SOLUTION OF HOMEWORK-9

(1) (20) $\int_{-2}^{-4} (x+4)^{10} dx = \int_{2}^{0} x^{10} dx$ (using theorem 1.18 and taking c=4)

So, $\int_{-2}^{-4} (x+4)^{10} dx = -\frac{2^{11}}{11}$

Similarly the others can be done!

(2) A cubic polynomial P satisfying P(0)=P(-2)=0 has to be in the form

$$P(x) = x(x+2)(ax+b)$$

And it also has the property that P(1)=15, $\int_{-2}^{0} P(x)dx = 4$

From the above two equations we will get a+b=5 and a-b=1.

By solving them we will get a=3 and b=2.

Hence the polynomial is $x^3 + 6x^2 + 8x$ and it is unique.

(3) If $a < b < c < d$ and f is integrable on [a,d] then for each $\epsilon > 0$ $\exists$ $s \leq f \leq t$ $\ni$ $\int_{a}^{d} t(x)dx - \int_{a}^{d} s(x)dx < \epsilon$.

Since $s \leq t$ on [a,d] therefore t-s ≥ 0 on [a,d].

So, $\int_{a}^{b} t - \int_{a}^{b} s$, $\int_{b}^{c} t - \int_{b}^{c} s$ and $\int_{c}^{d} t - \int_{c}^{d} s$ all are non-negative.

And also, $\int_{a}^{d} g = \int_{a}^{b} g + \int_{b}^{c} g + \int_{c}^{d} g$

Hence $\int_{b}^{c} t - \int_{b}^{c} s < \epsilon$. So, f is integrable on [b,c].

(4) Let f be defined on [-a,a] for a > 0 and f is integrable on [0,a].

Then $\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-x)dx$ (by using Theorem 1.19 from Apostol)

And $\int_{a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx$ (by definition)

Hence we have $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx$

So $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx$ (for even functions) and $\int_{-a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx$ (for odd functions)

Since the existence of one integral implies the existence of other, therefore $\int_{-a}^{0} f$ exists.

And we know, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

So, for even functions $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ and for odd functions $\int_{-a}^{a} f(x)dx = 0$.

(5) Let f is integrable on [a,b].

Then $\int_{a}^{b} f = sup\{\int_{a}^{b} s : s \leq f$ and s is a step function$\}$.

By definition of supremum for each ϵ $\exists$ a step function s $\ni \int_{a}^{b} f - \epsilon < \int_{a}^{b} s$.

Hence $\int_{a}^{b} f - \int_{a}^{b} s < \epsilon$.

(Harder variant): Now for $\epsilon > 0$ $\exists$ a step function $\ni$ $\int_{a}^{b} f - \int_{a}^{b} s < \frac{\epsilon}{2}$

For the step function s $\exists$ a partition $P := \{a = x_0, x_1, x_2, ..., x_{n+1} = b\}$ $\ni s(x) = s_i$ on$[x_{i-1}, x_i]$ for $1 \leq i \leq n+1$

Consider $\alpha := max_{1 \leq i \leq n+1} |s_i|$

Now consider the two points $(x_1 - \frac{\epsilon}{4n\alpha}, s_1)$, $(x_1 + \frac{\epsilon}{4n\alpha}, s_2)$ and join the straight line passing through this two points.

The equation of the straight line is

$$y = \frac{2n\alpha}{\epsilon}(s_2 - s_1)x + s_2 - \frac{2n\alpha}{\epsilon}(x_1 + \frac{\epsilon}{4n\alpha})(s_2 - s_1)$$

Again consider the two points $(x_2 - \frac{\epsilon}{4n\alpha}, s_2)$, $(x_2 + \frac{\epsilon}{4n\alpha}, s_3)$ and join the straight line passing through this two points.

The equation of the straight line is

$$y = \frac{2n\alpha}{\epsilon}(s_3 - s_2)x + s_3 - \frac{2n\alpha}{\epsilon}(x_2 + \frac{\epsilon}{4n\alpha})(s_3 - s_2)$$

Similarly, for each x_i we will get two points $(x_i - \frac{\epsilon}{4n\alpha}, s_i)$, $(x_i + \frac{\epsilon}{4n\alpha}, s_{i+1})$.

