} \cap \left(\cup_{|x| \leq t} \overline{N_{\gamma(x)}(\gamma;\sigma)}\right) \times \gamma[\partial\mathbb{B}_d(0;2r)] \to [0, \infty)$ by

$$\mathfrak{F}_t : (z, \xi) \mapsto F(\xi, z).$$

For a fixed $t \in (0, 2r)$, $\mathfrak{F}_t(z, \xi) > 0$, by convexity and by the foregoing choices for R and σ. Thus, there exists a $m_t > 0$ such that $\mathfrak{F}_t(z, \xi) \geq m_t \; \forall (z, \xi) \in \overline{\Omega} \cap \left(\cup_{|x| \leq t} \overline{N_{\gamma(x)}(\gamma;\sigma)}\right) \times \gamma[\partial\mathbb{B}_d(0;2r)]$. Write $s_t = \min\{m_t/2, \sigma\}$. Then

$$\text{(5.2)}\quad z \in \overline{\Omega} \cap \left(\cup_{|x| \leq t} \overline{N_{\gamma(x)}(\gamma;s_t)}\right)$$
$$\Rightarrow\; F(\zeta^z, z) = \operatorname{dist}[z, T_{\zeta^z}(\partial\Omega)] \leq s_t < F(\xi, z), \quad \forall \xi \in \gamma[\partial\mathbb{B}_d(0;2r)],$$

where $\zeta^z \in \gamma[\mathbb{B}_d(0;t)]$ such that $z \in N_{\zeta^z}(\gamma[B_d(0;R)])$. We define

$$\mathcal{U}(r) = \operatorname{int}\left[\cup_{t \in (0,2r)} \left\{\cup_{|x| \leq t} N_{\gamma(x)}(\gamma;s_t)\right\}\right],$$

and for each $z \in \overline{\Omega} \cap \mathcal{U}(r)$, we define y_r^z by

$$F(\gamma(y_r^z), z) = \inf_{x \in \mathbb{B}_d(0;2r)} F(\gamma(x), z).$$

If we could show that $y_r^z \notin \partial\mathbb{B}_d(0;2r)$, then we would be done. For each $z \in \mathcal{U}(r)$, let $x^z \in B_d(0;2r)$ be such that $z \in N_{\gamma(x^z)}(\gamma[B_d(0;R)])$. If $z \in \cup_{|x| \leq t} N_{\gamma(x)}(\gamma;s_t)$ for some $t \in (0, 2r)$, then $|x^z| \leq t$. In view of (5.2)

$$F(\gamma(y_r^z), z) \;\leq\; F(\gamma(x^z), z) \;<\; F(\gamma(s), z) \quad \forall s \in \partial\mathbb{B}_d(0;2r).$$

Hence, $y_r^z \notin \partial\mathbb{B}_d(0;2r)$, and we have our result. □

So far, we have not made use of the fact that $\gamma[B_d(0;R)]$ is complex-tangential. We shall do so in the next three lemmas.

Lemma 5.2. *Let Ω, γ, $y_r^z \in B_d(0;2r)$ and $\mathcal{U}(r)$ be as in Lemma 5.1. Then, for $z \in \overline{\Omega} \cap \mathcal{U}(r)$*

$$\operatorname{Re}\left\{\sum_{j,k=1}^{n} \partial^2_{jk}\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z)-z_j] + \partial^2_{j\overline{k}}\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z)-z_j]\right\} = 0, \quad \mu = 1, ..., d. \tag{5.3}$$

Proof. If $z \in \overline{\Omega} \cap \mathcal{U}(r)$, then y_r^z is a local minimum of the function

$$B_d(0;2r) \ni x \mapsto F(\gamma(x), z).$$

Therefore, taking the partial derivative of the above with respect to x_μ and evaluating at $x = y_r^z$, we get

$$\operatorname{Re}\left\{\sum_{j,k=1}^{n} \partial^2_{jk}\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z)-z_j] + \sum_{j,k=1}^{n}\partial^2_{j\overline{k}}\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z)-z_j] + \sum_{j=1}^{n}\partial_j\rho(\gamma(y_r^z))\frac{\partial\gamma_j}{\partial x_\mu}(y_r^z)\right\} = 0, \ \mu = 1, ..., d. \tag{5.4}$$

