PLURISUBHARMONIC POLYNOMIALS AND BUMPING

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Abstract. We wish to study the problem of bumping outwards a pseudoconvex, finite-type domain $\Omega \subset \mathbb{C}^n$ in such a way that pseudoconvexity is preserved and such that the lowest possible orders of contact of the bumped domain with $\partial\Omega$, at the site of the bumping, are explicitly realised. Generally, when $\Omega \subset \mathbb{C}^n,\ n \geq 3$, the known methods lead to bumpings with high orders of contact — which are not explicitly known either — at the site of the bumping. Precise orders are known for h-extendible/semiregular domains. This paper is motivated by certain families of *non-semiregular* domains in $\mathbb{C}^3$. These families are identified by the behaviour of the least-weight plurisubharmonic polynomial in the Catlin normal form. Accordingly, we study how to perturb certain homogeneous plurisubharmonic polynomials without destroying plurisubharmonicity.

1. Introduction

This paper is a part of a study of the boundary-geometry of bounded pseudoconvex domains of finite type. For such a domain in $\mathbb{C}^2$, one can demonstrate many nice properties that have major function-theoretic consequences. For example, Bedford and Fornaess [1] showed that every boundary point of such a domain with *real-analytic* boundary admits a holomorphic peak function. A similar conclusion was obtained by Fornaess and Sibony [8] in the finite-type case. Later, Fornaess [7] and Range [10] exploited a crucial ingredient needed in both [1] and [8] to obtain Hölder estimates for the $\overline{\partial}$-problem.

All of the mentioned results depend on the fact that the given domain in $\mathbb{C}^2$ has a good local bumping. If $\Omega \subset \mathbb{C}^n,\ n \geq 2$, is a smoothly bounded pseudoconvex domain and $\zeta \in \partial\Omega$, we say that Ω *admits a local bumping around* ζ if we can find a neighbourhood U_ζ of ζ and a smooth function $\rho_\zeta \in \mathsf{psh}(U_\zeta)$ such that

- i) $\rho_\zeta^{-1}\{0\}$ is a smooth hypersurface in U_ζ that is pseudoconvex from the side $U_\zeta^- := \{z : \rho_\zeta(z) < 0\}$; and
- ii) $\rho_\zeta(\zeta) = 0$, but $(\overline{\Omega} \setminus \{\zeta\}) \bigcap U_\zeta \subsetneq U_\zeta^-$.

We shall call the triple $(\partial\Omega, U_\zeta, \rho_\zeta)$ a local bumping of Ω around ζ.

Diederich and Fornaess [5] did show that if Ω is a bounded, pseudoconvex domain with real-analytic boundary, then local bumpings always exist around each $\zeta \in \partial\Omega$. The problem however — from the viewpoint of the applications mentioned above — is that the order of contact between $\partial\Omega$ and $\rho_\zeta^{-1}\{0\}$ at ζ might be very high. In fact, if $\Omega \subset \mathbb{C}^n,\ n \geq 3$, then this order of contact is much higher, in many cases, than the type of the point $\zeta \in \partial\Omega$. (See Catlin's [3] and D'Angelo's [2] for different notions

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of type.) Given these function-theoretic motivations, one would like to attempt to solve the following problem:

(*) With Ω and $\zeta \in \partial\Omega$ as above, construct a local bumping $(\partial\Omega, U_\zeta, \rho_\zeta)$ such that the orders of contact of $\partial\Omega \bigcap U_\zeta$ with $\rho_\zeta^{-1}\{0\}$ at ζ along various directions $V \in T_\zeta(\partial\Omega) \bigcap iT_\zeta(\partial\Omega)$ are the *lowest possible* and *explicitly known.*

Consider the following situation: given Ω and $\zeta \in \partial\Omega$ as above, we can find a local holomorphic coordinate system $(V_\zeta; w, z_1, \dots, z_{n-1})$, centered at ζ, such that

$$\Omega \bigcap V_\zeta \;=\; \left\{(w,z) \in V_\zeta : \mathfrak{Re}(w) + P_{2k}(z) + O(|z|^{2k+1}, |wz|, |w|^2) < 0\right\},$$

where P_{2k} is a plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ that is homogeneous of degree $2k$. The first result in $\mathbb{C}^n$, $n \geq 3$, to address (*) is due to Noell [9]. He showed that if P_{2k} is plurisubharmonic and is not harmonic along any complex line through 0, then Ω can be bumped homogeneously to order $2k$ around $0 \in \mathbb{C}^n$. So, one would like to know whether some form of Noell's result holds *without the "nonharmonicity" assumption on P_{2k}.*

A clearer connection between (*) and the boundary-geometry emerges when we look at the Catlin normal form for $\partial\Omega$ near a finite-type $\zeta \in \partial\Omega$. If $(1, m_1, \dots, m_{n-1})$ is the Catlin multitype of ζ (readers are once more referred to [3]), and we write $\Lambda := (m_1, \dots, m_{n-1})$, then, there exists a local holomorphic coordinate system $(V_\zeta; w, z_1, \dots, z_{n-1})$, centered at ζ, such that

$$\begin{aligned} &(1.1) \quad \Omega \bigcap V_\zeta \\ &= \left\{(w,z) \in V_\zeta : \mathfrak{Re}(w) + P(z) + O(|wz|, |w|^2) + (\textit{higher-weight terms in } z) < 0\right\}, \end{aligned}$$

where P is a Λ-homogeneous plurisubharmonic polynomial in $\mathbb{C}^{n-1}$ that has no pluriharmonic terms. We say that P is *Λ-homogeneous* if $P(t^{1/m_1}z_1, \dots, t^{1/m_{n-1}}z_n) = tP(z_1, \dots, z_{n-1})$ $\forall z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ and for every $t > 0$. Note that (*) would be completely solved if one could prove the existence of a Λ-homogeneous function $H \in \mathcal{C}^\infty(\mathbb{C}^{n-1})$ satisfying

$$H(z) \;\geq\; C\sum_{j=1}^{n-1} |z_j|^{m_j} \quad \forall z \in \mathbb{C}^{n-1},$$

for some $C > 0$, such that $(P - H)$ is strictly plurisubharmonic on $\mathbb{C}^{n-1} \setminus \{0\}$. Unfortunately, this plan does not work in general. Yu in [12], and Diederich and Herbort in [6], have independently shown that

(**) A Λ-homogeneous H of the sort described above exists only if there are no complex-analytic subvarieties of $\mathbb{C}^{n-1}$ of positive dimension along which P is harmonic.

There are certainly domains in $\mathbb{C}^3$ for which the condition in (**) fails. Our paper is inspired by the following examples where that condition fails:

Example 1:

$$\Omega_1 \;=\; \left\{(w,z) \in \mathbb{C} \times \mathbb{C}^2 : \mathfrak{Re}(w) + |z_1|^6|z_2|^2 + |z_1|^8 + \tfrac{15}{7}|z_1|^2\mathfrak{Re}(z_1^6) + |z_2|^{10} < 0\right\},$$

and

Example 2:

$$\Omega_2 \;=\; \left\{(w,z) \in \mathbb{C} \times \mathbb{C}^2 : \mathfrak{Re}(w) + |z_1z_2|^8 + \tfrac{15}{7}|z_1z_2|^2\mathfrak{Re}(z_1^6z_2^6) + |z_1|^{18} + |z_2|^{20} < 0\right\}.$$

The difficulty in achieving (*), in either case, is the existence of complex lines in $\mathbb{C}^2$ along which P_{2k} (in the notation used earlier) is harmonic. Note that the defining functions of Ω_1 and Ω_2 are modelled on the polynomial

$$G(z) \;=\; |z|^8 + \tfrac{15}{7}|z|^2\mathfrak{Re}(z^6),$$

that features in the well-known Kohn-Nirenberg example. Recalling the behaviour of G, we note that this *rules out the possibility of bumping* $\Omega_j,\ j=1,2,$ *by perturbing the higher-order terms* in (z_1, z_2).

This last remark suggests that we are committed to perturbing the lowest-order — or, more generally, the lowest-weight — polynomial in z in the defining function of (1.1). This is the first step towards obtaining the possible bumpings of (*). This, in itself, is difficult because we are treating the case where the polynomial P (in the terminology of (1.1)) is harmonic along certain complex subvarieties of $\mathbb{C}^{n-1}$. The structure of these exceptional varieties can be *very difficult* to resolve. However, we can handle a large class of polynomials associated to domains in $\mathbb{C}^3$, which includes *Example 1* and *Example 2*. To obtain the possible bumpings, we need to construct a non-negative, Λ-homogeneous function $H \in \mathcal{C}^\infty(\mathbb{C}^2)$ such that

$$P(z_1, z_2) - H(z_1, z_2) \;\; \textit{is plurisubharmonic} \; \forall (z_1, z_2),$$

and such that

$$H(z_1, z_2) \;\geq\; \varepsilon |P(z_1, z_2)| \quad \forall (z_1, z_2) \in \mathbb{C}^2$$

for some $\varepsilon > 0$ sufficiently small. That is the focus of this paper. The precise results are given in the next section. Since this focused task is already rather involved, its application to specific function-theoretic estimates will be tackled in a different article.

2. Statement of results

For clarity, we shall *initially* present our results in the setting of homogeneous plurisubharmonic polynomials. However, we begin with some notation that is relevant to the general setting. Thus, let P be a (m_1, m_2)-homogeneous plurisubharmonic polynomial on $\mathbb{C}^2$, and define:

$$\begin{aligned} \omega(P) \;&:=\; \{z \in \mathbb{C}^2 : \mathfrak{H}_{\mathbb{C}}(P)(z) \text{ is not strictly positive definite}\}, \\ \mathfrak{C}(P) \;&:=\; \text{the set of all irreducible complex curves } V \subset \mathbb{C}^2 \\ &\qquad\qquad \text{such that } P \text{ is harmonic along the smooth part of } V, \end{aligned}$$

where $\mathfrak{H}_{\mathbb{C}}(P)(z)$ denotes the complex Hessian of P at $z \in \mathbb{C}^2$. As already mentioned, we need to tackle the case when $\mathfrak{C}(P) \neq \emptyset$.

Let us now consider P to be homogeneous of degree $2k$ (plurisubharmonicity ensures that P is of even degree). If we assume that $\mathfrak{C}(P) \neq \emptyset$, then there is a non-empty collection of complex lines through the origin in $\mathbb{C}^2$ along which P is harmonic. This follows from the following observation by Noell:

Result 2.1 (Lemma 4.2, [9])**.** *Let P be a homogeneous, plurisubharmonic, non-pluriharmonic polynomial in $\mathbb{C}^n$, $n \geq 2$. Suppose there exist complex-analytic varieties of positive dimension in $\mathbb{C}^n$ along which P is harmonic. Then, there exist complex lines through the origin in $\mathbb{C}^n$ along which P is harmonic.*

This collection of complex lines will play a key role. Let us denote this non-empty collection of exceptional complex lines by $\boldsymbol{\mathcal{E}}(P)$. What is the structure of $\boldsymbol{\mathcal{E}}(P)$? An answer to this is provided by the following proposition which is indispensable to our construction, and which may be of independent interest:

Proposition 2.2. *Let P be a plurisubharmonic, non-pluriharmonic polynomial in $\mathbb{C}^2$ that is homogeneous of degree $2k$. There are at most finitely many complex lines passing through $0 \in \mathbb{C}^2$ along which P is harmonic.*

It is not hard to show that the set of complex lines passing through the origin in $\mathbb{C}^2$ along which P is harmonic describes a *real*-algebraic subset of $\mathbb{CP}^1$. Thus, it is possible *a priori* that the real dimension of this projective set equals 1. The non-trivial part of Proposition 2.2 is that this set is in fact zero-dimensional.

