ON ANALYTIC INTERPOLATION MANIFOLDS IN BOUNDARIES OF WEAKLY PSEUDOCONVEX DOMAINS

GAUTAM BHARALI

ABSTRACT. Let Ω be a bounded, weakly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real-analytic boundary. A real-analytic submanifold $\mathcal{M} \subset \partial\Omega$ is called an analytic interpolation manifold if every real-analytic function on $\mathcal{M}$ extends to a function belonging to $\mathcal{O}(\overline{\Omega})$. We provide sufficient conditions for $\mathcal{M}$ to be an analytic interpolation manifold. We give examples showing that neither of these conditions can be relaxed, as well as examples of analytic interpolation manifolds lying entirely within the set of weakly pseudoconvex points of $\partial\Omega$.

1. Introduction and Statement of Main Result

In this paper, we will work with bounded (weakly) pseudoconvex domains Ω in $\mathbb{C}^n$, $n \geq 2$, with real-analytic boundary. A real-analytic submanifold $\mathcal{M}$ of $\mathbb{C}^n$ contained in $\partial\Omega$ is called an **analytic interpolation manifold** if every real-analytic function on $\mathcal{M}$ extends to some function holomorphic in a neighbourhood of $\overline{\Omega}$ (this neighbourhood will, of course, depend on the prescribed function). This definition is due to Burns and Stout [3]. Their article proves the following result :

Theorem 1.1 (Burns-Stout)**.** *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. A real-analytic submanifold $\mathcal{M}$ of $\mathbb{C}^n$, $\mathcal{M} \subset \partial\Omega$, is an analytic interpolation manifold if and only if $T_p(\mathcal{M}) \subseteq H_p(\partial\Omega)$ $\forall p \in \mathcal{M}$.*

In the above result, $H(\partial\Omega)$ is the maximal complex sub-bundle of the tangent bundle $T(\partial\Omega)$. One could ask whether a real-analytic submanifold $\mathcal{M} \subset \partial\Omega$, given that $\partial\Omega$ is *not* strictly pseudoconvex along $\mathcal{M}$, is an analytic interpolation manifold if it is complex tangential, i.e. if $T_p(\mathcal{M}) \subseteq H_p(\partial\Omega)$ $\forall p \in \mathcal{M}$, and if some appropriately defined higher Levi-form is strictly positive definite at each point $p \in \mathcal{M}$. It will become clear below that complex-tangency is a necessary condition for $\mathcal{M}$ to be an analytic interpolation manifold, but the two aforementioned conditions (once precisely defined) are not sufficient for $\mathcal{M}$ to be an analytic interpolation manifold. To describe our result, we need the following definition.

Definition 1.2. Let $H^{\mathbb{C}}(\partial\Omega) = H^{1,0}(\partial\Omega) \oplus H^{0,1}(\partial\Omega)$ denote the complexification of $H(\partial\Omega)$ and $T^{\mathbb{C}}(\partial\Omega)$ denote the complexification of $T(\partial\Omega)$. For $p \in \partial\Omega$, let $\mathcal{L}_p^1(\partial\Omega) = H_p^{\mathbb{C}}(\partial\Omega)$, and for $j \geq 2$, let $\mathcal{L}_p^j(\partial\Omega)$ be the $\mathbb{C}$-vector space spanned by $H_p^{\mathbb{C}}(\partial\Omega)$ and all iterated commutators of length $\leq j$ formed by the elements of $H_p^{\mathbb{C}}(\partial\Omega)$. A point $p \in \partial\Omega$ is said to be of **Bloom-Graham type** M (or

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simply, of **type** M) if there exists an $M \in \mathbb{N}$ such that $\mathcal{L}_p^M(\partial\Omega) = T_p^{\mathbb{C}}(\partial\Omega)$ and $\mathcal{L}_p^j(\partial\Omega) \subsetneq T_p^{\mathbb{C}}(\partial\Omega)$ for $j < M$.

Our result below shows that if $\partial\Omega$ is of constant type (say M) along $\mathcal{M}$, if $\mathcal{M}$ is complex-tangential, and if the $(M-1)$th Levi-form (which is defined below) is positive definite on a certain subspace of $H_p(\partial\Omega)$ $\forall p \in \mathcal{M}$, then $\mathcal{M}$ is an analytic interpolation manifold. It is worthwhile noting that the $(M-1)$th Levi form at $p \in \mathcal{M}$ is *not* required to be strictly positive definite on all of $H_p(\partial\Omega)$. Furthermore, neither of the aforementioned conditions can be relaxed. We shall show this through examples in Section 4 below.

We now define the higher Levi-forms of $\partial\Omega$ that were mentioned above.

Definition 1.3. Let Ω be a smoothly bounded pseudoconvex domain and let $p \in \partial\Omega$. Suppose p is of type $k+1$. Then, we define the k**th Levi-form of** $\partial\Omega$ **at** p, $\mathfrak{L}_{\partial\Omega}^{(k)}(p\,;\boldsymbol{.}) : H_p^{1,0}(\partial\Omega) \to \mathbb{R}$ as follows :

(1) There exist holomorphic coordinates $(w_1, ..., w_n)$ near p such that $\partial\Omega$ is defined, in a neighbourhood of p, by

$$\varrho(w) = \sum_{\substack{|\alpha|+|\beta|=k+1 \\ 1\leq|\beta|<k+1}} A_{\alpha\beta}^{(p)} {w_*}^{\alpha} {\bar{w}_*}^{\beta} + \mathcal{E}_p(w_*, \mathfrak{im}(w_n)) - \mathfrak{re}(w_n), \tag{1.1}$$

where we write $w = (w_1, ..., w_n) \equiv (w_*, w_n)$ and where $\mathcal{E}_p$ is a smooth function with the property that $\mathcal{E}_p(0,0) = 0$, $\nabla\mathcal{E}_p(0,0) = 0$, and that any term of order $\leq k+1$ is a mixed term involving non-zero powers of $\mathfrak{im}(w_n)$, w_* and $\bar{w}_*$. This is a result from [2]. Let Φ_p be the biholomorphism associated with the above change of coordinate. Let $\mathbf{v} \in H_p^{1,0}(\partial\Omega)$; $d\Phi_p(p)(\mathbf{v})$ is an $(n-1)$-tuple $d\Phi_p(p)(\mathbf{v}) = (\zeta_1, ..., \zeta_{n-1})$. We define

$$\mathfrak{L}_{\partial\Omega}^{(k)}(p; \mathbf{v}) = \sum_{\substack{|\alpha|+|\beta|=k+1 \\ 1\leq|\beta|<k+1}} A_{\alpha\beta}^{(p)} \zeta^{\alpha} \bar{\zeta}^{\beta}.$$

(2) There is a canonical identification of $H_p(\partial\Omega)$, regarded as a $\mathbb{C}$-hyperplane in $\mathbb{C}^n$, with $H_p^{1,0}(\partial\Omega)$ given by

$$H_p(\partial\Omega) \ni (\xi_1, ..., \xi_n) \rightleftharpoons \sum_{l=1}^{n} \xi_l \frac{\partial}{\partial z_l}\bigg|_p \in H_p^{1,0}(\partial\Omega).$$

So, when we say that the Levi-form $\mathfrak{L}_{\partial\Omega}^{(k)}(p\,;\boldsymbol{.})$ acts on $(\xi_1, ..., \xi_n) \in H_p(\partial\Omega)$, it will mean the action of that Levi-form on $\sum_{l=1}^{n} \xi_l \dfrac{\partial}{\partial z_l}\bigg|_p \in H_p^{1,0}(\partial\Omega)$.