Since $\gamma[B_d(0;R)]$ is complex-tangential, we have

$$\sum_{j=1}^{n}\partial_j\rho(\gamma(y_r^z))\ \frac{\partial\gamma_j}{\partial x_\mu}(y_r^z) = 0, \ \mu = 1, ..., d. \tag{5.5}$$

The result follows from (5.4) and (5.5). $\square$

In the next lemma, we exploit the fact that $d\gamma(x)(\mathbb{R}^d) \cap \mathfrak{N}_{\gamma(x)} = \{0\}\ \forall x \in B_d(0;R)$.

Lemma 5.3. *Let Ω and γ be as described above. There exist uniform constants $\delta \equiv \delta(\gamma) > 0$ and $C \equiv C(\gamma) > 0$ such that*

$$\operatorname{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)]\right\} \geq C\ |x-y|^2\ \forall x, y \in B_d(0,\delta). \tag{5.6}$$

Proof. Let $\eta \in \partial\Omega$ and $z \in \overline{\Omega}$. We Taylor expand the function

$$\eta \mapsto \sum_{j=1}^{n}\partial_j\rho(\eta)(\eta_j - z_j)$$

about $\eta = \xi$ to get

$$\begin{aligned}\sum_{j=1}^{n}\partial_j\rho(\eta)(\eta_j-z_j) = {} & \sum_{j=1}^{n}\partial_j\rho(\xi)(\xi_j-z_j) + \sum_{j=1}^{n}\partial_j\rho(\xi)(\eta_j-\xi_j) + \sum_{j,k=1}^{n}\partial^2_{jk}\rho(\xi)(\xi_j-z_j)(\eta_k-\xi_k) \\ & + \sum_{j,k=1}^{n}\partial^2_{j\overline{k}}\rho(\xi)(\xi_j-z_j)(\overline{\eta_k}-\overline{\xi_k}) + \sum_{j,k=1}^{n}\partial^2_{jk}\rho(\xi)(\eta_j-\xi_j)(\eta_k-\xi_k) \\ & + \sum_{j,k=1}^{n}\partial^2_{j\overline{k}}\rho(\xi)(\eta_j-\xi_j)(\overline{\eta_k}-\overline{\xi_k}) + O(|\xi-z||\eta-\xi|^2,\ |\eta-\xi|^3).\end{aligned}$$

Substituting $z = \gamma(y)$, $\eta = \gamma(x)$ and $\xi = \gamma(y)$ in the above expression, we get

(5.7)
$$\begin{aligned}\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)] &= \sum_{j=1}^{n} \partial_j \rho(\gamma(y))[\gamma_j(x)-\gamma_j(y)] + \sum_{j,k=1}^{n} \partial^2_{jk}\rho(\gamma(y))[\gamma_j(x)-\gamma_j(y)][\gamma_k(x)-\gamma_k(y)]\\ &+ \sum_{j,k=1}^{n} \partial^2_{j\overline{k}}\rho(\gamma(y))[\gamma_j(x)-\gamma_j(y)][\overline{\gamma_k(x)}-\overline{\gamma_k(y)}] + O(|\gamma(x)-\gamma(y)|^3),\end{aligned}$$

for $x, y \in B_d(0;R)$. Taylor expanding γ around $y \in B_d(0;R)$, and substituting in (5.7), we have