The equation of the straight line is

$$y = \frac{2n\alpha}{\epsilon}(s_{i+1} - s_i)x + s_{i+1} - \frac{2n\alpha}{\epsilon}(x_i + \frac{\epsilon}{4n\alpha})(s_{i+1} - s_i)$$

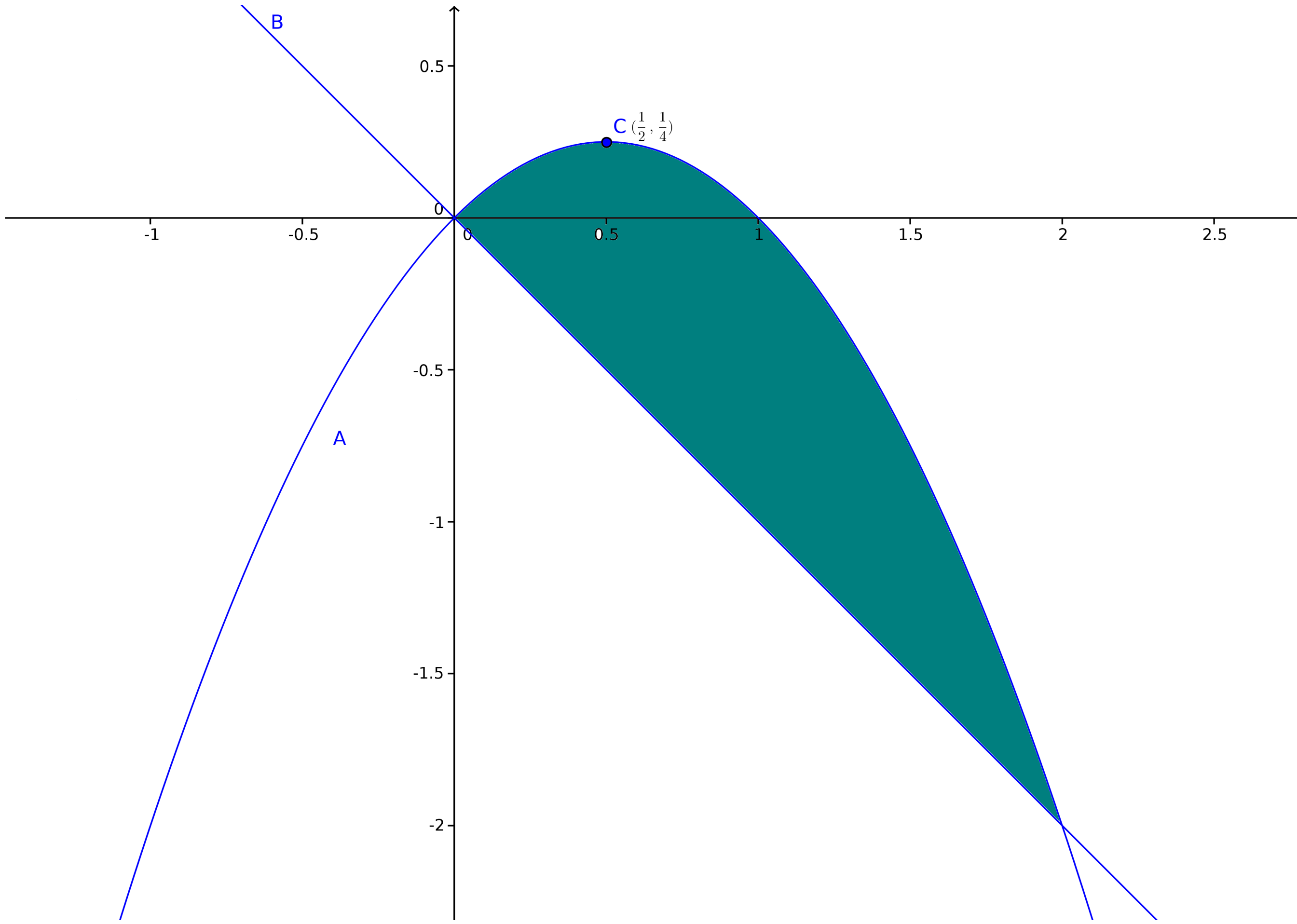
Consider,

$$g(x) = \begin{cases} s_1 & \text{if } x \in [a, x_1 - \frac{\epsilon}{4n\alpha}] \\ \frac{2n\alpha}{\epsilon}(s_2 - s_1)x + s_2 - \frac{2n\alpha}{\epsilon}(x_1 + \frac{\epsilon}{4n\alpha})(s_2 - s_1) & \text{if } x \in [x_1 - \frac{\epsilon}{4n\alpha}, x_1 + \frac{\epsilon}{4n\alpha}] \\ s_2 & \text{if} x \in [x_1 + \frac{\epsilon}{4n\alpha}, x_2 - \frac{\epsilon}{4n\alpha}] \\ \frac{2n\alpha}{\epsilon}(s_3 - s_2)x + s_3 - \frac{2n\alpha}{\epsilon}(x_2 + \frac{\epsilon}{4n\alpha})(s_3 - s_2) & \text{if} x \in [x_2 - \frac{\epsilon}{4n\alpha}, x_2 + \frac{\epsilon}{4n\alpha}] \\ \vdots & \\ \frac{2n\alpha}{\epsilon}(s_{i+1} - s_i)x + s_{i+1} - \frac{2n\alpha}{\epsilon}(x_i + \frac{\epsilon}{4n\alpha})(s_{i+1} - s_i) & \text{if} x \in [x_i - \frac{\epsilon}{4n\alpha}, x_i + \frac{\epsilon}{4n\alpha}] \\ \vdots & \\ s_n & \text{if} x \in [x_n + \frac{\epsilon}{4n\alpha}, b] \end{cases}$$

It can be easily seen that g is a continuous function on $[a, b]$.