We can now state our main result precisely ($\mathbb{J}$ below is the standard complex structure map on $\mathbb{C}^n$, and its effect on a vector is equivalent to multiplication by i) :

Theorem 1.4. *Let Ω be a bounded, weakly pseudoconvex domain with real-analytic boundary, and let $\mathcal{M}$ be a real-analytic, totally real submanifold of $\partial\Omega$. Assume that $T_p(\mathcal{M}) \subseteq H_p(\partial\Omega)$ for each $p \in \mathcal{M}$, $\partial\Omega$ is of constant type M along $\mathcal{M}$ and that the $(M-1)$th Levi-form of $\partial\Omega$ is positive definite on the real vector space $\mathbb{J}T_p(\mathcal{M}) \subseteq H_p(\partial\Omega)$ $\forall p \in \mathcal{M}$. Then, $\mathcal{M}$ is an analytic interpolation manifold.*

Remark 1.5. For $\mathcal{M}$ to be an analytic interpolation manifold, it is necessary for it to be totally real, since $\mathcal{M}$ must not admit any tangential Cauchy-Riemann equations induced by $\partial\Omega$. This condition on $\mathcal{M}$ is absent from Theorem 1.1 since it follows from strict pseudoconvexity. Additionally, the proof of Theorem 1.4 depends on showing that a complexification of $\mathcal{M}$ does not intersect $\overline{\Omega}$, in the sense of germs, off $\mathcal{M}$; we use the fact that $\mathcal{M}$ is totally real, in our proof, to construct a complexification of $\mathcal{M}$ that is naturally a complex submanifold of an open set in $\mathbb{C}^n$.

Remark 1.6. Given $p \in \partial\Omega$, we would like to find a formula for computing $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\centerdot)$ directly, without having to find holomorphic charts near each p in which the defining function has the form (1.1). Also, if p is of type M, Definition 1.3 only tells us what the $(M-1)$th Levi-form should be, whereas we would like to be able to compute $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\centerdot)$ for each k. Furthermore, we would like to define $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\centerdot)$ on $H_p(\partial\Omega)$ independently of the choice of local holomorphic coordinates. These issues are addressed in the next section.

2. Preliminary Lemmas

In this section, we state some general results concerning weakly pseudoconvex domains, which we shall use in Section 3 to prove Theorem 1.4.

Proposition 2.1. *Let Ω and $\mathcal{M}$ be as in Theorem 1.4. Let $p \in \mathcal{M}$. There exist an open neighbourhood $U(p) \subseteq \mathbb{C}^n$ of p and a smooth family $\{(\Phi_q;\omega_q)\}_{q\in U(p)\cap\mathcal{M}}$ of biholomorphisms*

$$\Phi_q : (\omega_q, q) \to (\Phi_q(\omega_q), 0)$$

such that $\Phi_q(\omega_q \cap \partial\Omega)$ is defined by

$$\varrho_q(w) = \sum_{\substack{|\alpha|+|\beta|=M \\ 1\leq|\beta|<M}} A^{(q)}_{\alpha\beta} {w_*}^{\alpha} {\bar{w}_*}^{\beta} + \mathcal{E}_q(w_*, \mathfrak{im}(w_n)) - \mathfrak{re}(w_n), \tag{2.1}$$

where we write $w = (w_1, ..., w_n) \equiv (w_*, w_n)$ and where $\mathcal{E}_q$ is a real-analytic function with the property that $\mathcal{E}_q(0,0) = 0,\ \nabla\mathcal{E}_q(0,0) = 0$, and that any term of order $\leq M$ is a mixed term involving non-zero powers of $\mathfrak{im}(w_n),\ w_*$ and $\bar{w}_*$.

Proof. The proof of this statement is standard and originates in [2]. ☐

We now provide a coordinate-free definition for the higher Levi forms, which allows us to compute $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\centerdot)$ given $p \in \partial\Omega$. To do this, we will need some preliminary notation. For a multi-index $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{N}^{n-1}$ and an integer $1 \leq \mu \leq (n-1)$ define the multi-index $\alpha * \mu$ by

$$\alpha * \mu = \begin{cases} (\alpha_1, ..., \alpha_{\mu-1}, \alpha_\mu - 1, \alpha_{\mu+1}, ..., \alpha_{n-1}), & \text{if } \alpha_\mu \geq 1 \\ (0, ..., 0), & \text{otherwise.} \end{cases}$$

Furthermore, if $\{\mathbf{S}_j\}_{j=1}^{n-1}$ is any local basis of vector fields for $H^{1,0}(\partial\Omega)$ near a point $p \in \partial\Omega$, we define $\mathbf{S}^\alpha$ as

$$\mathbf{S}^\alpha := \mathbf{S}_1^{\alpha_1} ... \ \mathbf{S}_{n-1}^{\alpha_{n-1}},$$

and we define $\overline{\mathbf{S}}^{\ \alpha}$ in an analogous way. Also, we will use angular brackets $\langle\ ,\ \rangle$ to denote contraction between a tangent vector and a form.

Definition 2.2 (This definition is due to Bloom, [1]). Let Ω be a bounded, weakly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let $p \in \partial\Omega$, and let ρ be a defining function for Ω.