(5.8)
$$\begin{aligned}\sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)] &= \sum_{j=1}^{n} \partial_j \rho(\gamma(y)) \left\{ \sum_{\mu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)(x_\mu - y_\mu) + \frac{1}{2} \sum_{\mu,\nu=1}^{d} \frac{\partial^2 \gamma_j}{\partial x_\mu \partial x_\nu}(y)(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &+ \sum_{j,k=1}^{n} \partial^2_{jk}\rho(\gamma(y)) \left\{ \sum_{\mu,\nu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \frac{\partial \gamma_k}{\partial x_\nu}(y)(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &+ \sum_{j,k=1}^{n} \partial^2_{j\overline{k}}\rho(\gamma(y)) \left\{ \sum_{\mu,\nu=1}^{d} \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \overline{\frac{\partial \gamma_k}{\partial x_\nu}(y)}(x_\mu - y_\mu)(x_\nu - y_\nu) \right\}\\ &+ O(|x-y|^3) \quad \forall x, y \in B_d(0;R).\end{aligned}$$

We now use the fact that $\gamma[B_d(0;R)]$ is complex-tangential. For $y \in B_d(0;R)$, we have

$$\sum_{j=1}^{n} \partial_j \rho(\gamma(y))\ \frac{\partial \gamma_j}{\partial x_\mu}(y) = 0, \quad \mu = 1, ..., d. \tag{5.9}$$

Differentiating the above expression with respect to x_ν and evaluating at $x = y$ yields

$$\begin{aligned}\sum_{j,k=1}^{n} \partial^2_{jk}\rho(\gamma(y)) \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \frac{\partial \gamma_k}{\partial x_\nu}(y) &+ \sum_{j,k=1}^{n} \partial^2_{j\overline{k}}\rho(\gamma(y)) \frac{\partial \gamma_j}{\partial x_\mu}(y)\ \overline{\frac{\partial \gamma_k}{\partial x_\nu}(y)}\\ &+ \sum_{j=1}^{n} \partial_j \rho(\gamma(y)) \frac{\partial^2 \gamma_j}{\partial x_\mu \partial x_\nu}(y) = 0, \quad \mu, \nu = 1, ..., d.\end{aligned} \tag{5.10}$$

From (5.8), (5.9) and (5.10), we get

$$\mathrm{Re}\left\{ \sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)] \right\} = \frac{1}{2} \langle\ d\gamma(y)(x-y) \mid (\mathfrak{H}\rho)(\gamma(y)) \mid d\gamma(y)(x-y)\ \rangle + O(|x-y|^3).$$

This statement, in conjunction with the strict positivity of $(\mathfrak{H}\rho)(\zeta)$ on $T_\zeta(\gamma[B_d(0;R)]) \subset T_\zeta(\partial\Omega)\ \forall \zeta \in \gamma[B_d(0;R)]$, allows us to infer that there are uniform constants $\delta \equiv \delta(\gamma) > 0$ and $C \equiv C(\gamma) > 0$ such that

$$\mathrm{Re}\left\{ \sum_{j=1}^{n} \partial_j \rho(\gamma(x))[\gamma_j(x)-\gamma_j(y)] \right\} \geq C\ |x-y|^2 \quad \forall x, y \in B_d(0;\delta).$$

□

Lemma 5.4. *Let Ω, γ and R be as described above. Then*

$$\lim_{\delta\to 0}\ \mathrm{Re}\left\{\frac{1}{\delta^2}\sum_{j=1}^{n}\partial_j\rho(\gamma(x+\delta v))[\gamma_j(x+\delta v)-\gamma_j(x)]\right\}$$
$$=\frac{1}{2}\sum_{j,k=1}^{n}\left(\partial^2_{j\overline{k}}\rho(\gamma(x))\ [d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k+\partial^2_{jk}\rho(\gamma(x))\ [d\gamma(x)v]_j\ [d\gamma(x)v]_k\right) \tag{5.11}$$

for any $x\in B_d(0;R)$ and $v\in\mathbb{R}^d$.