The interpretation of $\boldsymbol{\mathcal{E}}(P)$ for (m_1, m_2)-homogeneous polynomials, $m_1 \neq m_2$, is

$$\begin{aligned}\boldsymbol{\mathcal{E}}(P) \ := & \text{ the class of all curves of the form} \\ & \left\{(z_1, z_2) : z_1^{m_1/\gcd(m_1,m_2)} = \zeta z_2^{m_2/\gcd(m_1,m_2)}\right\}, \\ & \zeta \in \mathbb{C}_\infty, \text{ along which } P \text{ is harmonic}\end{aligned}$$

(with the understanding that $\zeta = \infty \Rightarrow P$ is harmonic along $\{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$). Note how $\boldsymbol{\mathcal{E}}(P)$ is just a collection of complex lines when $m_1 = m_2 = 2k$. As $\mathfrak{C}(P) \neq 0$, perturbing P in the desired manner becomes extremely messy. However, under certain conditions on $\omega(P)$, we can describe the desired bumping in a relatively brief and precise way. For this, we need one last definition. An (m_1, m_2)-*wedge in* $\mathbb{C}^2$ is defined to be a set $\boldsymbol{\mathcal{W}}$ having the property that if $(z_1, z_2) \in \boldsymbol{\mathcal{W}}$, then $(t^{1/m_1} z_1, t^{1/m_2} z_2) \in \boldsymbol{\mathcal{W}}\ \forall t > 0$. The terms *open* (m_1, m_2)*-wedge* and *closed* (m_1, m_2)*-wedge* will have the usual meanings. Note that when $m_1 = m_2 = 2k$ (the homogeneous case), an (m_1, m_2)-wedge is simply a cone.

The problem we wish to solve can be resolved very precisely if the Levi-degeneracy set $\omega(P)$ possesses either one of the following properties:

> ***Property (A):*** $\omega(P) \setminus \bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C$ contains no complex subvarieties of positive dimension and is *well separated* from $\bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C$. In more precise terms: there is a closed (m_1, m_2)-wedge $\overline{\boldsymbol{\mathcal{W}}} \subset \mathbb{C}^2$ that contains $\omega(P) \setminus \bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C$ and satisfies $\overline{\boldsymbol{\mathcal{W}}} \bigcap (\bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C) = \{0\}$.

OR

> ***Property (B):*** There exists an entire function $\mathcal{H}$ such that P is harmonic along the smooth part of *every* level-curve of $\mathcal{H}$, i.e. $\omega(P) = \mathbb{C}^2$ and is foliated by these level-curves.

Note that, in some sense, Property (A) and Property (B) represent the two extremes of the complex structure within $\omega(P)$, given that $\mathfrak{C}(P) \neq \emptyset$.

The reader is referred to Section 1 for an illustration of these properties. The set $\omega(P)$ for *Example 1* (resp. *Example 2*) has Property (A) (Property (B) resp.). We feel that our main results — both statements and proofs — are clearest in the setting of homogeneous polynomials. Thus, we shall first present our results in this setting.

Theorem 2.3. *Let $P(z_1, z_2)$ be a plurisubharmonic polynomial in $\mathbb{C}^2$ that is homogeneous of degree $2k$ and has no pluriharmonic terms. Assume that $\omega(P)$ possesses Property (A). Define $\boldsymbol{\mathcal{E}}(P) :=$ the set of all complex lines passing through $0 \in \mathbb{C}^2$ along which P is harmonic. Then:*
1) $\boldsymbol{\mathcal{E}}(P)$ consists of finitely many complex lines; and
2) There exist a constant $\delta_0 > 0$ and a $\mathcal{C}^\infty$-smooth function $H \geq 0$ that is homogeneous of degree $2k$ such that the following hold:

(a) $H^{-1}\{0\} = \bigcup_{L \in \boldsymbol{\mathcal{E}}(P)} L$.

(b) $(P - \delta H) \in \mathsf{psh}(\mathbb{C}^2)$ *and is strictly plurisubharmonic on* $\mathbb{C}^2 \setminus \bigcup_{L \in \boldsymbol{\mathcal{E}}(P)} L \ \forall \delta \in (0, \delta_0)$.

The next theorem is the analogue of Theorem 2.3 in the case when $\omega(P)$ possesses Property (B)

Theorem 2.4. *Let $P(z_1, z_2)$ be a plurisubharmonic, non-pluriharmonic polynomial in $\mathbb{C}^2$ that is homogeneous of degree $2k$ and has no pluriharmonic terms. Assume that $\omega(P)$ possesses Property (B). Then:*
1) There exist a subharmonic, homogeneous polynomial U, and a holomorphic homogeneous polynomial F such that $P(z_1, z_2) = U(F(z_1, z_2))$; and
2) Let $L_1, \dots, L_N$ be the complex lines passing through $0 \in \mathbb{C}^2$ that constitute $F^{-1}\{0\}$. There exists a $\mathcal{C}^\infty$-smooth function $H \geq 0$ such that the following hold:

(a) $H^{-1}\{0\} = \bigcup_{j=1}^N L_j$.

(b) $(P - \delta H) \in \mathsf{psh}(\mathbb{C}^2) \ \forall \delta : 0 < \delta \leq 1$.

Before moving on to the weighted case, let us make a few observations about the proofs of the above theorems. Part (1) of Theorem 2.4 follows simply after it is established that P is constant on the level curves of the function $\mathcal{H}$ occurring in the description of Property (B). Part (2) then follows by constructing a bumping of the *subharmonic* function U. The latter is well understood; the reader is referred, for instance, to [8, Lemma 2.4].

Proving Theorem 2.3 subtler. Essentially, it involves the following steps:

- *Step 1:* By Proposition 2.2, $\boldsymbol{\mathcal{E}}(P)$ is a finite set, say $\{L_1, L_2, \dots, L_N\}$. Let $L_j := \{(z_1, z_2) : z_1 = \zeta_j z_2 \ \forall z_2 \in \mathbb{C}\}$ for some $\zeta_j \in \mathbb{C}$. We fix a ζ_j, $j = 1, \dots, N$, and view P in (ζ, w)-coordinates given by the relations
$$z_1 = (\zeta + \zeta_j)w, \quad \text{and} \quad z_2 = w,$$
and we define $\widetilde{{}^jP}(\zeta, w) := P((\zeta + \zeta_j)w, w)$. We expand $\widetilde{{}^jP}(\zeta, w)$ as a sum of polynomials that are homogeneous in the first variable, with increasing degree in ζ.
- *Step 2:* By examining the lowest-degree terms, in the ζ-variable, of this expansion, one can find cones $\boldsymbol{\mathcal{K}}(\zeta_j; \sigma_j)$, and functions H_j that are smooth in $\boldsymbol{\mathcal{K}}(\zeta_j; \sigma_j)$, $j = 1, \dots, N$, such that $(P - \delta H_j)$ are bumpings of P inside the aforementioned cones for each $\delta : 0 < \delta \leq 1$.
- *Step 3:* Property (A) allows us to patch together all these bumpings $(P - \delta H_j)$, $j = 1, \dots, N$ — shrinking $\delta > 0$ sufficiently when necessary — to obtain our result.

For any $\zeta \in \mathbb{C}$, the notation $\boldsymbol{\mathcal{K}}(\zeta; \varepsilon)$, used above, denotes the open cone
$$\boldsymbol{\mathcal{K}}(\zeta; \varepsilon) \ := \ \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - \zeta z_2| < \varepsilon |z_2|\}.$$

Note that $\boldsymbol{\mathcal{K}}(\zeta;\varepsilon)$ is a conical neighbourhood of the punctured complex line $\{(z_1 = \zeta z_2, z_2) : z_2 \in \mathbb{C}\setminus\{0\}\}$. The details of the above discussion are presented in Sections 3 and 4 below.

Continuing with the theme of homogeneous polynomials, we present a result which — though it has no bearing on our Main Theorems below — we found in the course of our investigations relating to Theorem 2.4. Since it could be of independent interest, we present it as:

Theorem 2.5. *Let $P(z_1, z_2)$ be a plurisubharmonic, non-harmonic polynomial that is homogeneous of degree $2p$ in z_1 and $2q$ in z_2. Then, P is of the form*

$$P(z_1, z_2) \;=\; U(z_1^d z_2^D),$$

where $d, D \in \mathbb{Z}_+$ and U is a homogeneous, subharmonic, non-harmonic polynomial.

The reader will note the resemblance between the conclusion of the above theorem and Part (1) of Theorem 2.4. The proof of Theorem 2.5 is given in Section 5.

The reader will probably intuit that the bumping results for an (m_1, m_2)-homogeneous P are obtained by applying Theorem 2.3 and Theorem 2.4 to the pullback of P by an appropriate proper holomorphic mapping that homogenises the pullback. We now state our results in the (m_1, m_2)-homogeneous setting. The first of our main results — rephrased for (m_1, m_2)-homogeneous polynomials — is:

Main Theorem 2.6. *Let $P(z_1, z_2)$ be an (m_1, m_2)-homogeneous, plurisubharmonic polynomial in $\mathbb{C}^2$ that has no pluriharmonic terms. Assume that $\omega(P)$ possesses Property (A). Then:*

1) $\boldsymbol{\mathcal{E}}(P)$ consists of finitely many curves of the form

$$\left\{(z_1, z_2) : z_1^{m_1/\gcd(m_1,m_2)} = \zeta_j z_2^{m_2/\gcd(m_1,m_2)}\right\},$$

$j = 1, \dots, N$, where $\zeta_j \in \mathbb{C}$; and

2) There exist a constant $\delta_0 > 0$ and a $\mathcal{C}^\infty$-smooth (m_1, m_2)-homogeneous function $G \geq 0$ such that the following hold:

(a) $G^{-1}\{0\} = \bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C$.

(b) $(P - \delta G) \in \mathsf{psh}(\mathbb{C}^2)$ *and is strictly plurisubharmonic on* $\mathbb{C}^2 \setminus \bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C \; \forall \delta \in (0, \delta_0)$.

The next result tells us what happens when $\omega(P)$ possesses Property (B). However, in order to state this, we will need to refine a definition made in Section 1. A real or complex polynomial Q defined on $\mathbb{C}^2$ is said to be *(m_1, m_2)-homogeneous with weight r* if $Q(t^{1/m_1} z_1, t^{1/m_2} z_2) = t^r Q(z_1, z_2)$, $\forall z = (z_1, z_2) \in \mathbb{C}^2$ and for every $t > 0$. Our second result can now be stated as follows

Main Theorem 2.7. *Let $P(z_1, z_2)$ be an (m_1, m_2)-homogeneous, plurisubharmonic polynomial in $\mathbb{C}^2$ that has no pluriharmonic terms. Assume that $\omega(P)$ possesses Property (B). Then:*

1) There exist a holomorphic polynomial F that is (m_1, m_2)-homogeneous with weight $1/2\nu$, $\nu \in \mathbb{Z}_+$, and a subharmonic, polynomial U that is homogeneous of degree 2ν such that $P(z_1, z_2) = U(F(z_1, z_2))$; and

2) There exists a $\mathcal{C}^\infty$-smooth (m_1, m_2)-homogeneous function $G \geq 0$ such that the following hold:

(a) $G^{-1}\{0\} = \bigcup_{C \in \boldsymbol{\mathcal{E}}(P)} C$.

(b) $(P - \delta G) \in \mathsf{psh}(\mathbb{C}^2)\ \forall \delta : 0 < \delta \leq 1$.

3. Some technical propositions

The goal of this section is to state and prove several results of a technical nature that will be needed in the proof of Theorem 2.3. Key among these is Proposition 2.2, stated above. We begin with its proof.