(1) An alternate definition of the kth Levi-form of $\partial\Omega$ at p, $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\boldsymbol{\cdot}) : H^{1,0}_p(\partial\Omega) \to \mathbb{R}$ is as follows : Let $\{\mathbf{S}_j\}_{j=1}^{n-1}$ be any local basis of vector fields for $H^{1,0}(\partial\Omega)$ near p. Then, for any $\mathbf{v} \in H^{1,0}_p(\partial\Omega)$

$$\mathfrak{L}^{(k)}_{\partial\Omega}(p;\mathbf{v}) = \sum_{\substack{|\alpha|+|\beta|=k+1\\ 1\leq|\beta|<k+1}} \frac{\mathfrak{a}_{\alpha\beta}(p)}{\alpha!\ \beta!}\zeta^\alpha\bar{\zeta}^\beta,$$

where ζ_j are so defined that $\mathbf{v} = \sum_{j=1}^{n-1}\zeta_j\mathbf{S}_j|_p$, and where

$$\mathfrak{a}_{\alpha\beta}(p) = -\mathbf{S}^{\alpha*\mu}\overline{\mathbf{S}}^{\ \beta*\nu}\langle[\mathbf{S}_\mu, \overline{\mathbf{S}}_\nu]\ ,\ \overline{\partial}\rho\rangle(p),$$

μ, ν being so chosen that $\alpha * \mu$, $\beta * \nu \neq 0$ (the coefficient $\mathfrak{a}_{\alpha\beta}$ will be independent of the choice of μ, ν – this has been shown in [1]). We note here that $\mathfrak{L}^{(1)}_{\partial\Omega}(p\ ;\boldsymbol{\cdot})$ is just the usual Levi form of $\partial\Omega$ at p and that if p is a point of type M, then $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\boldsymbol{\cdot}) = 0$ if $k < M-1$.

(2) We would like to show that the foregoing definition is the same as Definition 1.3. This follows from the following result :

Theorem. *Let Ω be as in item (1), $p \in \partial\Omega$, and let p be of type M. Let Φ_p be as defined in Proposition 2.1. The $(M-1)$th Levi-form of $\Phi_q(\omega_q \cap \partial\Omega)$ at the origin, $\mathfrak{L}^{(M-1)}_{\Phi_q(\omega_q\cap\partial\Omega)}(0\ ;\boldsymbol{\cdot})$ is defined by*

$$\mathfrak{L}^{(M-1)}_{\Phi_q(\omega_q\cap\partial\Omega)}(0\ ;\boldsymbol{\cdot}) : \mathbb{C}^{n-1} \ni \zeta \mapsto \sum_{\substack{|\alpha|+|\beta|=M\\ 1\leq|\beta|<M}} A^{(q)}_{\alpha\beta}\zeta^\alpha\overline{\zeta}^\beta.$$

The above is a result of Bloom [1, Theorem 3.3]

Remark 2.3. Our definition of $\mathfrak{L}^{(k)}_{\partial\Omega}(p\ ;\boldsymbol{\cdot})$ (assume that p is a point of type M) differs from that in [1] by a sign. This is because, in that paper, the normal form for $\partial\Omega$ analogous to (2.1) above, in local coordinates, is taken to be

$$\varrho(w) = \mathfrak{re}(w_n) + \sum_{\substack{|\alpha|+|\beta|=M\\ 1\leq|\beta|<M}} A^{(p)}_{\alpha\beta}{w_*}^\alpha{\bar{w}_*}^\beta + \mathcal{E}_p(w_*, \mathfrak{im}(w_n)).$$

The reader can check that Definition 2.2(1), when applied to $\mathfrak{L}^{(1)}_{\partial\Omega}(p\ ;\boldsymbol{\cdot})$, gives us the usual Levi-form, which, if Ω is a pseudoconvex domain, is a positive semi-definite Hermitian form on $H^{1,0}_p(\partial\Omega)$.

Lemma 2.4. *Let Ω be a smoothly bounded pseudoconvex domain. Suppose that $0 \in \partial\Omega$ and suppose that, in a neighbourhood of 0, $\partial\Omega$ is defined by*

$$\rho(w) = h(w_*, \mathfrak{im}(w_n)) - \mathfrak{re}(w_n),$$

where h is a smooth function with $h(0) = 0$ and $\nabla h(0) = 0$ (and where we write $w = (w_1, ..., w_n) \equiv (w_*, w_n)$). Let D be an open neighbourhood of $0 \in \mathbb{R}^m$ and let $\gamma = (\gamma_1, ..., \gamma_n) : (D, 0) \to (\partial\Omega, 0)$ be a smooth imbedding. Assume that

(1) *$T_{\gamma(x)}(Image(\gamma)) \subseteq H_{\gamma(x)}(\partial\Omega)$ for every $x \in D$;*
(2) *The first Levi-form vanishes on $H^{1,0}_{\gamma(x)}(\partial\Omega)$ for each $\gamma(x)$.*

Then, $\gamma_n \equiv 0$.

Proof. $\{\mathbf{S}_j\}_{j=1}^{n-1}$ is a basis of $H^{1,0}(\partial\Omega)$ near the origin, where $\mathbf{S}_j$ have the form

$$\mathbf{S}_j|_w = \left.\frac{\partial}{\partial w_j}\right|_w + F_j(w_*, w_n)\left.\frac{\partial}{\partial w_n}\right|_w .$$

For any (1,0)-vector field given by $\mathbb{L}|_w = \sum_{j=1}^{n-1} A_j(w)\mathbf{S}_j|_w$, we compute :

$$[\mathbb{L}\ ,\ \overline{\mathbb{L}}] =\ 2i\,\Im\mathfrak{m}\left[\sum_{\mu,j=1}^{n-1} A_j(w)\frac{\partial\overline{A_\mu}}{\partial w_j}(w)\frac{\partial}{\partial\bar{w}_\mu} + \sum_{\mu,j=1}^{n-1} A_j(w)\left\{\frac{\partial\overline{A_\mu}}{\partial w_j}(w)\overline{F_\mu(w)} + \overline{A_\mu(w)}\frac{\partial\overline{F_\mu}}{\partial w_j}(w)\right\}\frac{\partial}{\partial\bar{w}_n}\right.$$
$$\left. + \sum_{\mu,j=1}^{n-1} A_j(w)F_j(w)\frac{\partial\overline{A_\mu}}{\partial w_n}(w)\frac{\partial}{\partial\bar{w}_\mu} + \sum_{\mu,j=1}^{n-1} A_j(w)\left\{\overline{A_\mu(w)}F_j(w)\frac{\partial\overline{F_\mu}}{\partial w_n}(w) + A_j(w)F_j(w)\overline{F_\mu(w)}\frac{\partial\overline{A_\mu}}{\partial w_n}(w)\right\}\frac{\partial}{\partial\bar{w}_n}\right].$$

Thus, we have

(2.2)
$$[\mathbb{L}\ ,\ \overline{\mathbb{L}}] =\ \mathbf{V} + 2i\ \Im\mathfrak{m}\left[\left(\sum_{\mu,j=1}^{n-1} A_j(w)\overline{A_\mu(w)}\frac{\partial\overline{F_\mu}}{\partial w_j}(w) + \sum_{\mu,j=1}^{n-1} A_j(w)\overline{A_\mu(w)}F_j(w)\frac{\partial\overline{F_\mu}}{\partial w_n}(w)\right)\frac{\partial}{\partial\bar{w}_n}\right],$$

where $\mathbf{V}$ is a section of $H^{\mathbb{C}}(\partial\Omega)$.