Proof. Follows from (5.8), (5.9) and (5.10) in the proof of Lemma 5.3 $\square$

Theorem 5.5. *Let $\Omega\subset\mathbb{C}^n$, $n\geq 2$, be a bounded, weakly convex domain having a smooth boundary that contains no line segments, and let $\gamma: B_d(0;R)\to\partial\Omega$ be a smooth imbedding whose image is complex-tangential. Also assume that $d\gamma(x)(\mathbb{R}^d)\cap\mathfrak{N}_{\gamma(x)}=\{0\}\ \forall x$. There exists a $\varrho\equiv\varrho(\gamma)>0$ such that if $f\in\mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$, then defining*

$$h_\delta(z)=\int_{B_d(0;\varrho)}\frac{\delta^d f(x)/G(x)\ dx}{\left\{\delta^2+\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-z_j]\right\}^d}\ ,\ z\in\overline{\Omega}, \tag{5.12}$$

where G is defined as

(5.13) $G(x)$

$$=\int_{\mathbb{R}^d}\left\{1+\frac{1}{2}\sum_{j,k=1}^{n}\left(\partial^2_{j\overline{k}}\rho(\gamma(x))\ [d\gamma(x)v]_j\ \overline{[d\gamma(x)v]}_k+\partial^2_{jk}\rho(\gamma(x))\ [d\gamma(x)v]_j\ [d\gamma(x)v]_k\right)\right\}^{-d}dv\ ,$$

we have

- *(i)* $\{h_\delta\}_{\delta>0}\subset A(\Omega)$ *and is uniformly bounded on* $\overline{\Omega}$,
- *(ii)* $\lim_{\delta\to 0}h_\delta(z)=0$ *if* $z\in\overline{\Omega}\setminus\gamma[B_d(0;\varrho)]$,
- *(iii)* $\lim_{\delta\to 0}h_\delta[\gamma(s)]=f(s)\ \forall s\in B_d(0;\varrho)$.

Proof. Consider any $r\in(0,R/3)$ and any $z\in\overline{\Omega}\cap\mathcal{U}(r)$, where $\mathcal{U}(r)$ is as described in Lemma 5.1. We first estimate the quantity

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y_r^z+x))-\partial_j\rho(\gamma(y_r^z))\right]\left[\gamma_j(y_r^z)-z_j\right]\right\},$$

(where y_r^z is as introduced in Lemma 5.1) given that $(y_r^z+x)\in B_d(0;R)$. Taylor expanding about y_r^z and using the complex-tangency hypothesis for $\gamma[B_d(0;R)]$, we have

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y_r^z+x))-\partial_j\rho(\gamma(y_r^z))\right]\left[\gamma_j(y_r^z)-z_j\right]\right\}$$
$$=\mathrm{Re}\left\{\sum_{\mu=1}^{d}\left[\sum_{j,k=1}^{n}\partial^2_{jk}\rho[\gamma(y_r^z)]\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)[\gamma_j(y_r^z)-z_j]+\partial^2_{j\overline{k}}\rho[\gamma(y_r^z)]\overline{\frac{\partial\gamma_k}{\partial x_\mu}(y_r^z)}[\gamma_j(y_r^z)-z_j]\right]x_\mu\right\}$$
$$+O(|x|^2)|\gamma(y_r^z)-z|.$$

In view of Lemma 5.2, we can find uniform constants $c > 0$ and $\varepsilon_* > 0$ such that

(5.14)
$$\left|\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y_r^z+x))-\partial_j\rho(\gamma(y_r^z))\right][\gamma_j(y_r^z)-z_j]\right\}\right| \le c|x|^2\,|\gamma(y_r^z)-z| \quad \forall r\in(0,R/3), \;\forall |x|\le\varepsilon_*, \;\forall z\in\overline{\Omega}\cap\mathcal{U}(r).$$

Let $r_* \in (0, R/3)$ be so small that for every $r \in (0, r_*]$,

$$|\gamma(x)-z| \le \frac{C(\gamma)}{2c} \quad \forall x\in B_d(0;r), \;\forall z\in\overline{\Omega}\cap\mathcal{U}(r),$$

where c is as in (5.14) and $C(\gamma)$ is the constant appearing in Lemma 5.3. We now define a constant

$$\varrho \equiv \varrho(\gamma) := \min\{1/2, r_*, \delta(\gamma)/2, \varepsilon_*/3\},$$

where $\delta(\gamma)$ is the constant appearing in Lemma 5.3. In what follows, we will use the notation $y^z \in B_d(0; 2\varrho)$ to mean $y^z = y^z_\varrho$. From the preceding estimate and (5.14), we have the estimate

$$\left|\mathrm{Re}\left\{\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y^z+x))-\partial_j\rho(\gamma(y^z))\right][\gamma_j(y^z)-z_j]\right\}\right| \le \frac{C(\gamma)}{2}\,|x|^2 \quad \forall |x|\le\varepsilon_*, \;\forall z\in\overline{\Omega}\cap\mathcal{U}(\varrho). \tag{5.15}$$

