Solutions of mid-sem examination

(1) Since X is non-empty and bounded above, X has the supremum in $\mathbb{R}$. Let $sup(X)$ be the supremum of X. Therefore,

$$x \leq sup(X), \ \forall x \in X$$

$$i.e., -x \geq -sup(X), \ \forall x \in \ X$$

hence for any $a \in \mathbb{R}$,

$$a - x \geq a - sup(X), \ \forall x \in \ X. \quad ...(1)$$

Since X is non-empty, therefore $a - X$ is non-empty. And also by the equation (1) above, $a - X$ has the infimum in $\mathbb{R}$ and $a - sup(X) \leq inf(a - X)$.

Let $M \in \mathbb{R}$ be such that $M > a - sup(X)$. Therefore $sup(X) > a - M$. Now by the definition of $sup(X)$, there exists a $x \in X$ such that $x > a - M$, i.e., $a - x < M$. Therefore M is not a lower bound of $a - X$. Which gives $a - sup(X)$ is the infimum of $a - X$.

(2) Given $a > 1$, the sequence $\frac{a^n}{n}$ is divergent.
To prove this, we will use problem no. 1 of Homework-2: An unbounded sequence can not have a limit i.e., divergent.
$\exists \ h > 0$ such that $a = 1 + h$. Now using binomial expansion(for $n \geq 3$) we have

$$\begin{aligned} &a^n \\ &= (1+h)^n \\ &= 1 + nh + \frac{n(n-1)h^2}{2} + ... \\ &\geq \frac{n(n-1)h^2}{2} \end{aligned}$$

So we have $\frac{a^n}{n} \geq \frac{(n-1)h^2}{2}$ for $n \geq 3$. Now the R.H.S$(\frac{(n-1)h^2}{2})$ is unbounded because $h^2/2$ is fixed(positive) and $\mathbb{N}$ is unbounded. So the given sequence is unbounded and we are done by problem no. 1 of Homework-2.

(3) $\sum_{n=1}^{\infty} \frac{(2-x)^n}{n(x+1)^n}$.
We will use Dirichlet test to show that the series is absolutely convergent and hence convergent when $x \in (1/2, \infty)$.
Note that $b_n = 1/n$ is a monotonically decreasing sequence which converges to 0. Also, for $x \in (1/2, \infty)$, $|\frac{(2-x)}{(x+1)}| < 1$. Hence, for $a_n = \frac{(2-x)^n}{n(x+1)^n}$, $\sum_{n=1}^{\infty} a_n$ is bounded. So, from Dirichlet test, the series converges absolutely.
When $x = 1/2$, the series is $\sum \frac{1}{n}$ which we know is divergent. For $x \in (0, 1/2)$, $\frac{(2-x)}{(x+1)} > 1$. So, $\frac{(2-x)^n}{n(x+1)^n} > \frac{1}{n}$. Hence, by comparison test, the series diverges.
Therefore, $(1/2, \infty)$ is the set of all non-negative values of x for which the above series converges.

Remark: One can use the ratio test or root test to prove convergence in $(1/2, 2]$ and divergence in $(0, 1/2)$. Then use Alternating test to show convergence in $(2, \infty)$ and argue the divergence at $1/2$ as above.

(4) Given that $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n > 0, b_n > 0$ and $a_n b_{n+1} > a_{n+1} b_n$ for all $n \geq 1$. So, we have

$$\frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}} \quad \forall n \geq 1.$$

Hence $\{\frac{a_n}{b_n}\}$ is a strictly decreasing sequence of positive real numbers. So,

$$c := \frac{a_1}{b_1} > \frac{a_n}{b_n} \quad \forall n > 1.$$

$$i.e., \; b_n > \frac{1}{c} a_n \quad \forall n > 1.$$

Hence if $\sum a_n$ diverges, then by **comparison test** $\sum b_n$ diverges.

(5)
Guess: $\lim_{x \to 0} \frac{x^2 - 2x}{x} = -2.$
Proof (using definition): Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon$. Now see that whenever $0 < |x - 0| = |x| < \delta$, we have

$$\left| \frac{x^2 - 2x}{x} - (-2) \right| = |x - 2 + 2| \; [since \; x \neq 0]$$

$$= |x| < \delta = \varepsilon.$$

Hence our guess is correct.

Problem 6 : Define continuity of a function f at a point a. If the function is continuous at a then prove that so is $|f|$.

Solution : A function f is said to be continuous at a point a if for any given $\epsilon > 0$ there exists a $\delta > 0$ (depending upon the given ϵ) such that

$$|f(x) - f(a)| < \epsilon \qquad \text{whenever } |x - a| < \delta$$

• It is given that the function f is continuous at a, we have to show that the function $|f|$ is continuous at a. Let $\epsilon > 0$ be given. Using the continuity of f we can say that there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \qquad \text{whenever } |x - a| < \delta \tag{1}$$

Also by *Triangle inequality* we have the following

$$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)| \tag{2}$$

Therefore by (1) and (2) we have

$$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)| < \epsilon \qquad \text{whenever } |x - a| < \delta$$

i.e.

$$\big||f(x)| - |f(a)|\big| < \epsilon \qquad \text{whenever } |x - a| < \delta$$

Therefore we have shown that $|f|$ is continuous at a.

