Quiz - 5

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Question: Let $a, b, c, d \in \mathbb{R}$. Find all conditions on a, b, c, d such that the function f below is continuous everywhere. Clearly state the limit theorems you are using at each step.

$$f(x) = \begin{cases} x^2 + 12x + b, & x > 0, \\ x + \frac{a}{x-1} + \frac{d}{x+7}, & x < 0, \\ c, & x = 0. \end{cases}$$

Solution: For $x > 0$, f is a polynomial, so it is continuous for all $x > 0$. Now $\frac{1}{x+7}$ is not defined at $x = -7$. So for f to be defined for $x < 0$, d has to be 0. So we get, $f(x) = x + \frac{a}{x-1}$, for $x < 0$. Since it is a rational function with non-zero denominator, it is continuous for all $x < 0$. Now,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 12x + b) = \lim_{x \to 0^+} x^2 + \lim_{x \to 0^+} 12x + \lim_{x \to 0^+} b = b$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x + \frac{a}{x-1} \right) = \lim_{x \to 0^-} x + \lim_{x \to 0^-} \left(\frac{a}{x-1} \right) = 0 + \frac{a}{\lim\limits_{x \to 0^-} (x-1)} = -a$$

For the function f to be continuous at 0,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} = f(0) = c$$

i.e.

$$b = -a = c$$

So f is continuous everywhere if $d = 0$ and $-a = b = c$.