3.1. The proof of Proposition 2.2. Assume that P has at least one complex line, say L, passing through $0 \in \mathbb{C}^2$ such that $P|_L$ is harmonic. Since P non-pluriharmonic, there exists a complex line $\Lambda \neq L$ passing through 0 such that $P|_\Lambda$ is subharmonic and non-harmonic. By making a complex-linear change of coordinate if necessary, let us work in global holomorphic coordinates (z, w) with respect to which

$$L = \{(z, w) \in \mathbb{C}^2 \mid z = 0\}, \qquad \Lambda = \{(z, w) \in \mathbb{C}^2 \mid w = 0\}.$$

Let M be the lowest degree to which z and $\overline{z}$ occur among the monomials constituting P. Let us write

$$P(z, w) = \sum_{j=M}^{2k} \sum_{\substack{\alpha+\beta=j \\ \mu+\nu=2k-j}} C^j_{\alpha\beta\mu\nu} z^\alpha \overline{z}^\beta w^\mu \overline{w}^\nu.$$

Notice that by construction

$$s(z) := \sum_{\alpha+\beta=2k} C^{2k}_{\alpha\beta 00} z^\alpha \overline{z}^\beta \quad \textit{is subharmonic and non-harmonic.}$$

We shall make use of this fact soon. We now study the restriction of P along the complex lines $L^\zeta := \{(z = \zeta w, w) : w \in \mathbb{C}\}$. Note that

$$P(\zeta w, w) = \sum_{m+n=2k} \left\{ \sum_{j=M}^{2k} \sum_{\alpha+\beta=j} C^j_{\alpha\beta,(m-\alpha),(n-\beta)} \zeta^\alpha \overline{\zeta}^\beta \right\} w^m \overline{w}^n. \tag{3.1}$$

Denoting the function $w \mapsto P(\zeta w, w)$ by $P_\zeta(w)$, let us use the notation

$$\triangle P_\zeta(w) \equiv \sum_{m+n=2k} \phi_{mn}(\zeta) w^{m-1} \overline{w}^{n-1}.$$

Note that

$$\{\zeta \in \mathbb{C} : P|_{L^\zeta} \text{ is harmonic}\} = \{\zeta \in \mathbb{C} : \phi_{mn}(\zeta) = 0\ \forall m, n \geq 0 \ \ni\ m + n = 2k\}. \tag{3.2}$$

Since P_ζ is subharmonic $\forall \zeta \in \mathbb{C}$, the coefficient of the $|w|^{2k}$ term occurring in (3.1) — i.e. the polynomial $\phi_{kk}(\zeta)/k^2$ — is non-negative, and must be positive at $\zeta \in \mathbb{C}$ whenever P_ζ is non-harmonic. To see this, assume for the moment that, for some $\zeta^* \in \mathbb{C}$, P_{ζ^*} is non-harmonic but $\phi_{kk}(\zeta^*) \leq 0$. Then, as $P_{\zeta^*}(w)$ is real-analytic, $\triangle P_{\zeta^*}(e^{i\theta}) > 0$ *except* at finitely many values of $\theta \in [0, 2\pi)$. Hence, we have

$$0 < \int_0^{2\pi} \triangle P_{\zeta^*}(e^{i\theta})\, d\theta = \int_0^{2\pi} \left\{ \sum_{m+n=2k} \phi_{mn}(\zeta^*) e^{i(m-n)\theta} \right\} d\theta = 2\pi\, \phi_{kk}(\zeta^*).$$

The assumption that $\phi_{kk}(\zeta^*) \leq 0$ produces a contradiction in the above inequality, whence our assertion. Thus $\phi_{kk} \geq 0$.

Let us study the zero-set of ϕ_{kk}. We first consider the highest-order terms of ϕ_{kk}, namely

$$\sum_{\alpha+\beta=2k} k^2 C^{2k}_{\alpha\beta,(k-\alpha),(k-\beta)} \zeta^\alpha \overline{\zeta}^\beta \;=\; k^2 C^{2k}_{kk00} |\zeta|^{2k}.$$

The tidy reduction on the right-hand side occurs because we must only consider those pairs of subscripts (α, β) such that $(k - \alpha) \geq 0$ and $(k - \beta) \geq 0$. Note that C^{2k}_{kk00} is the coefficient of the $|z|^{2k}$ term of $s(z)$, which is subharmonic and non-harmonic. Thus $C^{2k}_{kk00} > 0$. The nature of the highest-order term of ϕ_{kk} shows that $\phi_{kk} \neq 0$ when $|\zeta|$ is sufficiently large. We have thus inferred the following

Fact: *ϕ_{kk} is a real-analytic function such that $\phi_{kk} \geq 0$, $\phi_{kk} \not\equiv 0$, and such that $\phi^{-1}_{kk}\{0\}$ is compact.*

Since ϕ_{kk} is a real-analytic function on $\mathbb{C}$ that is not identically zero, $\dim_{\mathbb{R}}[\phi^{-1}_{kk}\{0\}] \leq 1$. Let us assume that $\dim_{\mathbb{R}}[\phi^{-1}_{kk}\{0\}] = 1$. We make the following

Claim: *The function ϕ_{kk} is subharmonic*

Consider the function

$$S(\zeta) \;:=\; \frac{k^2}{2\pi} \int_0^{2\pi} P(\zeta e^{i\theta}, e^{i\theta}) \; d\theta \; .$$

Notice that

$$S(\zeta) \;:=\; \frac{k^2}{2\pi} \int_0^{2\pi} \left\{ \sum_{m+n=2k} \frac{\phi_{mn}(\zeta)}{mn} e^{i(m-n)\theta} \right\} d\theta \;=\; \phi_{kk}(\zeta). \tag{3.3}$$

Furthermore, denoting the function $\zeta \mapsto P(\zeta w, w)$ by $P_w(\zeta)$, we see that

$$\frac{\partial^2 S}{\partial\zeta\partial\overline{\zeta}}(\zeta) \;=\; \frac{k^2}{2\pi} \int_0^{2\pi} \frac{\partial^2 P_{e^{i\theta}}}{\partial\zeta\partial\overline{\zeta}}(\zeta) \; d\theta \;=\; \frac{k^2}{2\pi} \int_0^{2\pi} \frac{\partial^2 P}{\partial z\partial\overline{z}}(\zeta e^{i\theta}, e^{i\theta}) \; d\theta \;\geq\; 0. \tag{3.4}$$

The last inequality follows from the plurisubharmonicity of P. By (3.3) and (3.4), the above claim is established.

By assumption, $\phi^{-1}_{kk}\{0\}$ is a 1-dimensional real-analytic variety, and we have shown that it is compact. Owing to compactness, there is an open, connected region $D \Subset \mathbb{C}$ such that ∂D is a piecewise real-analytic curve (or a disjoint union of piecewise real-analytic curves). Furthermore

$$\phi_{kk}(\zeta) \;>\; 0 \; \forall\zeta \in D, \qquad \phi_{kk}(\zeta) \;=\; 0 \; \forall\zeta \in \partial D.$$

But the above statement contradicts the Maximum Principle for ϕ_{kk}, which is subharmonic. Hence, our assumption must be wrong. Thus, $\phi^{-1}\{0\}$ is a discrete set; and being compact, it is a finite set. In view of (3.2), we have $\{\zeta \in \mathbb{C} : P|_{L^\zeta} \text{ is harmonic}\} \subseteq \phi^{-1}_{kk}\{0\}$, which establishes our result. $\square$

Our next result expands upon of the ideas summarised in *Step 1* and *Step 2* in Section 2 above. But first, we remind the reader that, given a function G of class $\mathcal{C}^2$ in an open set $U \subset \mathbb{C}^2$ and a vector $v = (v_1, v_2) \in \mathbb{C}^2$, the Levi-form $\mathfrak{L}G(z; v)$ is defined as

$$\mathfrak{L}G(z; v) := \sum_{j,k=1}^{2} \partial^2_{j\overline{k}} G(z) v_j \overline{v_k}.$$

A comment about the hypothesis imposed on P in the following result: the P below is the prototype for the polynomials $\widetilde{{}^jP}$ discussed in *Step 1* of Section 2.

Proposition 3.2. *Let $P(z_1, z_2)$ be a plurisubharmonic polynomial that is homogeneous of degree $2k$, and contains no pluriharmonic terms. Assume that $P(z_1, z_2)$ vanishes identically along $L := \{(z_1, z_2) : z_1 = 0\}$ and that there exists an $\varepsilon > 0$ such that P is strictly plurisubharmonic in the cone $(\boldsymbol{\mathcal{K}}(0;\varepsilon) \setminus L)$. There exist constants C_1 and $\sigma > 0$ — both of which depend only on P — and a non-negative function $H \in \mathcal{C}^\infty(\mathbb{C}^2)$ that is homogeneous of degree $2k$ such that:*

a) *$H(z_1, z_2) > 0$ when $0 < |z_1| < \sigma|z_2|$.*
b) *For any $\delta : 0 < \delta \le 1$:*

$$\mathfrak{L}(P - \delta H)(z; (V_1, V_2)) \;\ge\; C_1|z_1|^{2(k-1)} \left(|V_1|^2 \left(1 - \frac{\delta}{2}\right) + \left|\frac{z_1}{z_2} V_2\right|^2 \right)$$
$$\forall z : 0 \le |z_1| < \sigma|z_2|, \forall V \in \mathbb{C}^2.$$

Proof. Let us write

$$P(z_1, z_2) \;=\; \sum_{j=M}^{2k} Q_j(z_1, z_2),$$

where each Q_j is the sum of all monomials of P that involve powers of z_1 and $\overline{z}_1$ having total degree j, $M \le j \le 2k$. We make a Levi-form calculation. Let us define $\phi := \mathsf{Arg}(w)$ and $\alpha := \mathsf{Arg}(\zeta)$. With this notation, the Levi-form of P at points $(z_1, z_2) = (\zeta w, w)$ can be written as

$$\begin{aligned}(3.5)\quad & \mathfrak{L}P((\zeta w, w); V) \\ &:= \; |w|^{2(k-1)} \sum_{j=M}^{2k} |\zeta|^{j-2} \times (V_1 \quad \zeta V_2) \begin{pmatrix} T_{11}^{(j)}(\phi,\alpha) & T_{12}^{(j)}(\phi,\alpha) \\ \overline{T_{12}^{(j)}(\phi,\alpha)} & T_{22}^{(j)}(\phi,\alpha) \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ \overline{\zeta V_2} \end{pmatrix}.\end{aligned}$$

where $T_{11}^{(j)}$, $T_{12}^{(j)}$ and $T_{22}^{(j)}$ are trigonometric polynomials obtained when $\mathfrak{L}Q_j((\zeta w, w); V)$ is written out using the substitutions

$$w = |w|e^{i\phi} \qquad \text{and} \qquad \zeta \;=\; |\zeta|e^{i\alpha},$$

$j = M, \ldots, 2k$. Now, consider the matrix-valued functions $\mathsf{F}(r;\ \boldsymbol{\cdot}) : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{C}^{2\times 2}$ defined by

$$\mathsf{F}(r; \theta_1, \theta_2) \;:=\; \sum_{j=M}^{2k} r^{j-2} \begin{pmatrix} T_{11}^{(j)}(\theta_1,\theta_2) & T_{12}^{(j)}(\theta_1,\theta_2) \\ \overline{T_{12}^{(j)}(\theta_1,\theta_2)} & T_{22}^{(j)}(\theta_1,\theta_2) \end{pmatrix}.$$

Here $\mathbb{T}$ stands for the circle, and $\mathsf{F}(r;\ \boldsymbol{\cdot})$ is a periodic function in (θ_1, θ_2). Let us introduce the notation $\mu(M) := \min_{\lambda\in\sigma(M)} |\lambda|$ — i.e. the modulus of the least-magnitude eigenvalue of the matrix M. Since $\mathsf{F}(r;\ \boldsymbol{\cdot})$ takes values in the class of 2×2 *Hermitian matrices*, we get

$$(3.6) \qquad \mu\left[\mathsf{F}(r; \theta_1, \theta_2)\right] \;=\; \frac{|\det(\mathsf{F}(r; \theta_1, \theta_2))|}{\|\mathsf{F}(r; \theta_1, \theta_2)\|_2},$$

where the denominator represents the operator norm of the matrix $\mathsf{F}(r;\ \boldsymbol{\cdot})$. Since $P \in \mathsf{spsh}\,(\boldsymbol{\mathcal{K}}(0;\varepsilon) \setminus L)$, comparing $\mathsf{F}(r;\ \boldsymbol{\cdot})$ with the Levi-form computation (3.5), we see that, provided $r \in (0, \varepsilon)$

- $\mathsf{F}(r;\ \boldsymbol{\cdot})$ takes strictly positive-definite values on $\mathbb{T} \times \mathbb{T}$;
- in view of the above and by the relation (3.6), $\mu\left[\mathsf{F}(r;\ \boldsymbol{\cdot})\right]$ are continuous functions on $\mathbb{T} \times \mathbb{T}$; and

- owing to the preceding two facts, an estimate of $\mu\,[\mathsf{F}(r;\ \boldsymbol{\cdot})]$ using the quadratic formula tells us that there exists a $C_1 > 0$ such that
$$\mu\,[\mathsf{F}(r;\theta_1,\theta_2)] \ \geq\ C_1 r^{2(k-1)} \quad \forall(\theta_1,\theta_2)\in\mathbb{T}\times\mathbb{T} \quad \text{(provided } r: 0<r<\varepsilon).$$

Substituting $\theta_1 = \phi$ and $\theta_2 = \alpha$ above therefore gives us
$$\mathfrak{L}P((\zeta w, w); V) \ \geq\ C_1|\zeta|^{2(k-1)}|w|^{2(k-1)}\left(|V_1|^2 + |\zeta V_2|^2\right) \quad \forall\zeta: 0\leq|\zeta|<\varepsilon, \ \forall w\in\mathbb{C} \text{ and } \forall V\in\mathbb{C}^2.$$

Let us now define
$$H(z_1,z_2) \ :=\ \frac{C_1}{2k^2}|z_1|^{2k},$$
and fix a σ such that $0<\sigma<\varepsilon$. Then
$$\mathfrak{L}(P-\delta H)((\zeta w, w); V) \ \geq\ C_1|\zeta|^{2(k-1)}|w|^{2(k-1)}\left(|V_1|^2\left(1-\frac{\delta}{2}\right) + |\zeta V_2|^2\right) \quad \forall\zeta: 0\leq|\zeta|<\sigma, \ \forall w\in\mathbb{C} \text{ and } \forall V\in\mathbb{C}^2.$$
The last inequality is the desired result. □

Our next lemma is the first result of this section that refers to Λ-homogeneous polynomials for a general ordered pair Λ. This result will provide a useful first step towards tackling the theorems in Section 2 pertaining to plurisubharmonic polynomials that possess Property (B).