Now consider a (1,0)-vector field, that, restricted to Image(γ), is given by

$$\mathbb{L}|_{\gamma(x)} = \sum_{j=1}^{n-1}\left\{\sum_{k=1}^{m}\frac{\partial\gamma_j}{\partial x_k}(x)v_k\right\}\left.\frac{\partial}{\partial w_j}\right|_{\gamma(x)} + \left\{\sum_{j=1}^{n-1}\sum_{k=1}^{m}\frac{\partial\gamma_j}{\partial x_k}(x)F_j[\gamma(x)]v_k\right\}\left.\frac{\partial}{\partial w_n}\right|_{\gamma(x)},$$

where $(v_1, ..., v_m) \in \mathbb{C}^m$.

$$[\mathbb{L}\ ,\ \overline{\mathbb{L}}]\ (\text{mod}\ \ H^{\mathbb{C}}(\partial\Omega)) \equiv\ \mathfrak{a}(v;x)\left.\frac{\partial}{\partial\bar{w}_n}\right|_{\gamma(x)} - \overline{\mathfrak{a}(v;x)}\left.\frac{\partial}{\partial w_n}\right|_{\gamma(x)} = 0.$$

The last equality follows from the hypothesis (2) of the lemma. Using (2.2), we get

$$\mathfrak{a}(v;x) = \sum_{j,\mu=1}^{n-1}\sum_{k,\nu=1}^{m}\frac{\partial\gamma_j}{\partial x_k}(x)v_k\frac{\partial\overline{F_\mu}}{\partial w_j}[\gamma(x)]\overline{\frac{\partial\gamma_\mu}{\partial x_\nu}(x)}\overline{v_\nu} + \sum_{j,\mu=1}^{n-1}\sum_{k,\nu=1}^{m}\frac{\partial\gamma_j}{\partial x_k}(x)F_j[\gamma(x)]v_k\frac{\partial\overline{F_\mu}}{\partial w_n}[\gamma(x)]\overline{\frac{\partial\gamma_\mu}{\partial x_\nu}(x)}\overline{v_\nu}$$

In particular, observe that (ϵ_{k_0} below being the unit vector along the "v_{k_0}-axis")

$$\begin{aligned}\mathfrak{a}(\epsilon_{k_0};x) &= \sum_{\mu,j=1}^{n-1}\frac{\partial\overline{F_\mu}}{\partial w_j}[\gamma(x)]\frac{\partial\gamma_j}{\partial x_{k_0}}(x)\overline{\frac{\partial\gamma_\mu}{\partial x_{k_0}}(x)} + \sum_{\mu,j=1}^{n-1}\frac{\partial\gamma_j}{\partial x_{k_0}}(x)F_j[\gamma(x)]\frac{\partial\overline{F_\mu}}{\partial w_n}[\gamma(x)]\overline{\frac{\partial\gamma_\mu}{\partial x_{k_0}}(x)}\\ &= \sum_{\mu=1}^{n-1}\frac{\partial(\overline{F_\mu}\circ\gamma)}{\partial x_{k_0}}(x)\overline{\frac{\partial\gamma_\mu}{\partial x_{k_0}}(x)}\end{aligned}$$

since, by complex tangency, $\dfrac{\partial\gamma_n}{\partial x_{k_0}}(x) = \sum_{j=1}^{n-1} F_j[\gamma(x)]\dfrac{\partial\gamma_j}{\partial x_{k_0}}(x)$.

Therefore

$$\mathfrak{a}(v;x) =\ \sum_{k,\nu=1}^{m}\sum_{\mu=1}^{n-1}\frac{\partial(\overline{F_\mu}\circ\gamma)}{\partial x_k}(x)\overline{\frac{\partial\gamma_\mu}{\partial x_\nu}(x)}v_k\overline{v_\nu}\ =\ \langle v|[M_{jk}(x)][D_{jk}(x)]|v\rangle,$$

where

$$M_{jk} = \frac{\partial(\overline{F_k} \circ \gamma)}{\partial x_j}(x); \quad j = 1, ..., m; \; k = 1, ..., n-1$$

$$D_{jk} = \overline{\frac{\partial \gamma_j}{\partial x_k}}(x); \quad j = 1, ..., n-1; \; k = 1, ..., m$$

$$\langle v|[A_{jk}]|v\rangle = \sum_{j,k=1}^{m} v_j A_{jk} \overline{v_k}.$$

Consider the sesqui-linear forms

$$S_x : (u, v) \mapsto \langle u|[M_{jk}(x)][D_{jk}(x)]|v\rangle; \; x \in D.$$

Since $\mathfrak{a}(v; x) = 0$, $S_x(v, v) = 0 \in \mathbb{R}$ for each $v \in \mathbb{C}^m$ and for each $x \in D$. Thus, S_x are all Hermitian forms that are identically zero. Consequently, $[M_{jk}(x)][D_{jk}(x)] = 0$ for every x. Since $d\gamma(x)$ has maximal rank for each $x \in D$, we conclude that

$$\frac{\partial(\overline{F_\mu} \circ \gamma)}{\partial x_k} \equiv 0; \quad \forall \mu \leq n-1, \; \forall k \leq m.$$

This implies that, as $F_\mu[\gamma(0)] = 0$, $F_\mu \circ \gamma \equiv 0$, $\forall \mu \leq n-1$. So,

$$\nabla \gamma_n(x) \boldsymbol{.}\, v = \sum_{j=1}^{n-1} \sum_{k=1}^{m} \frac{\partial \gamma_j}{\partial x_k}(x) F_j[\gamma(x)] v_k = 0 \quad \forall x \in D, \; \forall v \in \mathbb{R}^m.$$

This implies that $\nabla \gamma_n \equiv 0$, whence $\gamma_n \equiv 0$ (since $\gamma_n(0) = 0$). ☐

3. Proof of theorem 1.4

. Without loss of generality, we may assume that $\partial\Omega$ is defined by a global defining function ρ that is defined in a neighbourhood $U \supseteq \partial\Omega$. Recall that $\partial\Omega$ is of type M along $\mathcal{M}$ and that $T_p(\mathcal{M}) \subseteq H_p(\partial\Omega)$ for each $p \in \mathcal{M}$. Pick a $p \in \mathcal{M}$. There exist a $V(p) \ni p$ open in $\partial\Omega$ and a real-analytic imbedding $\gamma : \mathbb{R}^m \supseteq (D, 0) \to (\mathcal{M} \cap V(p), p)$. It must be noted that $\gamma(\boldsymbol{.}) \equiv \gamma(\boldsymbol{.}\,; p)$, i.e. γ depends on p, but for purposes of notational convenience, we will suppress the dependence on p. It can easily be shown, using standard compactness and homogeneity arguments, that by our hypothesis on type along $\mathcal{M}$,

$$\mathfrak{L}^{(M-1)}_{\partial\Omega}(\gamma(x); i(d\gamma(x)\mathbf{v})) \geq C|\mathbf{v}|^M \;\; \forall \mathbf{v} \in \mathbb{R}^m, \tag{3.1}$$

and for each $p \in \mathcal{M}$, there exists a neighbourhood $V^*(p) \Subset V(p)$ such that (3.1) is true uniformly for all $\gamma(x) \in \text{Image}(\gamma) \cap V^*(p)$ with a uniform constant $C \equiv C(p)$.