Now, consider any $f \in \mathcal{C}_c[B_d(0;\varrho);\mathbb{C}]$. For each $\delta > 0$, define h_δ according to (5.12) above. We remark that by our assumption on γ, the form

$$\mathbb{R}^d \ni v \mapsto \mathrm{Re}\sum_{j,k=1}^{n}\left(\partial^2_{j\overline{k}}\rho(\gamma(x))\;[d\gamma(x)v]_j\;\overline{[d\gamma(x)v]}_k + \partial^2_{jk}\rho(\gamma(x))\;[d\gamma(x)v]_j\;[d\gamma(x)v]_k\right)$$

is strictly positive definite for each $x \in B_d(0;\varrho)$. Using this fact, it can be shown – see [10, Lemma 2.4] –that the integrals (5.13) converge, and that $G(x) \neq 0$. From the discussion at the beginning of this section, we conclude that the real part of $\sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]$, which occurs in the denominator of the integral in (5.12) is non-negative when $z \in \overline{\Omega}$. Thus, $h_\delta \in A(\Omega)$ for each $\delta > 0$.

Claim (i) $\{h_\delta\}_{\delta>0}$ is uniformly bounded on $\overline{\Omega}$.

We first consider the case when $z \in \overline{\Omega}\cap\mathcal{U}(\varrho)$. We indulge in a slight abuse of notation : we will define the integrand in (5.12) to be 0 when $x \notin B_d(0;\varrho)$, whence $B_d(0;\varrho)$ may be replaced by $\mathbb{R}^d$ in (5.12), but we will continue to refer to this extension to $\mathbb{R}^d$ by the expression of the integrand given above. Making a change of variable $x = y^z + \delta v$, we get

$$h_\delta(z) = \int_{\mathbb{R}^d}\frac{f(y^z+\delta v)/G(y^z+\delta v)\;dv}{\left\{1+\delta^{-2}\sum_{j=1}^n \partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\}^d}. \tag{5.16}$$

Observe that

$$\left|1+\delta^{-2}\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right| \geq 1+\delta^{-2}\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\}. \tag{5.17}$$

We compute

$$\begin{aligned}
&\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\} \\
&= \mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-\gamma(y^z)]+\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z)-z_j]\right\} \\
&= \mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-\gamma(y^z)]+\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z))[\gamma_j(y^z)-z_j]\right. \\
&\qquad \left.\sum_{j=1}^{n}\left[\partial_j\rho(\gamma(y^z+\delta v))-\partial_j\rho(\gamma(y^z))\right][\gamma_j(y^z)-z_j]\right\}
\end{aligned} \tag{5.18}$$

From the fact that $\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z))[\gamma_j(y^z)-z_j]\right\} \geq 0\ \forall z\in\overline{\Omega}$, and from Lemma 5.3 and (5.15), we have

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(y^z+\delta v))[\gamma_j(y^z+\delta v)-z_j]\right\} \geq \frac{C(\gamma)}{2}\,\delta^2|v|^2\ \forall z\in\overline{\Omega}\cap\mathcal{U}(\varrho). \tag{5.19}$$

From (5.17) and (5.19) we have

$$|h_\delta(z)| \leq \|f/G\|_\infty\int_{\mathbb{R}^d}\left\{1+\frac{C(\gamma)}{2}|v|^2\right\}^{-d}dv \quad \forall z\in\overline{\Omega}\cap\mathcal{U}(\varrho). \tag{5.20}$$