Lemma 3.3. *Let P be a (m_1,m_2)-homogeneous plurisubharmonic, non-pluriharmonic polynomial in $\mathbb{C}^2$ having Property (B). Then, there exists a rational number q^* and a complex polynomial F that is (m_1,m_2)-homogeneous with weight q^* such that P is harmonic along the smooth part of the level sets of F.*

Proof. Let us first begin by defining $M :=$ the largest positive integer μ such that there exists some $f\in\mathcal{O}(\mathbb{C}^2)$ and $f^\mu = \mathcal{H}$ (here, $\mathcal{H}$ is as given by Property (B)). Define F by the relation $F^M = \mathcal{H}$. Observe that the hypotheses of this lemma continue to hold when $\mathcal{H}$ is replaced by F.

Let $\mathcal{D}_t$ denote the dilations $\mathcal{D}_t : (z_1,z_2)\longmapsto(t^{1/m_1}z_1, t^{1/m_2}z_2)$, and define the set $\mathfrak{S}(P) := \{z\in\mathbb{C}^2 : \mathfrak{H}_{\mathbb{C}}(P)(z) = \mathbf{0}\}$. Since, by hypothesis, $\det[\mathfrak{H}_{\mathbb{C}}(P)]\equiv 0$, we have
$$\mathfrak{S}(P) \ =\ \{z\in\mathbb{C}^2 : (\partial^2_{1\overline{1}}P + \partial^2_{2\overline{2}}P)(z) = \operatorname{tr}(\mathfrak{H}_{\mathbb{C}}(P))(z) = 0\}.$$
As P is not pluriharmonic, $\dim_{\mathbb{R}}[\mathfrak{S}(P)]\leq 3$, and $\mathfrak{S}(P)$ is a closed subset of $\mathbb{C}^2$. Hence, by the open-mapping theorem
$$W(P) \ :=\ F(\mathbb{C}^2\setminus\mathfrak{S}(P))$$
is a non-empty open subset of $\mathbb{C}$. Pick any $c\in W(P)$ and set $V_c := F^{-1}\{c\}$. Then $V_c\bigcap\mathfrak{S}(P) = \emptyset$; and, in view of the transformation law for the Levi-form and the fact that P is (m_1,m_2)-homogeneous, $\mathcal{D}_t(V_c)\bigcap\mathfrak{S}(P) = \emptyset\ \forall t>0$. We now make the following

Claim. *Each $\mathcal{D}_t(V_c)$, $t>0$, is contained in some level set of F ($c\in W(P)$ as assumed above).*

To see this we note that by the transformation law for the Levi-form, P is harmonic along the smooth part of $\mathcal{D}_t(V_c)$ $\forall t>0$. If $\mathcal{D}_t(V_c)$, for some $t>0$, is not contained in any level set, then there would exist a non-empty, Zariski-open subset, say S, of the curve $\mathcal{D}_t(V_c)$ such that:

- for each $\zeta \in S$, there is some level set of F passing through ζ that is transverse to $\mathcal{D}_t(V_c)$ at ζ; and
- owing to the above, $\mathfrak{L}P(\zeta;\ \cdot) \equiv 0\ \forall\zeta \in S$.

But the second statement above cannot be true because $\mathcal{D}_t(V_c) \bigcap \mathfrak{S}(P) = \emptyset$. Hence the claim.

We now define the following function $\phi : W(P) \times (0, \infty) \longrightarrow W(P)$ defined by

$$\phi(c,t) \ := \ \text{the number } b \in \mathbb{C} \text{ such that } F^{-1}\{b\} \supset \mathcal{D}_t(V_c).$$

The above ϕ is well-defined in view of the claim above. Define the following two sets

$$\begin{aligned} \mathsf{Supp}(F) \ &:= \ \left\{\alpha \in \mathbb{N}^2 : \frac{\partial^{|\alpha|}F}{\partial z^\alpha}(0) \neq 0\right\}, \\ \mathcal{I}(F) \ &:= \ \left\{\frac{\alpha_1 m_2 + \alpha_2 m_1}{m_1 m_2} : (\alpha_1, \alpha_2) \in \mathsf{Supp}(F)\right\}. \end{aligned}$$

By construction of $\mathcal{I}(F)$, we can write

$$F(z) = \sum_{q \in \mathcal{I}(F)} P_q(z_1, z_2)$$

where each polynomial P_q is a regrouping of the terms in the Taylor expansion of F around $z = 0$ such P_q is (m_1, m_2)-homogeneous with weight q. The above discussion implies that

$$\begin{aligned} &\sum_{q \in \mathcal{I}(F)} P_q(t^{1/m_1} z_1, t^{1/m_2} z_2) \\ &\qquad = \ \phi(F(z), t) \sum_{q \in \mathcal{I}(F)} P_q(z_1, z_2)\ \forall(z_1, z_2) \in \mathbb{C}^2 \setminus \mathfrak{S}(P) \text{ and } \forall t > 0. \end{aligned} \tag{3.7}$$

On the other hand:

$$\sum_{q \in \mathcal{I}(F)} P_q(t^{1/m_1} z_1, t^{1/m_2} z_2) \ = \ \sum_{q \in \mathcal{I}(F)} t^q P_q(z_1, z_2)\ \forall(z_1, z_2) \in \mathbb{C}^2 \setminus \mathfrak{S}(P) \text{ and } \forall t > 0. \tag{3.8}$$

Since z varies through an open set in both the equations above, comparing (3.7) and (3.8) gives us

$$P_q \not\equiv 0 \implies \phi(F(z), t) = t^q\ \forall t > 0 \text{ and } \forall z \in \mathbb{C}^2 \setminus \mathfrak{S}(P).$$

From this, we conclude that there exists a $q^* \in \mathcal{I}(F)$ such that $P_q \equiv 0$ if $q \neq q^*$; and $F \equiv P_{q^*}$. Hence the result. $\square$

Our next result will play a key role in the proofs pertaining to polynomials having Property (B). It is a rephrasing of [8, Lemma 2.4]. Since it is an *almost* direct rephrasing, we shall not prove this result. Readers are, however, referred to the remark immediately following this lemma.

Lemma 3.4. *Let $U : \mathbb{C} \to \mathbb{R}$ be a real-analytic, subharmonic, non-harmonic function that is homogeneous of degree j. There exist a positive constant $C \equiv C(U)$ — i.e. depending only on U — and a 2π-periodic function $h \in \mathcal{C}^\infty(\mathbb{R})$ such that:*

a) $0 < h(x) \leq 1\ \forall x \in \mathbb{R}$.

b) $\triangle\left(U - \delta|\cdot|^j h \circ \mathsf{Arg}(\cdot)\right)(z) \geq \delta C|z|^{j-2}\ \forall z \in \mathbb{C}$ *and* $\forall\delta : 0 < \delta \leq 1$. *(Here* $\mathsf{Arg}(\cdot)$ *refers to any continuous branch of the argument.)*

Remark 3.5. The $\delta > 0$ appearing in the above lemma must not be confused for the δ appearing in the statement [8, Lemma 2.4]. The latter δ is a universal constant which is a component of the constant $C(U)$ in our notation. If we denote the δ of [8, Lemma 2.4] by δ_{univ}, then our $C(U)$ is a polynomial function of δ_{univ} and

$$\text{(in the notation of [8])} \ \ \|U\| \ := \ \sup_{|z|=1} |U(z)|.$$

4. Proofs of the theorems in the homogeneous case

We begin with the proof of Theorem 2.3, most of whose ingredients are now available from the previous section. However, we need one last result, which is derived from [9].

Result 4.1 (Version of Prop. 4.1 in [9])**.** *Let P be a plurisubharmonic polynomial on $\mathbb{C}^n$ that is homogeneous of degree $2k$. Let ω_0 be a connected component of $\omega(P)\setminus\{0\}$ having the following two properties:*

a) *There exist closed cones $\overline{\mathcal{K}}_1$ and $\overline{\mathcal{K}}_2$ such that*

$$\omega_0 \subset \ \text{int}(\overline{\mathcal{K}}_1) \subset \ \overline{\mathcal{K}}_1 \setminus \{0\} \subsetneq \ \text{int}(\overline{\mathcal{K}}_2),$$

and such that $\overline{\mathcal{K}}_2 \bigcap (\omega(P) \setminus \overline{\omega_0}) = \emptyset$.

b) *ω_0 does not contain any complex-analytic subvarieties of positive dimension along which P is harmonic.*

Then, there exist a smooth function $H \geq 0$ that is homogeneous of degree $2k$ and constants $C, \varepsilon_0 > 0$, which depend only on P, such that $\text{int}(\overline{\mathcal{K}}_2) = \{H > 0\}$ and such that for each $\varepsilon : 0 < \varepsilon \leq \varepsilon_0$, $\mathfrak{L}(P - \varepsilon H)(z; v) \geq C\varepsilon \|z\|^{2(k-1)} \ \|v\|^2 \ \forall (z, v) \in \overline{\mathcal{K}}_2 \times \mathbb{C}^n$.

The above result is not phrased in precisely these terms in [9, Proposition 4.1]. The proof of the latter proposition was derived from a construction pioneered by Diederich and Fornaess in [4]. A close comparison of the proof of [9, Proposition 4.1] with the Diederich-Fornaess construction reveals that incorporating the assumption (a) in Result 4.1 allows us to obtain the above “localised” version of [9, Proposition 4.1].

4.2. The proof of Theorem 2.3. Note that Part (1) follows simply from Proposition 2.2. Hence, let us denote the set $\boldsymbol{\mathcal{E}}(P)$ by $\boldsymbol{\mathcal{E}}(P) = \{L_1, \dots, L_N\}$. Let $\overline{\mathcal{K}}$ be the closed cone whose existence is guaranteed by Property (A). By assumption, we can find a slightly larger cone $\overline{\mathcal{K}}_*$ such that

$$\omega(P) \setminus (\bigcup_{j=1}^{N} L_j) \subset \ \text{int}(\overline{\mathcal{K}}) \subset \ \overline{\mathcal{K}} \setminus \{0\} \subsetneq \ \text{int}(\overline{\mathcal{K}}_*),$$

and such that $\overline{\mathcal{K}}_* \bigcap (\bigcup_{j=1}^{N} L_j) = \{0\}$. Hence, in view of Result 4.1, we can find a smooth function $H_0 \geq 0$ that is homogeneous of degree $2k$, and constants $C_0, \varepsilon_0 > 0$ such that

$$\begin{aligned} \{z : H_0 > 0\} \ &= \ \text{int}(\overline{\mathcal{K}}_*), \\ \mathfrak{L}(P - \delta H_0)(z; v) & \\ &\geq \ C_0\delta \|z\|^{2(k-1)} \ \|v\|^2 \quad \forall (z, v) \in \overline{\mathcal{K}}_* \times \mathbb{C}^2, \ \text{and} \ \forall \delta \in (0, \varepsilon_0). \end{aligned} \tag{4.1}$$

Without loss of generality, we may assume that each L_j is of the form $L_j = \{(\zeta_j z_2, z_2) : z_2 \in \mathbb{C}\}$. Applying Proposition 3.2 to

$$\widetilde{{}^jP}(z_1, z_2) \ := \ P(z_1 + \zeta_j z_2, z_2)$$

we can find:

