SOLUTION OF HOMEWORK-8

(1) For $x \in \mathbb{R}$ such that $|x| < 1$,

$$(1 - x^2)^{-1} = \sum_{n=0}^{\infty} x^{2n}.$$

Which clearly gives that $T_{2n+1}[(\frac{x}{1-x^2})] = \sum_{k=0}^{n} x^{2k+1}$.
Other parts are similar to the above problem.

(2) $T_3(\sin x) = x - \frac{x^3}{6}$. Hence the best approximate nonzero solution will be the nonzero root of the equation $x^2 = x - \frac{x^3}{6}$. Check that $\sqrt{15} - 3$ is a root of this equation.

3(c) If $x \in [-1, -\frac{1}{2})$ then $2x \in [-2, -1)$ and $\frac{x}{2} \in [-\frac{1}{2}, -\frac{1}{4})$. Therefore $[2x][\frac{x}{2}] = (-2)(-1) = 2$.
If $x \in [\frac{-1}{2}, 0)$ then $2x \in [-1, -0)$ and $\frac{x}{2} \in [-\frac{1}{4}, 0)$. Therefore $[2x][\frac{x}{2}] = (-1)(-1) = 1$.
If $x \in [0, 2)$ then $\frac{x}{2} \in [0, 1)$. Therefore $[2x][\frac{x}{2}] = 0$. If $x = 2$ then $[2x][\frac{x}{2}] = 4$. Now one can easily draw the graph of the function $[2x][\frac{x}{2}]$ in the given interval $[-1, 2]$.
Other parts are similar to the above part.

4(a) $$[2x] = \begin{cases} -2 & \text{for } x \in [-1, -\frac{1}{2}) \\ -1 & \text{for } x \in [-\frac{1}{2}, 0) \\ 0 & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 1) \\ \cdots\cdots \\ i & \text{for } x \in [\frac{i}{2}, \frac{i+1}{2}) \\ \cdots\cdots \\ 5 & \text{for } x \in [\frac{5}{2}, 3) \\ 6 & \text{for } x = 3 \end{cases}$$

Hence $\int_{-1}^{3} [2x] dx = \frac{1}{2}(-2 - 1 + 0 + 1 + \cdots + 5) = 6$.

4(b) $$[x] = \begin{cases} 0 & \text{for } x \in [0, 1) \\ 1 & \text{for } x \in [1, 2) \\ \cdots\cdots \\ i & \text{for } x \in [i, i+1) \\ \cdots\cdots \\ n-1 & \text{for } x \in [n-1, n) \\ n & \text{for } x = n \end{cases}$$

Hence $\int_0^n [x]dx = 1.(0 + 1 + 2 + \cdots + (n-2) + (n-1)) = \frac{(n-1)n}{2}$.

4(c) Note that $[-x] = \begin{cases} -[x] - 1 & \text{for } x \in \mathbb{R}/\mathbb{Z} \\ -[x] & \text{for } x \in \mathbb{Z} \end{cases}$

Hence $[x] + [-x] = \begin{cases} -1 & \text{for } x \in \mathbb{R}/\mathbb{Z} \\ 0 & \text{for } x \in \mathbb{Z} \end{cases}$

Let's make a partition of $[a, b]$ by $\{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$ where x_i for $i = 1, ..., n-1$ are consecutive integer points lies in the interval $[a, b]$. Also note that $[x] + [-x]$ takes constant value-1 on the interval (x_{i-1}, x_i) for $i = 1, ..., n$..

So $\int_a^b ([x] + [-x])dx = (-1) \sum_{i=1}^{n} (x_i - x_{i-1}) = -1(x_n - x_0) = a - b$.

Also using Theorem 1.2 we have

$\int_a^b [x]dx + \int_a^b [-x]dx = \int_a^b ([x] + [-x])dx = a - b$.

4(d) $[\sqrt{x}] = \begin{cases} 0 & \text{for } x \in [0, 1^2) \\ 1 & \text{for } x \in [1^2, 2^2) \\ \cdots\cdots \\ i & \text{for } x \in [i^2, (i+1)^2) \\ \cdots\cdots \\ n-1 & \text{for } x \in [(n-1)^2, n^2) \\ n & \text{for } x = n^2 \end{cases}$

$$\begin{aligned} \int_0^{n^2} [\sqrt{x}]dx &= \sum_{i=0}^{n-1} i((i+1)^2 - i^2) \\ &= \sum_{i=0}^{n-1} i(2i + 1 \\ &= 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} i \\ &= \frac{(n-1)n(2n-1)}{3} + \frac{(n-1)n}{2} \end{aligned}$$

(5) We will prove Th 1.2 and Th 1.7. Techniques for Proofs of other theorem are almost similar to these two.