We now consider the case when $z\in\overline{\Omega}\setminus\mathcal{U}(\varrho)$. Due to the fact that $\gamma[\mathbb{B}_d(0;\varrho)]\subsetneq\mathcal{U}(\varrho)$, we can find a uniform constant $c'>0$ such that

$$\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-z_j]\right\} \geq c'\,\mathrm{Re}\left\{\sum_{j=1}^{n}\partial_j\rho(\gamma(x))[\gamma_j(x)-\gamma_j(0)]\right\} \quad \forall x\in B_d(0;\varrho). \tag{5.21}$$

This time, we make a change of variable $x=\delta v$. For $z\in\overline{\Omega}\setminus\mathcal{U}(\varrho)$, this results in

$$h_\delta(z)=\int_{\mathbb{R}^d}\frac{f(\delta v)/G(\delta v)\ dv}{\left\{1+\delta^{-2}\sum_{j=1}^{n}\partial_j\rho(\gamma(\delta v))[\gamma_j(\delta v)-z_j]\right\}^d}. \tag{5.22}$$

From Lemma 5.3 and (5.21) we can deduce that

$$|h_\delta(z)| \leq \|f/G\|_\infty\int_{\mathbb{R}^d}\{1+c'|v|^2\}^{-d}\,dv \quad \forall z\in\overline{\Omega}\setminus\mathcal{U}\varrho. \tag{5.23}$$

We note here, in regard to the above and to the estimate (5.20), that $\|f/G\|_\infty < \infty$ in both those estimates because [10, Lemma 2.4] shows that $\|1/G\|_\infty < \infty$. Claim (i) now follows from (5.20) and (5.23).

The above argument actually yields the following observation, which we record.

Fact. *There exists a uniform constant $\kappa > 0$ such that the integrands occuring in* (5.16) *and* (5.22) *are dominated by the $\mathbb{L}^1$ function*

$$\mathbb{R}^d \ni v \mapsto \|f/G\|_\infty \{1 + \kappa|v|^2\}^{-d}.$$

Claim (ii) $\lim_{\delta\to 0} h_\delta(z) = 0$ if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$.

Notice that if $z \in \overline{\Omega} \setminus \gamma[B_d(0;\varrho)]$, then $\operatorname{Re}\Big\{\sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]\Big\} > 0 \ \forall x \in B_d(0;\varrho)$. Thus

$$\lim_{\delta\to 0} \frac{\delta^d f(x)/G(x)}{\Big\{\delta^2 + \sum_{j=1}^n \partial_j\rho(\gamma(x))[\gamma_j(x) - z_j]\Big\}^d} = 0.$$

In view of the fact recorded above, we can apply the dominated convergence theorem to (5.12). This results in Claim (ii).

Claim (iii) $\lim_{\delta\to 0} h_\delta[\gamma(s)] = f(s) \ \forall s \in B_d(0;\varrho)$.

Refer to Lemma 5.1. When $z = \gamma(s)$ in that lemma, $y^z = \gamma(s)$. Equation (5.16) reads as

$$h_\delta[\gamma(s)] = \int_{\mathbb{R}^d} \frac{f(s+\delta v)/G(s+\delta v)\ dv}{\Big\{1 + \delta^{-2}\sum_{j=1}^n \partial_j\rho(\gamma(s+\delta v))[\gamma_j(s+\delta v) - \gamma_j(s)]\Big\}^d}.$$

In view of 5.11, the integrands occuring above tend to

$$\frac{f(s)}{G(s)}\left\{1 + \frac{1}{2}\sum_{j,k=1}^n \Big(\partial^2_{j\overline{k}}\rho(\gamma(s))\ [d\gamma(s)v]_j\ \overline{[d\gamma(s)v]}_k + \partial^2_{jk}\rho(\gamma(s))\ [d\gamma(s)v]_j\ [d\gamma(s)v]_k\Big)\right\}^{-d},$$

as $\delta \to 0$. Once again, Claim (iii) follows from the dominated convergence theorem. □