Consider $\int\limits_a^b s - \int\limits_a^b g = \sum_{i=1}^n \{ \int\limits_{x_i - \frac{\epsilon}{4n\alpha}}^{x_i + \frac{\epsilon}{4n\alpha}} s - \int\limits_{x_i - \frac{\epsilon}{4n\alpha}}^{x_i + \frac{\epsilon}{4n\alpha}} g \} = \sum_{i=1}^n \{ (s_i + s_{i+1}) \frac{\epsilon}{4n\alpha} - (s_{i+1} - s_i) \frac{\epsilon}{4n\alpha} \}$
$= \sum_{i=1}^n \frac{s_i \epsilon}{2n\alpha} \leq \sum_{i=1}^n \frac{\epsilon}{2n} = \frac{\epsilon}{2}$

Therefore $\int\limits_a^b f - \int\limits_a^b g = \int\limits_a^b f - \int\limits_a^b s + \int\limits_a^b s - \int\limits_a^b g < \epsilon$.
Hence we are done.

(6) 2.4.4



We need to find the area of the region S between the graphs of f and g over the interval $[0, 2]$ where f and g are given by $f(x) = x - x^2$ and $g(x) = -x$. Since $f \geq g$ over the interval $[0, 2]$ we use Theorem 2.1 (from the book) to write

$$a(S) = \int_0^2 [f(x) - g(x)]dx = \int_0^2 (2x - x^2)dx$$

$$= 2\frac{2^2}{2} - \frac{2^3}{3} = \frac{4}{3}.$$

(7) The area enclosed by the ellipse is

$$2b \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}} dx.$$

Since, $\sqrt{1 - \frac{x^2}{a^2}}$ is continuous the area is measurable. After calculation, the area of the given ellipse is πab.

(8) 2.17.1
$\mathcal{A}(f) = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{3}(b^2 + ab + a^2)$
Similarly the others.