By Proposition 2.1, there exist an open subset of $\partial\Omega$, $W(p) \Subset V^*(p)$ and a smooth family of biholomorphisms, $\{(\varPhi_q, \omega_q)\}_{q \in W(p) \cap \text{Image}(\gamma)}$, having the effect that for each $q \in W(p) \cap \text{Image}(\gamma)$, $\varPhi_q(\omega_q \cap \partial\Omega)$ is defined by a ϱ_q as given in (2.1). Adopting coordinates $w = \varPhi_q(z)$, write

$$\varrho_q(w, \bar{w}) = \mathcal{P}_q(w_*, \bar{w}_*) + \mathcal{E}_q(\mathfrak{im}(w_n), w_*) - \mathfrak{re}(w_n),$$

where $\mathcal{P}_q$ is the polynomial occuring in (2.1). Let $x_q = \gamma^{-1}(q)$, consider the ball $B(x_q;\varepsilon_q) \subseteq \gamma^{-1}(\omega_q)$, and let $\tau_q : x \mapsto (x + x_q)$. Define $\psi_q = \Phi_q \circ \gamma \circ \tau_q$. Note that $\psi_q : (B(0;\varepsilon_q), 0) \to (\Phi_q(\omega_q \cap \partial\Omega), 0)$. Also $d\psi_q(x) = d\Phi_q(\gamma(x + x_q)) \circ d\gamma(x + x_q)$.

$$
\begin{aligned}
\mathcal{P}_q(i(d\psi_q(0)\mathbf{v}), -i(\overline{d\psi_q(0)\mathbf{v}})) &= \mathfrak{L}^{(M-1)}_{\partial\Omega}(q; i(d\gamma(x_q)\mathbf{v})) \\
&= \mathfrak{L}^{(M-1)}_{\partial\Omega}(\gamma(x_q); i(d\gamma(x_q)\mathbf{v})) \\
&\geq C|\mathbf{v}|^M \ \forall \mathbf{v} \in \mathbb{R}^m,
\end{aligned} \tag{3.2}
$$

and from (3.1), we can infer that the above inequality is true uniformly for $q \in W(p) \cap \mathrm{Image}(\gamma)$. Note that the first equality in (3.2) follows from Definition 2.2.

Let Ψ^q be the complexification of ψ_q (i.e. Ψ^q is defined, wherever the resultant power-series converges, by replacing the real variable x by the complex variable ζ in the power-series of ψ_q). Since $\{(\Phi_q;\omega_q)\}_{q\in W(p)\cap\mathrm{Image}(\gamma)}$ is a smooth family, choosing $W(p)$ appropriately, we can find a $\sigma \equiv \sigma(p)$ such that Ψ^q are all defined as holomorphic maps on $B(0;\varepsilon_q)+iB(0;\sigma)$ for each $q \in W(p)\cap\mathrm{Image}(\gamma)$. Shrinking σ if necessary, we define $u_q : B(0;\varepsilon_q) + iB(0;\sigma) \to \mathbb{R}$ by

$$
\begin{aligned}
u_q(\zeta) &= \rho \circ (\Gamma|_{B(x_q;\varepsilon_q)+iB(0;\sigma)})(\zeta + x_q) \\
&= \varrho_q \circ \Psi^q(\zeta)
\end{aligned}
$$

where Γ is the complexification of γ in an appropriately small neighbourhood of x_q.

In what follows, we will write $\zeta = \xi + i\eta$, and $\Psi^q \equiv (\Psi^q_*, \Psi^q_n)$. By Lemma 2.4, which says that $\Psi^q_n \equiv 0$ when $M > 2$, or by the normal form (2.1) in case $M = 2$, u_q has the series expansion

$$
\begin{aligned}
u_q(\zeta) &= \sum_{\substack{|\alpha|+|\beta|=M \\ 1\leq|\beta|<M}} A^{(q)}_{\alpha\beta} \Psi^q_*(\zeta)^\alpha\, \overline{\Psi^q_*(\zeta)}^\beta + O(|\zeta|^{(M+1)}) \\
&= \sum_{\substack{|\alpha|+|\beta|=M \\ 1\leq|\beta|<M}} A^{(q)}_{\alpha\beta} \prod_{k=1}^{n-1} \left(\sum_{l=1}^{m} \frac{\partial \Psi^q_k}{\partial \zeta_l}(0)\zeta_l \right)^{\alpha_k} \prod_{k=1}^{n-1} \left(\sum_{l=1}^{m} \overline{\frac{\partial \Psi^q_k}{\partial \zeta_l}(0)\zeta_l} \right)^{\beta_k} + O(|\zeta|^{(M+1)}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u_q(i\eta) &= \mathcal{P}_q(i(d\psi_q(0)\eta), -i(\overline{d\psi_q(0)\eta})) + O(|\eta|^{(M+1)}) \\
&\geq C|\eta|^M + O(|\eta|^{(M+1)}) \qquad \text{(by (3.2))}
\end{aligned}
$$

Since the above is true uniformly for all $q \in W(p) \cap \mathrm{Image}(\gamma)$ with a uniform constant $C \equiv C(p)$, there is a $\delta_p > 0$ such that

$$
u_q(i\eta) > 0; \quad 0 < |\eta| < \delta_p,\ \forall q \in W(p) \cap \mathrm{Image}(\gamma).
$$

Another way of saying this is that the complex analytic set $\Gamma(\gamma^{-1}(W(p)) + iB(0;\delta_p))$ meets $\overline{\Omega}$ precisely along $W(p) \cap \mathcal{M}$.