(7) We know that polynomials are continuous. Hence, as $cosx$ is continuous, so is $1 + x^2 + cosx$. Now, for $x \neq 0$, $x^2 > 0$. Also, $1 + cosx \geq 0$. So, $1 + x^2 + cosx > 0$. For $x = 0$, $1 + x^2 + cosx = 2 > 0$. Hence, $\frac{103}{1+x^2+cosx}$ is continuous. (Since, $f(x)$ and

$g(x)$ being continuous, $\frac{f(x)}{g(x)}$ is continuous whenever $g(x) \neq 0$). Hence, $x^5 + \frac{103}{1+x^2+cosx}$ is continuous.
Now, observe that for $x = 0$, $x^5 + \frac{103}{1+x^2+cosx} = \frac{103}{2} > 4$. For $x = -2$, $x^5 + \frac{103}{1+x^2+cosx} = -32 + \frac{103}{1+4+cos(-2)} \leq -32 + \frac{103}{4} < 4$. (Since $cos(-2) \geq -1$).
Hence, by intermediate value theorem, the given equation admits a solution in the interval $(-2, 0)$

8.

$f(x) = x\sqrt{x + \sqrt{x}}$

$\frac{d}{dx} f(x) = \frac{d}{dx}(x\sqrt{x + \sqrt{x}})$

$= x \cdot \frac{d}{dx}(\sqrt{x + \sqrt{x}}) + \sqrt{x + \sqrt{x}} \cdot \frac{dx}{dx}$ [Multiplicative rule: $\frac{d}{dx}(u(x)v(x)) = u(x) \cdot \frac{d}{dx}v(x) + v(x) \cdot \frac{d}{dx}u(x)$]

$= x \cdot \frac{d}{d(x+\sqrt{x})}(\sqrt{x + \sqrt{x}}) \cdot \frac{d}{dx}(x + \sqrt{x}) + \sqrt{x + \sqrt{x}}$ [chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$]

$= x \cdot \frac{1}{2\sqrt{x+\sqrt{x}}}(1 + \frac{1}{2\sqrt{x}}) + \sqrt{x + \sqrt{x}}$ [$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$]

$= \frac{\sqrt{x}(2\sqrt{x}+1)}{4\sqrt{x+\sqrt{x}}} + \sqrt{x + \sqrt{x}}$

$= \frac{2x+\sqrt{x}+4x+4\sqrt{x}}{4\sqrt{x+\sqrt{x}}}$

$= \frac{6x+5\sqrt{x}}{4\sqrt{x+\sqrt{x}}}$

(9) A particle is constrained to move along the parabola $y = \frac{x^2}{2}$.
Moving at the same rate means $\frac{dx}{dt} = \frac{dy}{dt}$. From the given equation we have $\frac{dy}{dt} = x\frac{dx}{dt}$.
If the same rate is not Zero(i.e., $\frac{dy}{dt} = \frac{dx}{dt} \neq 0$), then we have $x = 1$ which gives $y = \frac{1}{2}$.
So the coordinate of the(we will consider the rate zero case later) point is $(1, \frac{1}{2})$.
If the motion of the particle is such that $x(t) = t^3$. then we have

$$x = 1$$
$$\implies t^3 = 1$$
$$\implies t = 1$$

Now $\frac{dx}{dt} = 3t^2$ means the same rate is $\frac{dx}{dt} = \frac{dy}{dt} = 3$(we are not mentioning any unit because no unit were provided in the question).
The rate zero case is trivial. In that case the point is $(0, 0)$ and the rate is zero.

(10) Notice for any $m \in \mathbb{R}$,

$$p'(x) \neq 0, \ \forall x \in (0, 1).$$

By Rolle's theorem, p can not have two distinct roots in $(0, 1)$. p can not have a double root in $(0, 1)$ because at the double root, the derivative of the polynomial vanishes.

(11)

