SOLUTION OF QUIZ-9

Note: The question in the quiz was not correct. We will prove the following inequality instead $\frac{\sqrt{2}}{3} \leq \int\limits_0^1 \sqrt{\frac{x}{1+x}}dx \leq \frac{2}{3}$.

Solution:

Weighted Mean-Value Theorem For Integrals : Assume f and g are continuous on $[a, b]$. If g never changes sign in $[a, b]$ then, for some c in $[a, b]$ we have

$\int\limits_a^b f(x)g(x)dx = f(c)\int\limits_a^b g(x)dx$.

Moreover, if $m = minf(x)$ and $M = maxf(x)$ on $[a, b]$ and g non-negative, then we have

$m\int\limits_a^b g(x)dx \leq \int\limits_a^b f(x)g(x)dx \leq M\int\limits_a^b g(x)dx$.

We choose $f(x) = \frac{1}{\sqrt{1+x}}$ and $g(x) = \sqrt{x}$. It is easy to see that f, g satisfies the conditions of the above theorem on $[0, 1]$ and f is monotonically decreasing. The minimum of f is $\frac{1}{\sqrt{2}}$ and maximum is 1. Hence we have

$$\frac{1}{\sqrt{2}}\int\limits_0^1 g(x)dx \leq \int\limits_0^1 f(x)g(x)dx \leq \int\limits_0^1 g(x)dx$$

But $\int\limits_0^1 \sqrt{x}dx = \frac{2}{3}$. So, we have

$$\frac{1}{\sqrt{2}}\frac{2}{3} \leq \int\limits_0^1 f(x)g(x)dx \leq \frac{2}{3}$$

$$\Longrightarrow \frac{\sqrt{2}}{3} \leq \int\limits_0^1 \sqrt{\frac{x}{1+x}}dx \leq \frac{2}{3}$$