Solution Of Quiz 6

Question : Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable everywhere. Suppose $f(x) = 0$ for exactly three different values, namely, x_1, x_2 and x_3. Prove that there exists a point $c \in \mathbb{R}$ such that $f^{''}(c) = 0$

Solution : First of all, without loss of generality we can assume that $x_1 < x_2 < x_3$. It is given that $f(x_1) = f(x_2) = f(x_3) = 0$. Now consider the closed interval $[x_1, x_2]$. By *Rolle's Theorem* there exists $y_1 \in (x_1, x_2)$ such that $f^{'}(y_1) = 0$ (since $f(x_1) = f(x_2)$). Similarly considering the interval $[x_2, x_3]$ we can say that there exists $y_2 \in (x_2, x_3)$ such that $f^{'}(y_2) = 0$. Clearly $y_1 < y_2$ as $y_1 < x_2 < y_2$. We consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f^{'}(x)$ for all $x \in \mathbb{R}$. Since the function f is twice differentiable the function g becomes a differentiable function and $g^{'}(x) = f^{''}(x)$ for all $x \in \mathbb{R}$. Observe that $g(y_1) = g(y_2) = 0$. Therefore again by *Rolle's Theorem* there exists $c \in (y_1, y_2)$ such that $g^{'}(c) = 0$, i.e. $f^{''}(c) = 0$. Hence proved.