We can, therefore, find an open neighbourhood $U(p)$ of p in $\mathbb{C}^n$ and a complex submanifold $\widetilde{\mathcal{M}}_p$ of $U(p)$ which is the complexification of $\mathcal{M}$ near p. $\{U(p)\}_{p\in\mathcal{M}}$ is an open cover of $\mathcal{M}$. As $\mathcal{M}$ is compact, there exist $p_1, ..., p_N \in \mathcal{M}$ and a tubular neighbourhood $\mathcal{U}$ of $\mathcal{M}$ such that

(1) $\mathcal{M} \subseteq \cup_{k=1}^N U(p_k)$ and $\mathcal{U} \subseteq \cup_{k=1}^N U(p_k)$.
(2) $\widetilde{\mathcal{M}} = \cup_{k=1}^N (\widetilde{\mathcal{M}}_{p_k} \cap \mathcal{U})$ is a complexification of $\mathcal{M}$, and a complex submanifold of $\mathcal{U}$ such that $\widetilde{\mathcal{M}} \cap \overline{\Omega} = \mathcal{M}$.

Let f be the real-analytic function prescribed on $\mathcal{M}$. Shrinking $\mathcal{U}$ if necessary, we may assume that f extends to a holomorphic function $\tilde{f}$ on $\widetilde{\mathcal{M}}$.

$\overline{\Omega}$ has a basis of Stein neighbourhoods. This follows from results by Diederich and Fornaess [4], [5], and this is where the assumption about $\partial\Omega$ being real-analytic gets used. Choose a Stein domain $\mathfrak{D} \supset \overline{\Omega}$ such that $\tilde{f}$ is holomorphic on the complex submanifold $(\widetilde{\mathcal{M}} \cap \mathfrak{D})$ of $\mathfrak{D}$. By standard techniques, we can show that $\tilde{f}$ extends to a $F \in \mathcal{O}(\mathfrak{D})$. We remark that this last step reflects a technique used in [3] (which follows from theorems A and B of Cartan). Thus, $\mathcal{M}$ is an analytic interpolation manifold. $\square$

4. Examples

Before we present our examples, we would like to prove the following proposition. It can be inferred from [3, Theorem 1], but for the sake of completeness, we provide a proof.

Proposition 4.1. *Let Ω be as in Theorem 1.4. Let $\mathcal{M}$ be a real-analytic submanifold of $\partial\Omega$, let $\widetilde{\mathcal{M}}$ be its complexification, and let $p \in \mathcal{M}$. Suppose there is a curve $\gamma \subset \widetilde{\mathcal{M}}$ passing through p and an $\varepsilon_0 > 0$ such that $[\gamma \cap B(p;\varepsilon)] \cap \overline{\Omega} \neq \emptyset \ \forall \varepsilon \in (0, \varepsilon_0]$, then $\mathcal{M}$ is not an analytic interpolation manifold.*

Proof. We assume that $\mathcal{M}$ is an analytic interpolation manifold. By hypothesis, there exists a real-analytic imbedding $\psi : (S^1, 1) \to (\mathcal{M}, p)$ onto a simple closed curve $C \subseteq \mathcal{M}$, so that the following happens :

For $r > 0$ sufficiently small, ψ extends to a regular, injective, holomorphic map Ψ on $\mathrm{Ann}(0; 1-r, 1+r)$. There is a small disc $\triangle$, centered at 1, such that (defining $\triangle^- = (\triangle \cap \{\zeta : |\zeta| < 1\})$), without loss of generality, we have

$$\Psi(\triangle^-) \cap \overline{\Omega} \neq \emptyset,$$
$$\Psi^{-1}(\Psi(\triangle^-) \cap \overline{\Omega}) \text{ contains a curve } L \text{ tending to } 1.$$

Now choose some ζ_0 in L. Define

$$g(z) = \frac{1}{(\psi^{-1}(z) - \zeta_0)}$$

which is real-analytic on C, and so extends real-analytically to $\mathcal{M}$. This last conclusion follows from a result by Serre [7, Secn.19(b)]. By assumption, there exists a G holomorphic in a neighbourhood, call it D, of $\overline{\Omega}$, with $G|_C = g$. We can choose $\triangle$ above to be so small that $\Psi(\triangle) \subset D$. We can then define

$$H(\zeta) = G \circ \Psi(\zeta) - \frac{1}{\zeta - \zeta_0}.$$

Clearly $H \in \mathcal{O}(\triangle \setminus \{\zeta_0\})$ and $H|_{\triangle \cap S^1} \equiv 0$. The latter implies that $H \equiv 0$. Yet, $G \circ \Psi \in \mathcal{O}(\triangle)$ whereas $1/(\zeta - \zeta_0)$ has a pole at ζ_0. This is a contradiction. Our assumption that $\mathcal{M}$ is an analytic interpolation manifold must, therefore, be false. $\square$

We can now show that the assumptions on type and positivity in the statement of the Theorem 1.4 cannot be relaxed. We will do this by constructing real-analytic submanifolds $\mathcal{M}$ in $\partial\Omega$ such that $[\widetilde{\mathcal{M}}] \cap \overline{\Omega} \supsetneq \mathcal{M}$ (here, the notation $[\widetilde{\mathcal{M}}]$ denotes the germ of the complexification $\widetilde{\mathcal{M}}$ along $\mathcal{M}$). In view of Proposition 4.1, $\mathcal{M}$ would not, therefore, be an analytic interpolation manifold. We remark here that if at some $p \in \mathcal{M}$, $T_p(\mathcal{M}) \nsubseteq H_p(\partial\Omega)$, then $[\widetilde{\mathcal{M}}] \cap \Omega \neq \emptyset$ near p. Complex-tangency is, thus, a *necessary* condition for $\mathcal{M}$ to be an analytic interpolation manifold.

Example 4.2. *An example of a domain Ω and a complex-tangential, totally real submanifold $\mathcal{M}$ where $\partial\Omega$ is of varying type along $\mathcal{M}$.*

Let $\Omega = \{(w_1, w_2) \in \mathbb{C}^2 | \ |w_1|^2 + |w_2|^4 < 1\}$.

