QUIZ 4

<u>**Question**</u>: Prove that the series $\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n}$ converges. Does the series converges absolutely? Give a short justification to support your answer. $(6 + 4)$

<u>**Solution**</u>: We will use Dirichlet's Test to prove that the series converges.

Dirichlet's Test: Let $\sum a_n$ be a series of complex terms whose partial sums form a bounded sequence. Let $\{b_n\}$ be a decreasing sequence which converges to 0. Then the series $\sum a_n b_n$ converges.

Let $a_n = \sin(n\pi/3)$ and $b_n = \frac{1}{n}$. Observe that $\{b_n\}$ is decreasing and converges to 0.
The sequence $\{a_n\}$ is periodic with period 6 i.e., $a_{6+n} = a_n,\ \forall n \in \mathbb{N}$ and the first six terms are $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$. So if $\{A_n\}$ is the sequence of partial sums of $\{a_n\}$, then
$-4\frac{\sqrt{3}}{2} \leq A_n \leq 4\frac{\sqrt{3}}{2}$(Note that $A_{6k} = 0,\ \forall k \in \mathbb{N}$). So the sequence of partial sums is bounded. Now using Dirichlet's Test, we conclude that the given series converges.

NO, the given series does not converge absolutely. Let $\{c_n\} = \frac{\sin(n\pi/3)}{n}$, then $c_{3n} = 0$ and $|c_{3n+1}| = \frac{\sqrt{3}}{2(3n+1)}, |c_{3n+2}| = \frac{\sqrt{3}}{2(3n+2)}$.
Hence, $\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} |\frac{\sin(n\pi/3)}{n}| = \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty}[\frac{1}{3n+1} + \frac{1}{3n+2}] \geq \sqrt{3} \sum_{n=0}^{\infty} \frac{1}{3n+2}$(because $3n + 2 > 3n + 1$).
Now $\sum_{n=0}^{\infty} \frac{1}{3n+2} \geq \sum_{n=0}^{\infty} \frac{1}{3n+3} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{n+1}$. The series $\frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{n+1}$ is divergent. Hence, the given series is not absolutely convergent.