Theorem 1.2: Let s, t be two step function on the interval $[a, b]$. For s, t there exist partition P_s and P_t such that s, t takes constant value on the subinterval of corresponding partition. Now form a new partition by taking union of P_s and P_t. So there exist a partition $\{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$ for the interval $[a, b]$ and constants s_i for $i = 1, ..., n$ and t_i for $i = 1, ..., n$ so that

$$\begin{aligned}\text{For } i &= 1, ..., n \\ s(x) &= s_i \text{ for } x \in (x_{i-1}, x_i) \text{ and} \\ t(x) &= t_i \text{ for } x \in (x_{i-1}, x_i)\end{aligned}$$

clearly then we have that $(s+t)(x) = s(x) + t(x)$ takes the value $s_i + t_i$ on the interval (x_{i-1}, x_i) for $i = 1, ..., n$. Hence $s + t$ is also step function. And Note that

$$\begin{aligned}\int_a^b (s(x) + t(x))dx &= \sum_{i=1}^{n} (s_i + t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} s_i(x_i - x_{i-1}) + \sum_{i=1}^{n} t_i(x_i - x_{i-1}) \\ &= \int_a^b s(x)dx + \int_a^b t(x)dx\end{aligned}$$

Theorem 1.7: Let s be a step function on the interval $[a, b]$. So there exist a partition $\{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\}$ for the interval $[a, b]$. and constants s_i for $i = 1, ..., n$ so that

$$\begin{aligned}\text{For } i &= 1, ..., n \\ s(x) &= s_i \text{ for } x \in (x_{i-1}, x_i)\end{aligned}$$

Now let $t(x)$ be the function on the interval$[a + c, b + c]$ defined by

$$t(x) = s(x - c) \text{ for } x \in [a, b]$$

Now consider the partition $\{a + c = x_0 + c, x_1 + c, \cdots, x_{n-1} + c, x_n + c = b + c\}$ for the interval $[a + c, b + c]$. Note that as $x \in (x_{i-1} + c, x_i + c)$ we will have $x - c \in (x_{i-1}, x_i)$.Then clearly $t(x) = s(x - c) = s_i$ for $x \in (x_{i-1} + c, x_i + c)$. And this holds for each $i = 1, ..., n$.Hence t is a step function and we have

$$\begin{aligned}\int_{a+c}^{b+c} s(x - c)dx &= \int_{a+c}^{b+c} t(x)dx \\ &= \sum_{i=1}^{n} s_i((x_i + c) - (x_{i-1} + c)) \\ &= \sum_{i=1}^{n} s_i(x_i - x_{i-1}) \\ &= \int_a^b s(x)dx\end{aligned}$$

(6) Let s, t be two arbitrary step function on the interval $[a, b]$ satisfying $s(x) \leq g(x)$ for all $x \in [a, b]$ and $f(x) \leq t(x)$ for all $x \in [a, b]$. As $g(x) < f(x)$ for all $x \in [a, b]$ is given, we have

$s(x) < t(x)$ for all $x \in [a, b]$. As s, t are step function and $s(x) < t(x)$ on $[a, b]$, it is easy to verify that

$$\int_a^b s(x)dx < \int_a^b t(x)dx. \tag{1}$$

Now taking supremum over all step function s such that $s(x) \le g(x)$ on $[a, b]$, in the left side of equation (1) we get

$$\int_a^b g(x)dx \le \int_a^b t(x)dx. \tag{2}$$

Now taking infimum over all step function t such that $f(x) \le t(x)$ on $[a, b]$, in the right side of equation (2) we get

$$\int_a^b g(x)dx \le \int_a^b f(x)dx.$$

(7)

$$\begin{aligned}
\int_a^b f(x)dx &= (b-a)\int_{\frac{a}{b-a}}^{\frac{b}{b-a}} f((b-a)x)dx \quad \text{(using Th 1.19)} \\
&= (b-a)\int_{\frac{a}{b-a}-\frac{a}{b-a}}^{\frac{b}{b-a}-\frac{a}{b-a}} f((b-a)(x+\frac{a}{b-a}))dx \quad \text{(using Th 1.18)} \\
&= (b-a)\int_0^1 f(a+(b-a)x)dx
\end{aligned}$$