Ω is a bounded, pseudoconvex domain with real-analytic boundary. $\mathcal{M}$ will be a real analytic curve passing through $(-1, 0) \in \mathbb{C}^2$, and we will analyze $\mathcal{M}$ near $(-1, 0)$. To simplify calculations, we will work with a biholomorph of Ω. Define $\omega = \{(z_1, z_2) \in \mathbb{C}^2 | \ \mathfrak{re}(z_1) > |z_2|^4\}$. The two domains are related by a biholomorphism

$$\Phi(z_1, z_2) = (w_1, w_2) = \left(\frac{z_1 - 1}{z_1 + 1}, \frac{\sqrt{2} z_2}{\sqrt{z_1 + 1}} \right),$$

where $\Phi : (\omega, \partial\omega) \to (\Omega, (\partial\Omega \setminus \{(1, 0)\}))$ (**Note :** Since $\mathfrak{re}(z_1) > 0$ when $(z_1, z_2) \in \overline{\omega}$, we can choose an appropriate analytic branch of $z_1 \mapsto \sqrt{z_1 + 1}$.). Consider the real-analytic, complex-tangential curve $\gamma : \mathbb{R} \to \partial\omega$ (and write $\mathfrak{m} = \text{Image}(\gamma)$)

$$\gamma(t) = (t^4, t).$$

Define $\mathcal{M} = \overline{\Phi(\mathfrak{m})}$. It is easy to check that $\mathcal{M} \subseteq \partial\Omega$ is a real-analytic, complex-tangential submanifold. Notice that at $t = 0$, Image(γ) passes through a point of type 4, whereas it is of type 2 elsewhere. Consequently, the Bloom-Graham type of $\partial\Omega$ varies along $\mathcal{M}$.

Let Γ denote the complexification of γ. Writing $\zeta = Re^{i\theta}$

$$\begin{aligned} \mathfrak{re}[\Gamma_1(\zeta)] = &\ R^4 \cos 4\theta, \\ |\Gamma_2(\zeta)|^4 = &\ R^4. \end{aligned}$$

Observe that $\mathfrak{re}[\Gamma_1(Re^{i\theta})] = |\Gamma_2(Re^{i\theta})|^4$ for $\theta = 0,\ \pi/2,\ \pi$ and $3\pi/2$. Thus $[\widetilde{\mathfrak{m}}] \cap \overline{\omega} \supsetneq \mathfrak{m}$. Since the notion of type is invariant under biholomorphic transformations, we have found a real-analytic, complex-tangential $\mathcal{M} \subseteq \partial\Omega$ such that $\partial\Omega$ is of varying type along $\mathcal{M}$ and such that $[\widetilde{\mathcal{M}}] \cap \overline{\Omega} \supsetneq \mathcal{M}$.

We comment on the notation used in the next three examples. In all of the equations involving Levi-forms, we will use the identification between $H_p^{1,0}(\partial\Omega)$ and $H_p(\partial\Omega)$ that was introduced in Definition 1.3(2).

Example 4.3. *An example of a domain Ω and a complex-tangential, totally real submanifold $\mathcal{M}$ where $\partial\Omega$ is of constant type along $\mathcal{M}$, but where the positivity condition fails.*

Let $\Omega = \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 | \ |w_1|^2 + \sum_{k=2}^4 |w_k|^4 < 1\}$.

Ω is a bounded, pseudoconvex domain with real-analytic boundary. As in Example 4.2, we will work with a biholomorph of Ω. Define $\omega = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 | \ \mathfrak{re}(z_1) > \sum_{k=2}^4 |z_k|^4\}$. The two domains are related by a biholomorphism

$$\Phi(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4) = \left(\frac{z_1 - 1}{z_1 + 1}, \frac{\sqrt{2} z_2}{\sqrt{z_1 + 1}}, \frac{\sqrt{2} z_3}{\sqrt{z_1 + 1}}, \frac{\sqrt{2} z_4}{\sqrt{z_1 + 1}} \right),$$

where $\Phi : (\omega, \partial\omega) \to (\Omega, (\partial\Omega \setminus \{(1,0,0,0)\}))$. Consider the real-analytic, complex-tangential curve $\gamma : [-\pi, \pi] \to \partial\omega$ (and write Image$(\gamma) = \mathfrak{m}$)

$$\gamma(\theta) = (\sin^4\theta + \cos^4\theta + 1,\ 1,\ \sin\theta,\ \cos\theta).$$

Let $\mathcal{M} = \Phi(\mathfrak{m})$. $\mathcal{M} \subseteq \partial\Omega$ is clearly a real-analytic, complex-tangential submanifold.

Notice that $H(\partial\omega)$ is spanned at every point by the vector fields

$$\mathbb{L}_1|_{(z_1,z_2,z_3,z_4)} = (4\overline{z_2}|z_2|^2, 1, 0, 0), \qquad \mathbb{L}_2|_{(z_1,z_2,z_3,z_4)} = (4\overline{z_3}|z_3|^2, 0, 1, 0),$$
$$\mathbb{L}_3|_{(z_1,z_2,z_3,z_4)} = (4\overline{z_4}|z_4|^2, 0, 0, 1).$$

Observe further that the complex Hessian of ρ (where ρ is the defining function of $\partial\omega$) is given by

$$(\mathfrak{H}_{\mathbb{C}}\rho)(z_1, z_2, z_3, z_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4|z_2|^2 & 0 & 0 \\ 0 & 0 & 4|z_3|^2 & 0 \\ 0 & 0 & 0 & 4|z_4|^2 \end{pmatrix}.$$

Let $\mathfrak{L}^{(1)}_{\partial\omega}(p\,;\boldsymbol{\cdot})$ denote the Levi-form at $p \in \partial\omega$. Notice that $\mathfrak{L}^{(1)}_{\partial\omega}(\boldsymbol{\cdot}\,;\mathbb{L}_1) \neq 0$ along $\mathfrak{m}$. Thus $\partial\omega$ is of constant Bloom-Graham type 2 along $\mathfrak{m}$. This implies that $\partial\Omega$ is of constant Bloom-Graham type 2 along $\mathcal{M}$. But

$$\mathfrak{L}^{(1)}_{\partial\omega}(\gamma(\theta);\ i\gamma'(\theta)) = 8\cos^2\theta\,\sin^2\theta$$

which vanishes at $\gamma(0)$, whence the positivity condition along $\mathfrak{m}$ fails.

Let Γ denote the complexification of γ. Writing $\zeta = \xi + i\eta$, we observe

$$\begin{aligned} \mathfrak{re}[\Gamma_1(i\eta)] &= \cosh^4\eta + \sinh^4\eta + 1, \\ |\Gamma_2(i\eta)|^4 + |\Gamma_3(i\eta)|^4 + |\Gamma_4(i\eta)|^4 &= \cosh^4\eta + \sinh^4\eta + 1. \end{aligned}$$

Thus $[\widetilde{\mathfrak{m}}] \cap \overline{\omega} \supsetneq \mathfrak{m}$, whence, arguing exactly as in Example 4.2 above, we have a real-analytic, complex-tangential $\mathcal{M} \subseteq \partial\Omega$ such that $\partial\Omega$ is of constant type along $\mathcal{M}$, that the positivity condition fails and such that $[\widetilde{\mathcal{M}}] \cap \overline{\Omega} \supsetneq \mathcal{M}$.