- a constant $B_1 > 0$ that depends only on P;
- constants $\sigma_j > 0,\ j = 1, \dots, N$, that depend only on P and j; and
- functions $H_j \in \mathcal{C}^\infty(\mathbb{C}^2)$ that are homogeneous of degree $2k$;

such that

$$\begin{aligned}
&\mathfrak{L}(P - \delta H_j)(z; v) \\
&\qquad\qquad \geq \ \delta B_1 |z_1 - \zeta_j z_2|^{2(k-1)} \left(|v_1 - \zeta_j v_2|^2 \left(1 - \frac{\delta}{2}\right) + \left| \frac{z_1 - \zeta_j z_2}{z_2} v_2 \right|^2 \right) \\
&\qquad\qquad \forall (z, v) \in \left[\overline{\boldsymbol{\mathcal{K}}(\zeta_j; \sigma_j)} \setminus \{0\} \right] \times \mathbb{C}^2 \text{ and } \forall \delta : 0 < \delta \leq 1.
\end{aligned}$$

The reader is reminded that $\boldsymbol{\mathcal{K}}(\zeta_j; \sigma_j)$ denotes an open cone, as introduced in Section 2, and that the right-hand side above is finite. Let $\widetilde{\sigma_j} > 0,\ j = 1, \dots, N$, be so small that

$$\begin{aligned}
2\widetilde{\sigma_j} \ &\leq \ \sigma_j, \quad j = 1, \dots, N, \\
(\overline{\boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_j})} \cap S^3) \bigcap (\overline{\boldsymbol{\mathcal{K}}_*} \cap S^3) \ &= \ \phi \quad \forall j \leq N, \\
(\overline{\boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_j})} \cap S^3) \bigcap (\overline{\boldsymbol{\mathcal{K}}(\zeta_k; 2\widetilde{\sigma_k})} \cap S^3) \ &= \ \phi \quad \text{if } j \neq k.
\end{aligned} \tag{4.2}$$

Here, S^3 denotes the unit sphere in $\mathbb{C}^2$. We introduce these new parameters in order to patch together all the above "localised" bumpings.

Let us now define

$$V_j \ := \ \overline{\boldsymbol{\mathcal{K}}(\zeta_j; \widetilde{\sigma_j})} \cap S^3, \quad \text{and} \quad U_j \ := \ \overline{\boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_j})} \cap S^3.$$

Let $\chi_j : S^3 \longrightarrow [0, 1]$ be a smooth cut-off function such that $\chi_j|_{V_j} \equiv 1$ and $\operatorname{supp}(\chi_j) \subset U_j,\ j = 1, \dots, N$. Let us define $\Psi_j(z) := \chi_j(z/\|z\|)\ \forall z \in \mathbb{C}^2 \setminus \{0\}$. Finally, let us use the expression $\Psi_j H_j$ to denote the homogeneous function defined as

$$\Psi_j H_j(z) \ := \ \begin{cases} \Psi_j(z) H_j(z), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Note that $\Psi_j H_j \in \mathcal{C}^\infty(\mathbb{C}^2)$ and is homogeneous of degree $2k$. Let us now estimate the Levi-form of $(P - \delta \Psi_j H_j)$ on $\boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_j}) \setminus \overline{\boldsymbol{\mathcal{K}}(\zeta_j; \widetilde{\sigma_j})}$. Since, by construction, $(P - \delta \Psi_j H_j)$ is strictly plurisubharmonic on $\boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_j}) \setminus \overline{\boldsymbol{\mathcal{K}}(\zeta_j; \widetilde{\sigma_j})}$, and strict plurisubharmonicity is an open condition, we infer from continuity and homogeneity:

(4.3) *$\exists \varepsilon \ll 1$ such that $(P - \delta \Psi_j H_j)$ is strictly pseudoconvex on $\boldsymbol{\mathcal{K}}(\zeta_j; \widetilde{\sigma_j} + \varepsilon)\ \forall j = 1, \dots, N$, and for each $\delta : 0 < \delta \leq 1/2$.*

For the moment, let us *fix* $j \leq N$. Note that, by construction, we can find a $\beta_j > 0$ such that

$$(1 - \Psi_j)(z) \ \geq \ \beta_j \quad \forall z \in \boldsymbol{\mathcal{K}}(\zeta_j; 2\widetilde{\sigma_2}) \setminus \overline{\boldsymbol{\mathcal{K}}(\zeta_j; \widetilde{\sigma_j} + \varepsilon)}.$$

Furthermore, since $(\widetilde{\sigma_j}+\varepsilon)|z_2| < |z_1-\zeta_j z_2| < 2\widetilde{\sigma_j}|z_2|$ in the cone $\mathcal{K}(\zeta_j;2\widetilde{\sigma_2}) \setminus \overline{\mathcal{K}(\zeta_j;\widetilde{\sigma_j}+\varepsilon)}$, applying this to (4.2) gives us small constants $\gamma_j, c_j>0$ such that

$$\begin{aligned} \mathfrak{L}(P-\delta H_j)(z;v) \ &\geq \ \gamma_j\|z\|^{2(k-1)}\|v\|^2 \quad \forall z\in\mathcal{K}(\zeta_j;2\widetilde{\sigma_2})\setminus\overline{\mathcal{K}(\zeta_j;\widetilde{\sigma_j}+\varepsilon)}, \\ &\qquad\qquad \forall v\in\mathbb{C}^2, \text{ and } \forall\delta: 0<\delta\leq 1/2; \end{aligned} \tag{4.4}$$

$$\mathfrak{L}P(z;v) \ \geq \ c_j\|z\|^{2(k-1)}\|v\|^2 \quad \forall(z,v)\in(\mathcal{K}(\zeta_j;2\widetilde{\sigma_2})\setminus\overline{\mathcal{K}(\zeta_j;\widetilde{\sigma_j}+\varepsilon)})\times\mathbb{C}^2. \tag{4.5}$$

From the estimates (4.4) and (4.5), we get

$$\begin{aligned} \mathfrak{L}(P-\delta\Psi_jH_j)(z;v) \ &= \ \Psi_j(z)\mathfrak{L}(P-\delta H_j)(z;v)+(1-\Psi_j)(z)\mathfrak{L}P(z;V) \\ &\qquad -\delta H_j(z)\mathfrak{L}\Psi_j(z;v)-2\delta\mathfrak{Re}\left[\sum_{\mu,\nu=1}^{2}\partial_\mu\Psi_j(z)\partial_{\overline{\nu}}\Psi_k(z)v_\mu\overline{v_\nu}\right] \\ &\geq \ \|z\|^{2(k-1)}\|v\|^2\left(\gamma_j\Psi_j(z)+c_j\beta_j\right) \\ &\qquad -\delta\left(2\left|\sum_{\mu,\nu=1}^{2}\partial_\mu\Psi_j(z)\partial_{\overline{\nu}}\Psi_k(z)v_\mu\overline{v_\nu}\right|+|H_j(z)\mathfrak{L}\Psi_j(z;v)|\right) \\ &\qquad\qquad \forall z\in\mathcal{K}(\zeta_j;2\widetilde{\sigma_2})\setminus\overline{\mathcal{K}(\zeta_j;\widetilde{\sigma_j}+\varepsilon)} \text{ and } \forall v\in\mathbb{C}^2. \end{aligned}$$

Finally, we can find constants $K_1,K_2,K_3>0$ and a $\delta_j>0$ that is so small that, in view of the above calculation, we can make the following estimates:

$$\begin{aligned} \mathfrak{L}(P-\delta\Psi_jH_j)(z;v) \ &\geq \ \|z\|^{2(k-1)}\|v\|^2\left(c_j\beta_j-2\delta K_1\right)-2\delta K_2|H_j(z)|\|v\|^2 \\ &\geq \ \frac{c_j\beta_j}{2}\|z\|^{2(k-1)}\|v\|^2 \\ &\geq \ \delta K_3\|z\|^{2(k-1)}\|v\|^2 \\ &\qquad \forall z\in\mathcal{K}(\zeta_j;2\widetilde{\sigma_2})\setminus\overline{\mathcal{K}(\zeta_j;\widetilde{\sigma_j}+\varepsilon)}, \\ &\qquad \forall v\in\mathbb{C}^2, \text{ and } \forall\delta: 0<\delta\leq\delta_j. \end{aligned} \tag{4.6}$$

Let us now set

$$\begin{aligned} \widetilde{H} \ &:= \ H_0+\sum_{j=1}^{N}\Psi_jH_j, \\ \delta_0 \ &:= \ \min(\varepsilon_0,\delta_1,\ldots,\delta_N). \end{aligned}$$

So far, in view of (4.3) and (4.6), we have accomplished the following:

i) $(P-\delta\widetilde{H})\in\mathsf{psh}(\mathbb{C}^2)$, and $(P-\delta\widetilde{H})$ is *strictly* plurisubharmonic on $\mathbb{C}^2\setminus\bigcup_{j=1}^N L_j$ $\forall\delta\in(0,\delta_0)$.
ii) $\{z:\widetilde{H}>0\}=\mathrm{int}(\overline{\mathcal{K}}_*)\bigcup(\bigcup_{j=1}^N(\mathcal{K}(\zeta_j;2\widetilde{\sigma_j})\setminus L_j))$.

All we now have to do is make a perturbation of $\widetilde{H}$ to get an H that is strictly positive where desired. To carry this out, let:

$$\begin{aligned} W_0 &:= S^3 \bigcap \left(\mathrm{int}(\overline{\mathcal{K}}_*) \bigcup \left(\bigcup_{j=1}^{N} \mathcal{K}(\zeta_j; 2\widetilde{\sigma_j}) \right) \right)^{\mathsf{C}}, \\ W_1 &= \text{some small } S^3\text{-neighbourhood of } \overline{W_0} \text{ such that } \Big(\bigcup_{j=1}^{N} L_j\Big) \bigcap W_1 = \emptyset. \end{aligned}$$

Now let $\chi^* : S^3 \longrightarrow [0, \alpha]$ be a smooth cut-off function on S^3 such that $\chi^*|_{W_0} \equiv \alpha$ and $\mathrm{supp}(\chi^*) \subset W_1$, where $\alpha > 0$ is so small that if we define

$$H(z) := \widetilde{H}(z) + \|z\|^{2k} \chi^* \left(\frac{z}{\|z\|} \right),$$

then — in view of (i) above — Part (b) of this theorem follows without altering the conclusion of (i) above when $\widetilde{H}$ is replaced by H. Hence (a) follows. □

Next, we provide:

4.3. The proof of Theorem 2.4. Let us first begin by defining $M :=$ the largest positive integer μ such that there exists some $f \in \mathcal{O}(\mathbb{C}^2)$ and $f^\mu = \mathcal{H}$. Define F by the relation $F^M = \mathcal{H}$. Observe that the hypotheses of Theorem 2.4 continue to hold when $\mathcal{H}$ is replaced by F.

Step I. *The function F is a homogeneous polynomial*

This is a straightforward application of Lemma 3.3. Note that our preliminary construction of F is precisely the F provided by Lemma 3.3 applied to $(m_1, m_2) = (2k, 2k)$.

Step II. *To show that P is constant on each level-set of F*

First we note that, without loss of generality, we may assume that $F|_{z_1=0} \not\equiv 0$. If not, we carry out the following change of coordinates

$$\begin{aligned} Z_1 &:= z_1 - \zeta_0 z_2 \\ Z_2 &:= z_2, \end{aligned}$$

where $\zeta_0 \in \mathbb{C} \setminus \{0\}$ is so chosen that $F|_{z_1=\zeta_0 z_2} \not\equiv 0$. If we define

$$\widetilde{P}(Z_1, Z_2) := P(Z_1 + \zeta_0 Z_2, Z_2) \quad \text{and} \quad \widetilde{F}(Z_1, Z_2) := F(Z_1 + \zeta_0 Z_2, Z_2),$$

then it is easy to check that

- $\widetilde{P}$ is harmonic along the smooth part of each level curve of $\widetilde{F}$; and
- $\widetilde{F}|_{Z_1=0} \not\equiv 0$.

Hence, we may as well assume that F satisfies the desired condition. Then, by homogeneity of F, $F(0, \cdot)$ is non-constant. By the Fundamental Theorem of Algebra, then

$$\{z \in \mathbb{C}^2 : z_1 = 0\} \bigcap F^{-1}\{c\} \neq \emptyset \quad \forall c \in \mathbb{C}. \tag{4.7}$$

Let us now assume that P is non-constant on $F^{-1}\{c\}$ for some $c \in \mathbb{C}$. The fact that P being non-constant on $F^{-1}\{c\}$ is an open condition in $c \in \mathbb{C}$ implies, in conjunction with (4.7), that we can find a c_0 close to c and a $w_0 \in \mathbb{C}$ such that

- P is non-constant along $F^{-1}\{c_0\}$;
- the point $q_0 = (0, w_0)$ lies on $F^{-1}\{c_0\}$; and

- q_0 is not a singular point of $F^{-1}\{c_0\}$.

