Solutions of Mid-sem 2
Analysis and Linear Algebra I (Autumn 2018)
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1.

A function $s(x)$ on a closed interval $[a, b]$ is called a **step function** if there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ such that s is constant on each open subinterval of P. That is to say, for each $k = 1, 2, ..., n$ there is a real number s_k, such that $s(x) = s_k$ whenever $x_{k-1} < x < x_k$.

Let s be a step function on $[a, b]$ and let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$ such that s is constant on the open subintervals of P. Let $s(x) = s_k$ whenever $x_{k-1} < x < x_k$. Then

$$\int_a^b s(x)dx = \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1})$$

Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of the interval $[a, b]$ such that s is constant on the open subintervals of P. Assume that $s(x) = s_i$ if $x_{i-1} < x < x_i$. Let $t(x) = s(x/k)$ if $ka \le x \le kb$. Then $t(x) = s_i$ if x lies in the open interval (kx_{i-1}, kx_i); hence $P = \{kx_0, kx_1, \cdots, kx_n\}$ is a partition of $[ka, kb]$ and t is constant on the open subintervals of P. Therefore t is a step function whose integral is

$$\int_{ka}^{kb} t(x)dx = \sum_{i=1}^{n} s_i \cdot (kx_i - kx_{i-1}) = k\sum_{i=1}^{n} s_i \cdot (x_i - x_{i-1}) = k\int_a^b s(x)dx$$

Hence

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)dx = k\int_a^b s(x)dx$$

2.

$f(x) = x^3 - 3x - 1$ and $g(x) = -2x^2 - 1$. The graphs of f and g meet whenever $f(x) = g(x)$, i.e.

$$x^3 - 3x - 1 = -2x^2 - 1$$

i.e.

$$x^3 + 2x^2 - 3x = 0$$

i.e.

$$x(x-1)(x+3) = 0$$

i.e. $x = -3$ or $x = 0$ or $x = 1$. Notice $f(x) - g(x) = x(x-1)(x+3)$.
Now, if $x \in (-3, 0)$ then $(x+3) > 0$ and $(x-1) < 0$. So $f(x) - g(x) > 0$.
If $x \in (0, 1)$ then $(x+3) > 0$ and $(x-1) < 0$. So $f(x) - g(x) < 0$.
So the area of the regions enclosed by the graphs of $f(x)$ and $g(x)$ is

$$\int_{-3}^{1} |f(x) - g(x)| dx$$

$$= \int_{-3}^{0} (f(x) - g(x)) dx + \int_{0}^{1} (g(x) - f(x)) dx$$

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x) dx + \int_{0}^{1} (3x - 2x^2 - x^3)) dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2} \right]_{x=-3}^{x=0} + \left[\frac{3x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} \right]_{x=0}^{x=1}$$

$$= \tfrac{45}{4} + \tfrac{7}{12} = \tfrac{71}{6}$$

3.

Let f be a bounded function on $[a, b]$. Let us define

$$S := \left\{ \int_a^b s(x) dx \mid s : [a, b] \to \mathbb{R} \text{ is a step function with } s(x) \le f(x)\, \forall x \in [a, b] \right\}$$

$$T := \left\{ \int_a^b t(x) dx \mid t : [a, b] \to \mathbb{R} \text{ is a step function with } f(x) \le t(x)\, \forall x \in [a, b] \right\}$$

Since f is bounded, S is bounded above and T is bounded below. Sup$S$ is called the **lower integral** and Inf T is called the **upper integral** of f on $[a, b]$.
f is continuous and bounded on $[a, b]$ except at $b \in (a, c)$. Hence f is continuous on $[a, b)$ and (b,c]. So f is integrable on [a,b] and [b,c] as continuous functions are integrable. Let us define

$$S := \left\{ \int_a^c s(x) dx \mid s : [a, c] \to \mathbb{R} \text{ is a step function with } s(x) \le f(x)\, \forall x \in [a, c] \right\}$$

$$S_1 := \left\{ \int_a^b s_1(x)dx \mid s_1 : [a,b] \to \mathbb{R} \text{ is a step function with } s_1(x) \le f(x)\ \forall x \in [a,b] \right\}$$

$$S_2 := \left\{ \int_b^c s_2(x)dx \mid s_2 : [b,c] \to \mathbb{R} \text{ is a step function with } s_2(x) \le f(x)\ \forall x \in [b,c] \right\}$$