We would now like to give examples of analytic interpolation manifolds. In Example 4.4, our manifold $\mathcal{M}$ passes through weakly pseudoconvex points. In Example 4.5, $\mathcal{M}$ runs through points of Bloom-Graham type 4, although $\mathfrak{L}^{(3)}_{\partial\Omega}(p\,;\boldsymbol{\cdot})$ is not strictly positive definite on $H^{1,0}_p(\partial\Omega)$ for any $p \in \mathcal{M}$.

Example 4.4. *An example of a weakly pseudoconvex domain and an analytic interpolation manifold.*

Let Ω be exactly as in Example 4.3. All the notation used below will have the same meanings as in Example 4.3. As in that example, we will work with

$$\omega = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 |\ \mathfrak{re}(z_1) > \sum_{k=2}^{4} |z_k|^4\},$$

a biholomorph of Ω. Consider the real-analytic, complex-tangential curve $\gamma : [-\pi, \pi] \to \partial\omega$ (and write Image$(\gamma) = \mathfrak{m}$)

$$\gamma(\theta) = ((2 + \cos\theta)^4 + (2 + \sin\theta)^4,\ 0,\ 2 + \sin\theta,\ 2 + \cos\theta).$$

As before, define $\mathcal{M} = \Phi(\mathfrak{m})$, which is a complex-tangential, real-analytic submanifold of $\partial\Omega$.

As in Example 4.3 let $\mathfrak{L}^{(1)}_{\partial\omega}(p\ ;\boldsymbol{.})$ denote the Levi-form at $p \in \partial\omega$.

$$\mathfrak{L}^{(1)}_{\partial\omega}(\gamma(\theta);\ \mathbb{L}_3|_{\gamma(\theta)}) = 4(2+\cos\theta)^2 \neq 0,$$

whence $\partial\omega$ is of constant type 2 along $\mathfrak{m}$. Consequently, $\partial\Omega$ is of type 2 along $\mathcal{M}$, although *every* point on $\mathcal{M}$ is a weakly pseudoconvex point. To see this, observe that $\mathfrak{L}^{(1)}_{\partial\omega}(\boldsymbol{.}\ ;\mathbb{L}_1) = 0$ along $\mathfrak{m}$. But

$$\mathfrak{L}^{(1)}_{\partial\omega}(\gamma(\theta);\ i\gamma'(\theta)) = 4\cos^2\theta(2+\sin\theta)^2 + 4\sin^2\theta(2+\cos\theta)^2 > 0.$$

Since biholomorphisms preserve the positivity of the Levi-form and since Φ is biholomorphic in a neighbourhood of $\mathfrak{m}$, our positivity condition is preserved for $\mathcal{M}$. So $\mathcal{M} \subseteq \partial\Omega$ is a complex-tangential, real-analytic submanifold of $\partial\Omega$ that satisfies our positivity condition. By Theorem 1.4, therefore, $\mathcal{M}$ is an analytic interpolation manifold.

Example 4.5. *Another example of a weakly pseudoconvex domain and an analytic interpolation manifold $\mathcal{M}$. In this example, each point of $p \in \mathcal{M}$ is a point of type 4 and, in fact, $\mathfrak{L}^{(3)}_{\partial\Omega}(p\ ;\boldsymbol{.})$ is negative in certain directions in $H^{1,0}(\partial\Omega)$.*

Let $\Omega = \{(z,w) \in \mathbb{C}^2 |\ |w + e^{i\log(z\bar{z})}|^2 + C[\log(z\bar{z})]^4 < 1\}$.

This example is taken from [6]. For an appropriate $C > 0$, Ω is a pseudoconvex domain. We define

$$\mathcal{M} = \{(z,w) \in \partial\Omega |\ |z| = 1, w = 0\}.$$

$\mathcal{M}$ has a real-analytic parametrization $\gamma : [-\pi,\pi] \to \partial\Omega$ given by $\gamma(\theta) = (e^{i\theta}, 0)$. As in [6], we can show that each point of $\mathcal{M}$ is of type > 2.

In what follows, we will write $A(z) = e^{i\log(z\bar{z})}$, $B(z) = e^{-i\log(z\bar{z})}$. $H^{1,0}(\partial\Omega)$ is spanned by the vector field

$$\mathbb{L}|_{(z,w)} =\ z[\bar{w} + B(z)]\ \frac{\partial}{\partial z} - [i\bar{w}A(z) - iwB(z) + 4C\{\log(z\bar{z})\}^3]\ \frac{\partial}{\partial w}.$$

We can now see that $\gamma'(\theta) \in H_{(e^{i\theta},0)}(\partial\Omega)$. In fact, by the identification introduced in Definition 1.3(2)

$$i\gamma'(\theta)\ \rightleftharpoons\ -\mathbb{L}|_{(e^{i\theta},0)}.$$

We get the equation

$$\begin{aligned}(4.1)\quad & \mathfrak{L}^{(3)}_{\partial\Omega}((e^{i\theta},0);\ i\gamma'(\theta)) \\ &= -\left\{(-1)^3(-1)\frac{\mathbb{L}^2}{3!} + (-1)^2(-1)^2\frac{\mathbb{L}\overline{\mathbb{L}}}{2!\ 2!} + (-1)(-1)^3\frac{\overline{\mathbb{L}}^2}{3!}\right\}\langle[\mathbb{L},\overline{\mathbb{L}}]\ ,\overline{\partial}\rho\rangle(e^{i\theta},0).\end{aligned}$$

We compute to find that

$$(4.2)\quad -\langle[\mathbb{L},\overline{\mathbb{L}}]\ ,\overline{\partial}\rho\rangle(z,w) =\ 12C[\log(z\bar{z})]^2 + O([\log(z\bar{z})]^3,\ wB(z),\ \bar{w}A(z),\ |w|^2,\ |w|[\log(z\bar{z})]^2).$$

It can easily be shown from (4.1) and (4.2) that

$$\mathfrak{L}^{(3)}_{\partial\Omega}((e^{i\theta},0);\ i\gamma'(\theta)) = 14C > 0.$$

Similarly, we can show that $\mathfrak{L}^{(3)}_{\partial\Omega}((e^{i\theta},0);\ i\mathbb{L}) = -2C < 0$. So, $\mathfrak{L}^{(3)}_{\partial\Omega}((e^{i\theta},0)\ ;\boldsymbol{.})$ is actually negative in certain directions in $H^{1,0}_{(e^{i\theta},0)}(\partial\Omega)$. Yet, by Theorem 1.4, $\mathcal{M}$ is an analytic interpolation manifold (in this case, we can also check very easily that $[\widetilde{\mathcal{M}}] \cap \overline{\Omega} = \mathcal{M}$).

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Mathematics Department, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706
E-mail address: `bharali@math.wisc.edu`