Then, by construction, there exists an $\varepsilon > 0$ and a holomorphic map on the unit disc, $\Psi = (\psi_1, \psi_2) : \mathbb{D} \longrightarrow \mathbb{B}(q_0;\varepsilon)$ such that $\Psi(0) = q_0$ and Ψ parametrises $F^{-1}\{c_0\} \bigcap \mathbb{B}(q_0;\varepsilon)$.

Let us adopt the notations from the proof of Proposition 2.2 and write

$$P(z,w) \ = \ \sum_{j=M}^{2k} \sum_{\substack{\alpha+\beta=j \\ \mu+\nu=2k-j}} C_{\alpha\beta\mu\nu} z^\alpha \overline{z}^\beta w^\mu \overline{w}^\nu.$$

By hypothesis, the function

$$v(\xi) \ := \ \sum_{j=M}^{2k} \sum_{\substack{\alpha+\beta=j \\ \mu+\nu=2k-j}} C^j_{\alpha\beta\mu\nu} \psi_1(\xi)^\alpha \overline{\psi_1(\xi)}^\beta \psi_2(\xi)^\mu \overline{\psi_2(\xi)}^\nu, \quad \xi \in \mathbb{D},$$

is harmonic on the unit disc. Since, by hypothesis, P has no pluriharmonic terms, and the requirement of harmonicity forces v to have only harmonic terms in its Taylor expansion about $\xi = 0$, we have:

$$\begin{aligned} &(4.8) \quad v(\xi) \\ &:= \sum_{j=M}^{2k} \sum_{\substack{\alpha+\beta=j \\ \mu+\nu=2k-j}} C^j_{\alpha\beta\mu\nu} \left\{ \psi_1(\xi)^\alpha \psi_2(\xi)^\mu \overline{\psi_1(0)}^\beta \overline{\psi_2(0)}^\nu + \psi_1(0)^\alpha \psi_2(0)^\mu \overline{\psi_1(\xi)}^\beta \overline{\psi_2(\xi)}^\nu \right\}. \end{aligned}$$

However, recall that $\psi_1(0) = 0$. In view of (4.8), this forces the conclusion $v \equiv 0$. But this results in a contradiction because, by real-analyticity, $v = P \circ \Psi \equiv 0$ would force P to vanish on $F^{-1}\{c_0\}$. This establishes Step II.

Step III. *The proof of Part (1)*
Let us define the function $U : \mathbb{C} \longrightarrow \mathbb{R}$ as

$$U(c) \ := \ P(z_1^{(c)}, z_2^{(c)}),$$

where $(z_1^{(c)}, z_2^{(c)})$ is any point lying in $F^{-1}\{c\}$. We would be done if we could show that U is real-analytic. Let us outline our strategy before tackling the details. The strategy may be summarised as follows:

1) We shall choose a complex line $\Lambda_\tau := \{(z_1, z_2 = \tau z_1) : z_1 \in \mathbb{C}\}$ such that $F|_{\Lambda_\tau}$ is non-constant. We can then show that for *almost every* $c_0 \in \mathbb{C}$, we can find a function u^{c_0} that is holomorphic in a neighbourhood $V^{c_0} \ni c_0$ and parametrises a designated root of the equation $F|_{\Lambda_\tau} = c$ as c varies through V^{c_0}. In other words:
$$u^{c_0} : c \longmapsto (u^{c_0}(c), \tau u^{c_0}(c)) \in \bigcup_{\zeta\in\mathbb{C}} \left(\Lambda_\tau \cap F^{-1}\{\zeta\}\right),$$
$$(u^{c_0}(c), \tau u^{c_0}(c)) \in F^{-1}\{c\} \quad \text{as } c \text{ varies through } V^{c_0}.$$
2) Clearly, $U|_{V^{c_0}} = P(u^{c_0}, \tau u^{c_0})$. As real-analyticity is a local property, we would be done if the conclusions of (1) could be established in a neighbourhood of *every* $c_0 \in \mathbb{C}$. This can be achieved by repeating the above analysis on a different complex line $\Lambda_\eta \neq \Lambda_\tau$. Subharmonicity would follow from a Levi-form calculation.

The details follow.

Accordingly, choose any $\tau \in \mathbb{C}$ such that

$$F_\tau : z \longmapsto F(z, \tau z) \textit{ is a non-constant polynomial.}$$

Recall that — given a complex, univariate polynomial p — the map

$$\mathsf{dsc}_p(c) := \text{the discriminant of the polynomial } p(z) - c$$

has the following two properties:

i) $\mathsf{dsc}_p(c)$ is a complex polynomial in c.
ii) $\mathsf{dsc}_p(c) = 0 \iff$ the equation $p(z) - c = 0$ has repeated roots.

The reader is referred to van der Waerden's book [11] for an exposition on the discriminant. With this information in mind, let us define

$$\mathsf{dsc}_\tau(c) := \text{the discriminant of the polynomial } F_\tau(z) - c.$$

Then, by (ii) above, $\mathsf{dsc}_\tau^{-1}\{0\}$ is a finite set, and if $c_0 \in \mathbb{C} \setminus \mathsf{dsc}_\tau^{-1}\{0\}$, then there exists an open disc $D(c_0;\delta) \subset \mathbb{C} \setminus \mathsf{dsc}_\tau^{-1}\{0\}$ such that the equation $F_\tau(z) - c = 0$ has $\deg(F_\tau)$ simple roots for each $c \in D(c_0;\delta)$. In fact, we can find a $u^{c_0} \in \mathcal{O}(D(c_0;\delta))$ such that

$$F_\tau(u^{c_0}(c)) - c = 0 \quad \forall c \in D(c_0;\delta).$$

Note that by the above equation and our hypothesis on P, we have

$$U(c) = P(u^{c_0}(c), \tau u^{c_0}(c)) \quad \forall c \in D(c_0;\delta). \tag{4.9}$$

Since c_0 was arbitrarily chosen from $\mathbb{C} \setminus \mathsf{dsc}_\tau^{-1}\{0\}$, and real-analyticity is a local property, we have just shown that $U \in \mathcal{C}^\omega(\mathbb{C} \setminus \mathsf{dsc}_\tau^{-1}\{0\})$. But we can now repeat the above argument with some $\eta \neq \tau$, with the property that

$$\left(\mathbb{C} \setminus \mathsf{dsc}_\tau^{-1}\{0\}\right) \bigcup \left(\mathbb{C} \setminus \mathsf{dsc}_\eta^{-1}\{0\}\right) = \mathbb{C},$$

replacing τ. We then get a version of equation (4.9) with η in place of τ. This establishes that $U \in \mathcal{C}^\omega(\mathbb{C})$. By construction, $P = U \circ F$. Given the homogeneity of P and F (from Step I), it is obvious that U is homogeneous. Now, performing a Levi-form computation, we get

(4.10)

$$\begin{aligned} \mathfrak{L}P(z_1, z_2; (V_1, V_2)) = {} & \frac{1}{4} \triangle U(F(z_1, z_2)) \\ & \times (V_1 \ \ V_2) \begin{pmatrix} |\partial_1 F|^2 & \partial_1 F \, \partial_{\overline{2}} \overline{F} \\ \partial_{\overline{1}} \overline{F} \, \partial_2 F & |\partial_2 F|^2 \end{pmatrix}_{(z_1, z_2)} \begin{pmatrix} \overline{V}_1 \\ \overline{V}_2 \end{pmatrix}. \end{aligned}$$

Since $\mathfrak{L}P(z_1, z_2; \,\boldsymbol{\cdot}\,)$ must be positive semi-definite at every $(z_1, z_2) \in \mathbb{C}^2$, this forces the conclusion $\triangle U \geq 0$. Hence, U is subharmonic, and Part (1) is thus established.

Step IV. *The proof of Part (2)*
Write $2\nu := \deg(U)$ (the degree of U is even due to subharmonicity). We apply Lemma 3.4 to the subharmonic U to obtain the smooth function h that satisfies the conclusions of that lemma. Now define

$$H(z_1, z_2) := \begin{cases} |F(z_1, z_2)|^{2\nu} h(\mathsf{Arg}(F(z_1, z_2))), & \text{if } (z_1, z_2) \notin F^{-1}\{0\}, \\ 0, & \text{if } (z_1, z_2) \in F^{-1}\{0\}, \end{cases}$$

where $\mathsf{Arg}(\cdot)$ refers to any continuous branch of the argument. Now, a Levi-form computation reveals that

(4.11)

$$\begin{aligned}\mathfrak{L}(P-\delta H)(z_1,z_2;(V_1,V_2)) \;&=\; \frac{1}{4}|F(z_1,z_2)|^{2(\nu-1)}\,\triangle\left(U-\delta|\cdot|^{2\nu}h\circ\mathsf{Arg}(\cdot)\right)(F(z_1,z_2)) \\ &\quad\times\;(V_1\;\;V_2)\begin{pmatrix}|\partial_1F|^2 & \partial_1F\;\partial_{\overline{2}}\overline{F} \\ \partial_{\overline{1}}\overline{F}\;\partial_2F & |\partial_2F|^2\end{pmatrix}_{(z_1,z_2)}\begin{pmatrix}\overline{V}_1\\ \overline{V}_2\end{pmatrix}.\end{aligned}$$

In view of Lemma 3.4/(b), $(P-\delta H)$ is clearly plurisubharmonic $\forall\delta\in(0,1)$. Furthermore, note that, by the properties of h,

$$H\;\geq\;0\quad\text{and}\quad H(z_1,z_2)=0\iff F(z_1,z_2)=0.$$

This establishes Part (2). $\square$

5. The proof of Theorem 2.5

To avoid confusion resulting from subscripts, we shall write $z:=z_1$ and $w:=z_2$. We shall also adopt several of the conventions and facts that feature in the proof of Proposition 2.2. Accordingly, let us write

$$Q(z,w)\;=\;\sum_{\alpha,\beta\geq 0}C_{\alpha\beta}z^{\alpha}\overline{z}^{2p-\alpha}w^{\beta}\overline{w}^{2q-\beta}, \tag{5.1}$$

As before, let us consider the complex lines $L^{\zeta}:=\{(z=\zeta w,w):w\in\mathbb{C}\}$ and examine $Q|_{L^{\zeta}}$. As in Proposition 2.2, we write

$$Q(\zeta w,w)\;=\;\sum_{m+n=2(p+q)}\left\{\sum_{\alpha+\beta=m}C_{\alpha\beta}\zeta^{\alpha}\overline{\zeta}^{2p-\alpha}\right\}w^m\overline{w}^n\;\equiv\;\sum_{m+n=2(p+q)}\phi_{mn}(\zeta)w^m\overline{w}^n.$$

Recall from Proposition 2.2 that $\phi_{p+q,p+q}$ is a subharmonic function, $\phi_{p+q,p+q}\geq 0$, and $\phi_{p+q,p+q}\not\equiv 0$. All of this implies that (note that $\phi_{p+q,p+q}$ is homogeneous)

$$0\;<\;\int_0^{2\pi}\phi_{p+q,p+q}(e^{i\theta})d\theta\;=\;C_{pq}.$$

We have just concluded that in the expansion 5.1, the term $|z|^{2p}|w|^{2q}$ occurs with a positive coefficient. Let us thus decompose Q as

$$Q(z,w)\;=\;C_{pq}|z|^{2p}|w|^{2q}+R(z,w)\;\equiv\;A(z,w)+R(z,w).$$

Note that A is harmonic along the varieties $V_c:=\{(z,w)\in\mathbb{C}^2:z^pw^q=c\}$. Since, generically in V_c, $T^{\mathbb{C}}_{(z,w)}(V_c)=\mathsf{span}_{\mathbb{C}}[(qz,-pw)]$, we have

$$\mathfrak{L}A((z,w);v)\;=\;0\quad\forall v\in\mathsf{span}_{\mathbb{C}}[(qz,-pw)]\text{ and }\forall(z,w)\in\mathbb{C}^2. \tag{5.2}$$