6. The proof of Theorem 1.1

For a $p \in \boldsymbol{M}$, let $\mathcal{M}$ stand for an arbitrary d-dimensional stratum, $d \geq 1$, of the local stratification (4.1) of $\boldsymbol{M}$ near p. Let $q \in \mathcal{M}$ and let $\gamma : (B_d(0;R), 0) \to (\mathcal{M}, q)$ be a non-singular, real-analytic parametrization of $\mathcal{M}$ near q. Notice that by the definition of $\mathcal{M}$, γ satisfies the hypotheses of Theorem 5.5. In view of Bishop's theorem (refer back to Theorem 2.1), it would suffice to show that for any compact $K \subset B_d(0;\varrho)$ and any annihilating measure $\mu \perp A(\Omega)$, $\mu[\gamma(K)] = 0$, where $\varrho \equiv \varrho(\gamma) > 0$ is the constant introduced in Theorem 5.5.

Now, given a compact $K \subset B_d(0;\varrho)$, let $\{\mathfrak{D}_\nu\}_{\nu\in\mathbb{N}}$ be a shrinking family of compact subsets such that

(a) $\mathfrak{D}_\nu \subset B_d(0;\varrho)$,
(b) $\mathfrak{D}_{\nu+1} \subset \operatorname{int}(\mathfrak{D}_\nu)$,
(c) $\cap_{\nu\in\mathbb{N}} = K$.

Let $\chi_\nu \in \mathcal{C}^\infty(B_d(0;\varrho);[0,1])$ be a bump function with

$$\chi_\nu|_{\mathfrak{D}_{\nu+1}} \equiv 1, \qquad \text{supp } \chi_\nu \subseteq \mathfrak{D}_\nu.$$

We define $h_\delta^\nu \in A(\Omega)$ by taking $f = \chi_\nu$ in the equation (5.12). Let $\mu \perp A(\Omega)$. By Theorem 5.5 and the bounded convergence theorem, we have

$$0 = \lim_{\delta\to 0} \int h_\delta^\nu d\mu = \int_{\gamma[B_d(0;\varrho)]} \widetilde{\chi_\nu} d\mu,$$

where $\widetilde{\chi_\nu}$ are given by the equations $\widetilde{\chi_\nu}[\gamma(x)] = \chi_\nu(x)\ \forall x \in B_d(0;\varrho)$. Another passage to the limit gives $\mu[\gamma(K)] = 0$, and this is true for any $\mu \perp A(\Omega)$.

We have just shown that each $\mathcal{M}$ is a countable union of peak-interpolation sets for $A(\Omega)$. Since each of the finitely many points in M_0 (M_0 as given by (4.1)) are peak points for $A(\Omega)$ (since Ω is convex) , $\boldsymbol{M}$ is a compact subset of $\partial\Omega$ that is a countable union of peak-interpolation sets for $A(\Omega)$. Using Bishop's theorem again, we conclude that $\boldsymbol{M}$ is a peak-interpolation set for $A(\Omega)$.

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References

[1] E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc. **13** (1962), 140-143.

[2] T. Bloom and I. Graham, *On type conditions for generic real submanifolds in* $\mathbb{C}^n$, Invent. Math. **40** (1984) 217-43.

[3] D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math.(2) **120** (1984), 529-586.

[4] J.P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Annals of Math. **115** (1982), 615-637.

[5] G. M. Henkin and A.E. Tumanov, *Interpolation submanifolds of pseudoconvex manifolds*, Translations Amer. Math. Soc. **115** (1980), 59-69.

[6] S. Łojasiewicz, *Sur le problème de la division*, Studia Math. **18** (1959), 87-136.

[7] J.D. McNeal, *Convex domains of finite type*, J. Funct. Anal. **108** (1992), 361-373.

[8] A. Nagel, *Smooth zero sets and interpolation sets for some algebras of holomorphic functions on strictly pseudoconvex domains*, Duke Math. J. **43** (1976), 323-348.

[9] A. Nagel and W. Rudin, *Local boundary behavior of bounded holomorphic functions*, Can. J. Math. **30** (1978), 583-592.

[10] W. Rudin, *Peak-interpolation sets of class* $\mathcal{C}^1$, Pacific J. Math. **75** (1978), 267-279.

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