We claim that $S = S_1 + S_2$.
Let $p \in S$. So $p = \int_a^c s(x)dx$ for some below step function s.
Notice that $s|_{[a,b]}$ and $s|_{[b,c]}$ are also below step functions respectively on $[a,b]$ and $[b,c]$. Now
$p = \int_a^c s(x)dx = \int_a^b s(x)dx + \int_b^c s(x)dx = \int_a^b s|_{[a,b]}(x)dx + \int_b^c s|_{[b,c]}(x)dx$
Observe that $\int_a^b s|_{[a,b]}(x)dx \in S_1$ and $\int_b^c s|_{[b,c]}(x)dx \in S_2$
So, $p \in S_1 + S_2$ and hence $S \subset S_1 + S_2$.
Let $q \in S_1 + S_2$. Then $\exists$ step functions $s_1 : [a,b] \to \mathbb{R}$ with $s_1(x) \le f(x) \forall x \in [a,b]$ and $s_2 : [b,c] \to \mathbb{R}$ with $s_2(x) \le f(x) \forall x \in [b,c]$ satisfying $q = \int_a^b s_1(x)dx + \int_b^c s_2(x)dx$.
Define $s(x) = s_1(x)$ if $x \in [a,b)$
$= f(b)$ if $x = b$
$= s_2(x)$ if $x \in (b,c]$
Here s is also a below step function on $[a,c]$ and $\int_a^c s(x)dx = \int_a^b s(x)dx + \int_b^c s(x)dx = \int_a^b s_1(x)dx + \int_b^c s_2(x)dx$. So $q \in S$ and hence $S_1 + S_2 \subset S$.
Hence we prove our claim that $S = S_1 + S_2$.
Since f is bounded above, S, S_1, S_2 all are bounded above. Also Sup$S$=SupS_1+SupS_2 [follows from the fact that if $A, B, C \subset \mathbb{R}$, bounded above and $A = B + C$ then Sup$A$=Sup$B$+Sup$C$]........................(1)
Similarly define

$$T := \left\{ \int_a^c t(x)dx \mid t : [a,c] \to \mathbb{R} \text{ is a step function with } f(x) \le t(x)\ \forall x \in [a,c] \right\}$$

$$T_1 := \left\{ \int_a^b t_1(x)dx \mid t_1 : [a,b] \to \mathbb{R} \text{ is a step function with } f(x) \le t_1(x)\ \forall x \in [a,b] \right\}$$

$$T_2 := \left\{ \int_b^c t_2(x)dx \mid t_2 : [b,c] \to \mathbb{R} \text{ is a step function with } f(x) \le t_2(x)\ \forall x \in [b,c] \right\}$$

We can similarly show that $T = T_1 + T_2$
f is bounded below implies T, T_1, T_2 all are bounded below and Inf T=Inf T_1+Inf T_2........................(2)
Now f is integrable on $[a,b]$ and $[b,c]$. Hence SupS_1=Inf T_1 and SupS_2=InfT_2
From (1) and (2), we get Sup$S$=Inf T
Hence f is integrable on $[a,c]$.

4 a.
Let $G(x)$ be the primitive of $\frac{x^6}{1+x^4}$. Now using the second fundamental

theorem, we get

$$f(x) = \int_{x^3}^{x^2} \frac{t^6}{1+t^4} dt = \int_0^{x^2} \frac{t^6}{1+t^4} dt - \int_0^{x^3} \frac{t^6}{1+t^4} dt = G(x^2) - G(x^3)$$

Hence

$$\begin{aligned} f'(x) &= G'(x^2)\cdot 2x - G'(x^3)\cdot 3x^2 && \text{[Chain rule]} \\ &= \frac{(x^2)^6}{1+(x^2)^4}\cdot 2x - \frac{(x^3)^6}{1+(x^3)^4}\cdot 3x^2 \\ &= \frac{2x^{13}}{1+x^8} - \frac{3x^{20}}{1+x^{12}} \end{aligned}$$