In other words, equation (5.2) holds true for every $(z,w)\in\mathbb{C}^2$, independent of the variety V_c to which (z,w) belongs. By plurisubharmonicity of Q, we infer that

$$\mathfrak{L}R((z,w);(qz,-pw))\;\geq\;0\quad\forall(z,w)\in\mathbb{C}^2. \tag{5.3}$$

Let us now write $z = re^{i\theta}$ and $w = se^{i\tau}$. In this notation, we get

$$\begin{aligned} R(z,w) &= |z|^{2p}|w|^{2q}T(\theta,\tau), \\ \text{where} \quad T(\theta,\tau) &:= \sum_{(\alpha,\beta)\neq(p,q)} C_{\alpha\beta}e^{i(2\alpha-2p)\theta}e^{i(2\beta-2q)\tau}. \end{aligned} \tag{5.4}$$

It is a routine matter to check that

$$\begin{aligned} 4\,\frac{\partial^2}{\partial z\partial\overline{z}} &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \\ 4\,\frac{\partial^2}{\partial w\partial\overline{w}} &= \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial\tau^2} \\ 4\,\frac{\partial^2}{\partial z\partial\overline{w}} &= e^{i(\tau-\theta)}\left[\frac{\partial^2}{\partial r\partial s} + \frac{1}{rs}\,\frac{\partial^2}{\partial\theta\partial\tau} - i\left\{\frac{1}{r}\,\frac{\partial^2}{\partial\theta\partial s} - \frac{1}{s}\,\frac{\partial^2}{\partial r\partial\tau}\right\}\right] \end{aligned}$$

Using these differential operators in the inequality (5.3) gives us

$$\begin{aligned} &r^{2p}s^{2q} \\ &\times [q \;\; -p]\begin{bmatrix} 4p^2T + T_{\theta\theta} & 4pqT + T_{\theta\tau} + i(2pT_\tau - 2qT_\theta) \\ 4pqT + T_{\theta\tau} - i(2pT_\tau - 2qT_\theta) & 4q^2T + T_{\tau\tau} \end{bmatrix}\begin{bmatrix} q \\ -p \end{bmatrix} \\ &\geq\; 0 \quad \forall r,s\geq 0,\ \forall(\theta,\tau)\in[-\pi,\pi)\times[-\pi,\pi). \end{aligned}$$

Simplifying the above gives us

$$\begin{aligned} &(q^2T_{\theta\theta} - 2pqT_{\theta\tau} + p^2T_{\tau\tau})(\theta,\tau) \\ &\quad= \left(q\frac{\partial}{\partial\theta} - p\frac{\partial}{\partial\tau}\right)\left(q\frac{\partial}{\partial\theta} - p\frac{\partial}{\partial\tau}\right)T(\theta,\tau) \;\geq\; 0 \quad \forall(\theta,\tau)\in[-\pi,\pi)\times[-\pi,\pi). \end{aligned}$$

The above inequality simply tells us that for every line on the $\theta\tau$-plane having tangent vector $(q,-p)$, i.e. for every line $\ell^C := \{(\theta,\tau)\in\mathbb{R}^2 : p\theta + q\tau = C\}$,

$$T|_{\ell^C} \text{ is convex for each } C\in\mathbb{R}.$$

However, the definition of T in (5.4) above reveals that T is a real-analytic function as well as 2π-periodic. For such a T to be convex, necessarily

$$T|_{\ell^C} \;\equiv\; \text{const. } \textit{for each line } \ell^C\subset\mathbb{R}^2.$$

Hence, T must have the form $T(\theta,\tau) = G(p\theta + q\tau)$, where G is a periodic function. This means that T must have the form

$$T(\theta,\tau) \;=\; \sum_{M\in\mathfrak{F}} C_M e^{iM(p\theta+q\tau)}, \quad C_M\neq 0\ \forall M\in\mathfrak{F}, \tag{5.5}$$

where $\mathfrak{F}\subset\mathbb{Z}$ is a finite subset of integers. Comparing (5.4) with (5.5), we infer that

$$C_{\alpha\beta}\neq 0 \implies \exists M\in\mathfrak{F} \text{ such that } \frac{2\alpha-2p}{p} = M = \frac{2\beta-2q}{q}. \tag{5.6}$$

If we define $d := \gcd(\alpha : C_{\alpha\beta}\neq 0)$, we can immediately infer the following facts:

- Since R is real-valued, $C_{\alpha,\beta}\neq 0 \implies C_{2p-\alpha,2q-\beta}\neq 0$ — whence $d|(2p-\alpha)$ for any α such that $C_{\alpha\beta}\neq 0$; and
- Owing to (5.6)

$$\beta_0\in\{\beta : C_{\alpha\beta}\neq 0\} \iff \exists\alpha_0\in\{\alpha : C_{\alpha\beta}\neq 0\} \text{ such that } \beta_0 = q\alpha_0/p.$$

Therefore $D := \gcd(\beta : C_{\alpha\beta} \neq 0) = qd/p$, whence we can find a real-valued polynomial r that is homogeneous of degree $2p/d$ such that $R(z,w) = r(z^d w^D)$. Now set

$$U(\xi) \ := \ C_{pq}|\xi|^{2p/d} + r(\xi) \quad \forall \xi \in \mathbb{C}.$$

Clearly, $Q(z,w) = U(z^d w^D)$. We compute the Levi-form of Q one last time. In the process, we get

$$\begin{aligned}
&\mathfrak{L}Q((z,w);v) \\
&= \ U_{\xi\bar{\xi}}(z^d w^D)|z|^{2(d-1)}|w|^{2(D-1)} \times (v_1 \ \ v_2) \begin{pmatrix} d^2|w|^2 & Dd\bar{z}w \\ & \\ Ddz\overline{w} & D^2|z|^2 \end{pmatrix} \begin{pmatrix} \overline{v_1} \\ \\ \overline{v_2} \end{pmatrix} \ \geq \ 0 \\
&\forall (z,w) \in \mathbb{C}^2, \ \forall v \in \mathbb{C}^2.
\end{aligned}$$

Hence,

$$(\triangle U)(z^d w^D) \ \geq \ 0 \quad \forall (z,w) \in \mathbb{C}^2.$$

Since $z^d w^D$ attains every value in $\mathbb{C}$ as (z,w) varies through $\mathbb{C}^2$, we infer that $\triangle U \geq 0$, i.e. that U is subharmonic. This final fact completes the proof. $\square$

6. Proofs of the Main Theorems

We are now ready to provide a proof of Main Theorem 2.6. The basic idea — i.e. of examining the pullback of P by a suitable proper holomorphic map such that the pullback is homogeneous — is a simple one. The following argument provides the details.

6.1. The proof of Main Theorem 2.6. Define $K := \operatorname{lcm}(m_1, m_2)$ (i.e. the least common multiple of m_1 and m_2) and write $\sigma_j := K/m_j$, $j = 1, 2$. Define the proper holomorphic map $\Psi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ by $\Psi(z_1, z_2) := (z_1^{\sigma_1}, z_2^{\sigma_2})$, and write $Q = P \circ \Psi$. Fix a point $(z_1^0, z_2^0) \in \mathbb{C}^2 \setminus \{(z_1, z_2) : z_1 z_2 = 0\}$. Then, there exist neighbourhoods $U^j \ni z_j^0$ such that the functions $(\xi \mapsto \xi^{\sigma_j})|_{U^j}$ are injective, $j = 1, 2$. Therefore $\Psi|_{U^1 \times U^2}$ is a biholomorphism, whence $Q|_{U^1 \times U^2} \in \mathsf{psh}(U_1 \times U_2)$. Since plurisubharmonicity is a local property, we infer that $Q \in \mathsf{psh}(\mathbb{C}^2 \setminus \{(z_1, z_2) : z_1 z_2 = 0\})$. Finally, since Q is smooth and $\{(z_1, z_2) : z_1 z_2 = 0\}$ is a pluripolar set, we infer that $Q \in \mathsf{psh}(\mathbb{C}^2)$. Furthermore:

$$Q(t^{1/K} z_1, t^{1/K} z_2) \ = \ P((t^{1/K} z_1)^{\sigma_1}, (t^{1/K} z_2)^{\sigma_2}) \ = \ tQ(z_1, z_2) \ \forall t > 0,$$

whence Q is a plurisubharmonic polynomial that is homogeneous of degree K. By hypothesis, Q has no pluriharmonic terms. Furthermore, we observe that

$$\frac{m_1}{\gcd(m_1, m_2)} \ = \ \sigma_2, \quad \frac{m_2}{\gcd(m_1, m_2)} \ = \ \sigma_1,$$

and hence note that for any $\zeta \in \mathbb{C}$ such that $\{(w_1, w_2) : w_1^{\sigma_2} = \zeta w_2^{\sigma_1}\} \in \boldsymbol{\mathcal{E}}(P)$, Q is forced to be harmonic along each of the complex lines that make up the set

$$\mathsf{L}(\zeta) \ := \ \bigcup_{l=0}^{\sigma_1\sigma_2 - 1} \left\{ (z_1, z_2) : z_1 = |\zeta|^{1/\sigma_1\sigma_2} \exp\left(\frac{2\pi i l + i\mathsf{Arg}(\zeta)}{\sigma_1\sigma_2} \right) z_2 \right\}$$

(here Arg denotes some branch of the argument). But since, by Proposition 2.2, there are only finitely many complex lines along which Q can be harmonic, this implies Part (1) of our theorem.

Now, let $\zeta_1, \dots, \zeta_N$ be as in Part (1) of the statement of this theorem. Suppose there is some $z^0 \neq 0$ and a germ of a complex variety V^0 at z^0 such that

$$V_0 \not\subseteq \mathbf{L}(\zeta_j) \ \forall j = 1, \dots, N, \text{ and,} \\ V_0 \not\subseteq \{(z_1, z_2) : z_2 = 0\} \text{ if } \{(z_1, z_2) : z_2 = 0\} \in \boldsymbol{\mathcal{E}}(Q). \tag{6.1}$$

By assumption, we have the following situations for V^0:

Case (i) $V^0 \not\subseteq \{(z_1, z_2) : z_1 z_2 = 0\}$
In this case, we can select a domain $\Omega \subset \mathbb{C}^2$ such that

- $\Omega \bigcap \{(z_1, z_2) : z_1 z_2 = 0\} = \emptyset$ and $\Psi|_\Omega$ is a biholomorphism;
- $\Omega \bigcap V^0$ is a smooth subvariety of Ω; and
- we can find a regular parametrisation $\varphi = (\varphi_1, \varphi_2) : \mathbb{D} \longrightarrow (V^0 \bigcap \Omega)$ of $V^0 \bigcap \Omega$.

Here, $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. We now compute that

$$\begin{aligned} \triangle(P \circ \Psi \circ \varphi)(\xi) &= \mathfrak{L}P(\Psi \circ \varphi(\xi); \Psi_* \left|_{\varphi(\xi)} (\varphi'(\xi))\right) \\ &= \mathfrak{L}Q \left(\varphi(\xi); \left(\Psi^{-1}\right)_* \left|_{\Psi \circ \varphi(\xi)} \left\{\Psi_* \left|_{\varphi(\xi)} (\varphi'(\xi))\right.\right\}\right.\right) \\ &= \triangle \left(Q|_{V^0 \cap \Omega}\right) \ = \ 0, \end{aligned} \tag{6.2}$$

i.e., we conclude that P is harmonic along $\Psi(V^0 \bigcap \Omega)$. Yet, by the assumption (6.1), $\Psi(V^0 \bigcap \Omega)$ is not contained in any curve belonging to $\boldsymbol{\mathcal{E}}(P)$. But this contradicts the hypothesis that P has Property (A), whence this case cannot arise.

Case (ii) *Either* $V^0 \subset \{(z_1, z_2) : z_1 = 0\}$ *or* $V^0 \subset \{(z_1, z_2) : z_2 = 0\}$.
A *much* simpler variant of the above argument shows us that these cases will not arise depending on whether $\{(z_1, z_2) : z_1 = 0\} \notin \boldsymbol{\mathcal{E}}(Q)$, or $\{(z_1, z_2) : z_2 = 0\} \notin \boldsymbol{\mathcal{E}}(Q)$, respectively.

