Quiz - 3

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Question: Determine all nonnegetive values of x for which the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n3^n}$ converges.
Solution: Suppose $a_n := \frac{x^{n-1}}{n3^n}$ $\forall n \in \mathbb{N}$. We need to determine all nonnegetive values of x for which the series $\sum_n a_n$ converges. First note that for $x = 0$ the series converges trivially. Assume $x > 0$. Observe that

$$\frac{a_{n+1}}{a_n} = \frac{nx}{3(n+1)} \to \frac{x}{3} \; as \; n \to \infty. \Big(\; since \; \lim_{n\to\infty} \frac{n}{1+n} = 1 \Big)$$

Now using ratio test we can say that the above series converges when $\frac{x}{3} < 1$ i.e., when $x < 3$ and diverges when $\frac{x}{3} > 1$ i.e., when $x > 3$. Note that for $x = 3$ we have $a_n = \frac{1}{3n}$ $\forall n \in \mathbb{N}$. But we know that the harmonic series $\sum_n \frac{1}{n}$ diverges. So, $\sum_n \frac{1}{3n}$ diverges. Hence the given series converges for $x \in [0, 3)$.