4 b.

$$\int_0^x f(t)dt = f(x)^2 + C$$

Taking derivative on both sides w.r.t. x and using the first fundamental theorem we get,

$$f(x) = 2f(x)f'(x)$$
$$\implies f'(x) = \frac{1}{2}$$

Since, f is a nonconstant function, we discard the case $f \equiv 0$. Let us take $f(x) = \frac{x}{2}$. Now

$$\int_0^x f(t)dt = \int_0^x \frac{t}{2} dt = \frac{1}{2}\cdot\frac{x^2}{2} = \left(\frac{x}{2}\right)^2 = f(x)^2$$

So, $f(x) = \frac{x}{2}$ satisfies the given condition for $C = 0$.

5 a.

$$I = \int \frac{1}{\sqrt{x + x^{3/2}}} dx = \int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx$$

Let $1 + \sqrt{x} = y$. Then $\frac{dx}{\sqrt{x}} = 2dy$. Substituting these, we get,

$$I = \int \frac{2dy}{\sqrt{y}} = 4\sqrt{y} + C = 4\sqrt{1+\sqrt{x}} + C$$

where C is an arbitrary constant.

5 b.

We will use integral by parts to evaluate the integral, which says, if f and g are two continuously differentiable function then the following holds :

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx + C$$

In our case let $f(x) = x^2, g(x) = -\cos x$. Then $g'(x) = \sin x$ and we have

$$\int x^2 \sin x \cdot dx = -x^2 \cos x + \int 2x \cos x dx + C_1$$

$= -x^2 \cos x + 2[x \sin x - \int \sin x dx] = -x^2 \cos x + 2x \sin x + 2 \cos x + C$, where C is an arbitrary constant. To evaluate the 2nd integral again we applied integral by parts, taking $f(x) = x$ and $g(x) = \sin x$. So we get

$$\int x^2 \sin x \cdot dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

where C is an arbitrary constant.

6.

Let $f(x) = x^2 - \sin x$

$f'(x) = 2x - \cos x;\ f''(x) = 2 + \sin x;\ f'''(x) = \cos x;\ f^{(iv)}(x) = -\sin x$

Taking the 3rd degree Taylor polynomial around 0, we get

$T_3(f(x), 0) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$\quad = -x + x^2 + \frac{x^3}{6}$

Now $T_3(f(x), 0) = 0 \implies -x + x^2 + \frac{x^3}{6} = 0 \implies x^2 + 6x - 6 = 0 \qquad [x \neq 0]$

We get $x = \frac{-6 \pm \sqrt{36+24}}{2} = \pm\sqrt{15} - 3$. But $(-3 - \sqrt{15})$ can not be an approximation to the root since $(-3 - \sqrt{15})^2 > 9$ but $\sin x \leq 1\ \forall x \in \mathbb{R}$.

We know $f(x) = T_3(f(x), 0) + E_3(x)$ where E_3 is the error. But $T_3(f(x), 0) = 0$ implies $|f(x)| = |\sin x - x^2| = |E_3(x)|$ Now,

$$E_3(r) = \frac{1}{3!} \int_0^r (r-x)^3 f^{(iv)}(x) dx$$

So,

$$\begin{aligned}
|E_3(r)| &= \frac{1}{3!} \left| \int_0^r (r-x)^3 f^{(iv)}(x) dx \right| \\
&\leq \frac{1}{6} \int_0^r |(r-x)^3 f^{(iv)}(x)| dx \\
&\leq \frac{1}{6} \int_0^r |(r-x)^3| dx \qquad [|f^{(iv)}(x)| = |\sin x| \leq 1] \\
&= \frac{1}{6} \int_0^r (r-x)^3 dx \qquad [r - x \geq 0\ \forall\ x \in [0, r]] \\
&= -\frac{1}{6} \left[\frac{(x-r)^4}{4} \right]_{x=0}^{x=r} = \frac{r^4}{24} \\
&\leq \frac{(0.9)^4}{24} \leq 0.027 \qquad\qquad [\text{Given } r < 0.9]
\end{aligned}$$