We have therefore established the following fact:

(6.3) *Q is a plurisubharmonic polynomial that is homogeneous of degree K, and*

$$\mathfrak{C}(Q) \ = \ \textit{the set of all complex lines belonging to } \boldsymbol{\mathcal{E}}(Q).$$

In fact, in view of the above fact, it is completely routine to infer that

$$P \textit{ has Property (A)} \implies Q \textit{ has Property (A)}.$$

Now consider the unitary transformations $R^{lm} : (z_1, z_2) \longrightarrow (e^{2\pi i l/\sigma_1} z_1, e^{2\pi i m/\sigma_2} z_2)$ and compute:

$$\begin{aligned} \mathfrak{L}Q(R^{lm}(z); R^{lm}(V)) &= \mathfrak{L}P \circ \Psi \left(R^{lm}(z); \Psi_* \Big|_{R^{lm}(z))} (R^{ln} V)\right) \\ &= \begin{pmatrix} \sigma_1 z_1^{\sigma_1 - 1} e^{-2\pi i l/\sigma_1} (R^{lm} V)_1 & \sigma_2 z_2^{\sigma_2 - 1} e^{-2\pi i m/\sigma_2} (R^{lm} V)_2 \end{pmatrix} \\ &\qquad \times \mathfrak{H}_{\mathbb{C}}(P)|_{\Psi(R^{lm}(z))} \begin{pmatrix} \sigma_1 \overline{z}_1^{\sigma_1 - 1} e^{2\pi i l/\sigma_1} \overline{(R^{lm} V)_1} \\ \sigma_2 \overline{z}_2^{\sigma_2 - 1} e^{2\pi i m/\sigma_2} \overline{(R^{lm} V)_2} \end{pmatrix} \\ &= \mathfrak{L}P \left(\Psi(z); \Psi_* |_z \, (V)\right) \\ &= \mathfrak{L}Q(z; V). \end{aligned} \tag{6.4}$$

So, if we define $\mathfrak{N}_Q(z)$ to be the null-space of $\mathfrak{L}Q(z;\ \bullet)$, then the computation (6.4) reveals that

$$\begin{aligned}(6.5)\quad & z \in \omega(Q) \text{ and } V \in \mathfrak{N}_Q(z)\\ \iff\ & R^{lm}(z) \in \omega(Q) \text{ and } R^{lm}(V) \in \mathfrak{N}_Q(R^{lm}(z))\ \forall l,m : 1 \leq l \leq \sigma_1,\ 1 \leq m \leq \sigma_2.\end{aligned}$$

In particular (6.5) implies that if, in the notation borrowed from the proof of Theorem 2.3, we can find a constant $\alpha_j > 0$ and a $H_j \in \mathcal{C}^\infty(\mathbb{C}^2)$ that is homogeneous of degree K such that

- $H_j = 0$ on $L_j := \{z : z_1 = |\zeta_j|^{1/\sigma_1\sigma_2} \exp(i\mathsf{Arg}(\zeta_j)/\sigma_1\sigma_2)z_2\}$,
- $(Q - \delta H_j) \in \mathsf{spsh}[\boldsymbol{\mathcal{K}}(|\zeta_j|^{1/\sigma_1\sigma_2} \exp(i\mathsf{Arg}(\zeta_j)/\sigma_1\sigma_2); \alpha_j) \setminus L_j]$ for each $\delta : 0 < \delta \leq 1$,

then the above remains true with

- H_j replaced by $H_j^{(lm)} := H_j \circ (R^{lm})^{-1}$,
- L_j replaced by
$$L_j^{(lm)} := \left\{z : z_1 = |\zeta_j|^{1/\sigma_1\sigma_2} \exp\left(\frac{2\pi i(\sigma_1 m - \sigma_2 l) + i\mathsf{Arg}(\zeta_j)}{\sigma_1\sigma_2}\right) z_2\right\},$$
- the cone $\boldsymbol{\mathcal{K}}(|\zeta_j|^{1/\sigma_1\sigma_2} \exp(i\mathsf{Arg}(\zeta_j)/\sigma_1\sigma_2); \alpha_j)$ is replaced by its image under R^{lm},

for any $l, m : 1 \leq 1 \leq \sigma_1,\ 1 \leq m \leq \sigma_2$.

Since Q has Property (A), Result 4.1 is applicable. A careful examination of its proof reveals that Noell's construction of the bumping is local. Then, in view of (6.5) and the preceding discussion, we can — by selecting our cut-off functions in Theorem 2.3 to be equivariant with respect to R^{lm} — construct our bumping H (of the polynomial Q) to have the property

$$\begin{aligned}(6.6)\qquad H(z_1, z_2) &= H(e^{2\pi il/\sigma_1}z_1, e^{2\pi im/\sigma_2}z_2)\ \forall z \in \mathbb{C}^2 \text{ and}\\ &\qquad \forall l, m : 1 \leq 1 \leq \sigma_1,\ 1 \leq m \leq \sigma_2.\end{aligned}$$

Now define

$$\begin{aligned}&G(z)\\ &:= \frac{1}{\sigma_1\sigma_2}\sum_{j=1}^{\sigma_1}\sum_{k=1}^{\sigma_2} H\left(|z_1|^{1/\sigma_1}\exp\left(\frac{2\pi ij + i\mathsf{Arg}(z_1)}{\sigma_1}\right), |z_2|^{1/\sigma_2}\exp\left(\frac{2\pi ik + i\mathsf{Arg}(z_2)}{\sigma_2}\right)\right).\end{aligned}$$

Observe, however, that by the definition of Q, P satisfies

$$\begin{aligned}(6.7)\quad &P(z)\\ &:= \frac{1}{\sigma_1\sigma_2}\sum_{j=1}^{\sigma_1}\sum_{k=1}^{\sigma_2} Q\left(|z_1|^{1/\sigma_1}\exp\left(\frac{2\pi ij + i\mathsf{Arg}(z_1)}{\sigma_1}\right), |z_2|^{1/\sigma_2}\exp\left(\frac{2\pi ik + i\mathsf{Arg}(z_2)}{\sigma_2}\right)\right).\end{aligned}$$

Let $\delta_0 > 0$ be as given by Theorem 2.3 applied to Q. Now, as in the beginning of this proof, fix $(z_1^0, z_2^0) \in \mathbb{C}^2 \setminus \{(z_1, z_2) : z_1 z_2 = 0\}$ and let $U^j \ni z_j^0,\ j = 1, 2$, be neighbourhooods such that $(\xi \mapsto \xi^{\sigma_j})|_{U^j}$ are injective and such that $(U^1 \times U^2) \bigcap \{(z_1, z_2) : z_1 z_2 = 0\} = \emptyset$. Write $V^1 \times V^2 := \Psi(U^1 \times U^2)$. Note that by the definition of G, and by (6.6) and (6.7), we have

$$(P - \delta G)|_{V^1 \times V^2} = (Q - \delta H) \circ (\Psi|_{U^1 \times U^2})^{-1}.$$

Then, whenever $0 < \delta \le \delta_0$, we have the Levi-form computation

$$\begin{aligned}\mathfrak{L}(P - \delta G)(w; V) &= \mathfrak{L}(Q - \delta H)\left((\Psi|_{U^1\times U^2})^{-1}(w); \left((\Psi|_{U^1\times U^2})^{-1}\right)_* |_w (V)\right) \\ &\ge 0 \quad \forall w \in V^1 \times V^2,\ \forall V \in \mathbb{C}^2.\end{aligned}$$

The above argument establishes that whenever $0 < \delta \le \delta_0$, $(P - \delta G) \in \mathsf{psh}[\mathbb{C}^2 \setminus \{w : w_1 w_2 = 0\}]$. Since $(P - \delta G) \in \mathcal{C}^2(\mathbb{C}^2)$, we infer that $(P - \delta G) \in \mathsf{psh}(\mathbb{C}^2)$ by exactly the same argument as in the first paragraph of this proof. Finally, given the relationship between the sets $\bigcup_{j=1}^N \mathbf{L}(\zeta_j)$ and $\bigcup_{C\in\boldsymbol{\mathcal{E}}(P)} C$, Part (2) follows. $\square$

6.2. The proof of Main Theorem 2.7. We will re-use the ideas in the preceding proof, but we shall be brief. Let $\mathcal{H}$ be as in the hypothesis of the theorem and, as before, define $M :=$ the largest positive integer μ such that there exists some $g \in \mathcal{O}(\mathbb{C}^2)$ and $g^\mu = \mathcal{H}$. Define F by the relation $F^M = \mathcal{H}$. Our hypotheses continue to hold when $\mathcal{H}$ is replaced by F and, by Lemma 3.3, F is (m_1, m_2)-homogeneous. Let K (i.e. the least common multiple of m_1 and m_2), σ_1, σ_2, Q, and the proper holomorphic map $\Psi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be *exactly* as in the proof of Main Theorem 2.6. We recall, in particular, that:

$$\begin{aligned}\Psi(z_1, z_2) &:= (z_1^{\sigma_1}, z_2^{\sigma_2}), \\ Q &:= P \circ \Psi.\end{aligned}$$

And as before, Q is homogeneous of degree K.

Furthermore, if we define $f := F \circ \Psi$, we get:

$$\begin{aligned}&(z_1, z_2) \in f^{-1}\{c\} \\ \Rightarrow\ &\Psi^{-1}\{(z_1, z_2)\} \subset F^{-1}\{c\} \\ &\qquad\qquad \Rightarrow Q \text{ is harmonic along the smooth part of } f^{-1}\{c\}.\end{aligned}$$

Furthermore, we leave the reader to verify that Q has no pluriharmonic terms. Therefore, by applying Theorem 2.4, we obtain a homogeneous, subharmonic polynomial U such that $Q = U \circ f$. Let us write $2\nu := \deg(U)$. Recall that by applying Lemma 3.4 to this U to obtain the h as stated in that lemma, and defining $\widetilde{\varkappa}(z) := |z|^{\deg(U)} h(\mathsf{Arg}(z))$, we get

$$P - \delta(\widetilde{\varkappa} \circ f) \in \mathsf{psh}(\mathbb{C}^2) \quad \forall \delta : 0 < \delta \le 1. \tag{6.8}$$

(The above $\widetilde{\varkappa} \circ f$ is precisely what we called H in Theorem 2.4.) It is clear that $P = U \circ F$. This establishes Part (1) of the theorem. The fact that F is (m_1, m_2)-homogeneous *with weight* $1/2\nu$ follows from degree considerations; since P is (m_1, m_2)-homogeneous, F must be homogeneous with weight $1/\deg(U) = 1/2\nu$.

Now define

$$G(z_1, z_2) := \widetilde{\varkappa} \circ F(z_1, z_2) \quad \forall (z_1, z_2) \in \mathbb{C}^2.$$

By (6.8), and by a repetition of the argument in the second half of the last paragraph of the proof of Main Theorem 2.6, we infer

$$Q - \delta G \in \mathsf{psh}(\mathbb{C}^2) \quad \forall \delta : 0 < \delta \le 1.$$

$\square$

References

[1] E. Bedford and J.E. Fornaess, *A construction of peak functions on weakly pseudoconvex domains*, Ann. of Math. **107** (1978), 555-568.

[2] J.P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Annals of Math. **115** (1982), 615-637.

[3] D. Catlin, *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, Ann. of Math. **126** (1987), 131-191.

[4] K. Diederich and J.E. Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129-141.

[5] K. Diederich and J.E. Fornaess, *Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary*, Ann. of Math. **110** (1979), 575-592.

[6] K. Diederich and G. Herbort, *Pseudoconvex domains of semiregular type*, Contributions to Complex Analysis and Analytic Geometry, Aspects Math. (1994), Vieweg, Braunschweig, pp. 127-161.

[7] J.E. Fornaess, *Sup-norm estimates for $\overline{\partial}$ in $\mathbb{C}^2$*, Ann. of Math. **123** (1986), 335-345.

[8] J.E. Fornaess and N. Sibony, *Construction of p.s.h. functions on weakly pseudoconvex domains*, Duke Math. J. **58** (1989), 633-655.

[9] A. Noell, *Peak functions for pseudoconvex domains*, Several Complex Variables - Proceedings of the Mittag-Leffler Institute, 1987-1988, Princeton University Press, Princeton, NJ, 1993, pp. 529-541.

[10] R.M. Range, *Integral kernels and Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$*, Math. Ann. **288** (1990), 63-74.

[11] B.L. van der Waerden, *Algebra Vol. 1 (translated from the German by F. Blum & J.R. Schulenberger)*, Frederick Ungar Publishing Co., 1970.

[12] J.Y. Yu, *Peak functions on weakly pseudoconvex domains*, Indiana Univ. Math. J. **43** (1994), 1271-1295.

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