$f(x) = x^3 - 4x^2 + 4x + 1; \quad f(0) = 1$

$f'(x) = 3x^2 - 8x + 4 = 3x^2 - 6x - 2x + 4 = 3x(x-2) - 2(x-2) = (x-2)(3x-2)$

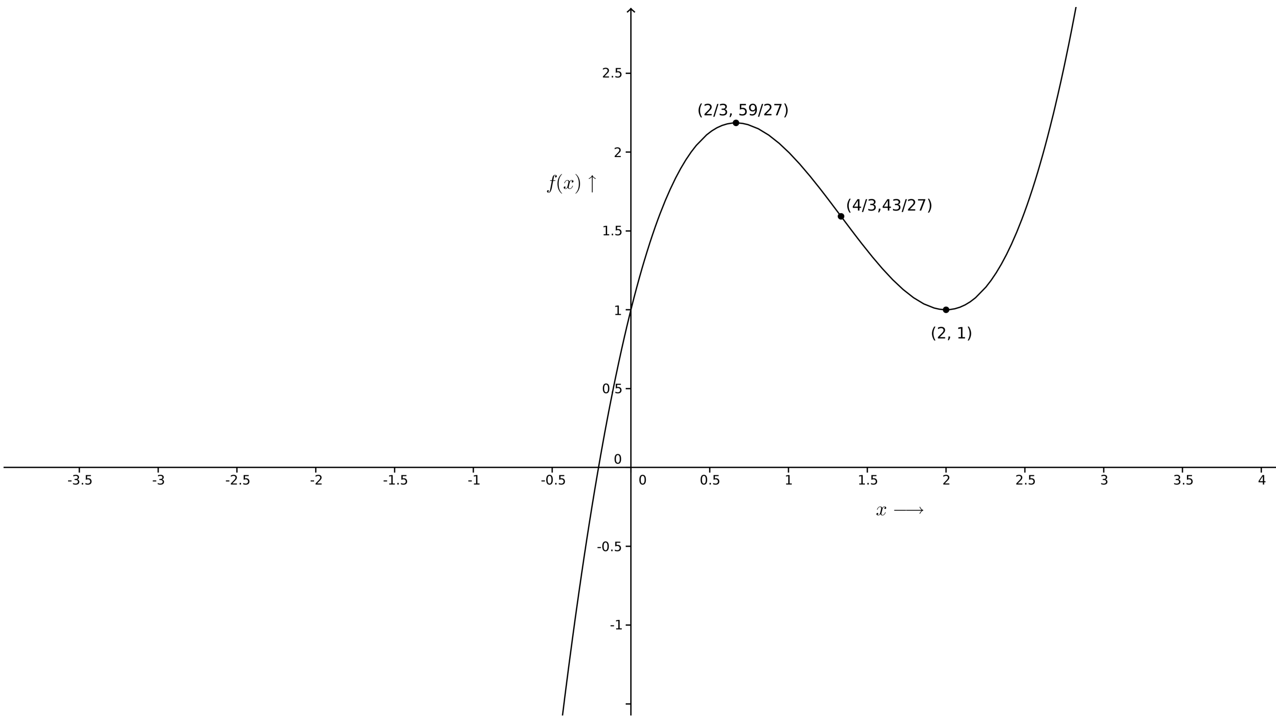
Now $f'(x) = 0 \implies x = 2, \frac{2}{3}$. So, $x = 2$ and $x = \frac{2}{3}$ are the critical points of the function f.

$f''(x) = 6x - 8$; $f''(x) = 0 \implies x = \frac{4}{3}$. It is the point of inflection of f. $f''(2) = 4 > 0$

and $f''(2/3) = -4 < 0$, So f has a local minima at $x = 2$ and local maxima at $x = 2/3$.

Also $f''(x) > 0$ for $x > \frac{4}{3}$ and $f''(x) < 0$ for $x < \frac{4}{3}$. So f is concave down in $(-\infty, \frac{4}{3})$ and

concave up in $(\frac{4}{3}, +\infty)$. $f(2/3) = \frac{59}{27}$, $f(2) = 1$ and $f(4/3) = \frac{43}{27}$. Also $f(0) = 1$, $f(-1) = -8$. Using Bolzano's theorem we get, there is a root of f between -1 and 0.



graph of $f(x) = x^3 - 4x^2 + 4x + 1$

Problem 12 : Define the Taylor polynomial $T_n f(x;a)$ of a function f at a point a of degree n. If f is n-times differentiable, then prove that $(T_n f(x;0))' = T_{n-1} f'(x;0)$.

Solution : The n-degree Taylor polynomial of a function f (atleast n-times differentiable) at a point a is a polynomial which takes the same value as f at a and the derivatives of both at a coincide upto order n. It turns out that

$$T_n f(x;a) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k$$

- So we have

$$T_n f(x;0) = \sum_{k=0}^{n} \frac{f^k(0)}{k!}x^k$$

Therefore

$$\begin{aligned}
(T_n f(x;0))^{'} &= (f(0) + \textstyle\sum_{k=1}^{n} \frac{f^k(0)}{k!}x^k)^{'} \\
&= 0 + \textstyle\sum_{k=1}^{n} \frac{f^k(0)}{k!}kx^{k-1} \\
&= \textstyle\sum_{k=1}^{n} \frac{f^k(0)}{(k-1)!}x^{k-1} \\
&= \textstyle\sum_{m=0}^{n-1} \frac{f^{m+1}(0)}{m!}x^m
\end{aligned}$$

The second step follows from the fact $(g+h)^{'} = g^{'} + h^{'}$, where g and h both are differentiable. And in the last step we replace $k-1$ by m. See that

$$\begin{aligned}
T_{n-1}f^{'}(x;0) &= \textstyle\sum_{m=0}^{n-1} \frac{(f^{'})^m(0)}{m!}x^m \\
&= \textstyle\sum_{m=0}^{n-1} \frac{f^{m+1}(0)}{m!}x^m
\end{aligned}$$

Therefore we have shown that $(T_n f(x;0))^{'} = T_{n-1}f^{'}(x;0)$.