**Quiz 12 solution**

Let $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$. Then

$$T(e_1) = (1,3,2) = 1.e_1 + 3.e_2 + 2.e_3$$

$$T(e_2) = (2,1,3) = 2.e_1 + 1.e_2 + 3.e_3$$

$$T(e_3) = (3,2,1) = 3.e_1 + 2.e_2 + 1.e_1$$

Therefore the matrix $m(T)$ in the standard basis is

$$m(T) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

**Claim: $T$ is onto-** If we show that the null-space of $T$ is zero then by the Rank-Nullity theorem, the dimension of the range of $T$ will be 3. Therefore range of $T$ will be $\mathbb{R}^3$, $T$ will be onto.

Let $(x, y, z) \in \mathbb{R}^3$ be such that $T(x, y, z) = (0, 0, 0)$. So we have the system of linear equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now apply the Gauss-Jordan elimination process. In following paragraph $R_i - cR_j$ denote the operation, namely, substracting c times $j^{th}$ row from $i^{th}$ row of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3\text{-}2R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2\text{-}3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3\text{-}\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & -\frac{18}{5} & 0 \end{bmatrix}$$

So we can get the solution from the following system of equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives us the solution $x = 0, y = 0, z = 0$. Which proves that the null space of $T$ is zero .