SOLUTION OF HOMEWORK-7

(1) (d) $g^{'}(x) = \frac{d}{df(x^2)} f(f(x^2)) \frac{d}{dx^2} f(x^2) \frac{d}{dx} x^2$ $=2xf^{'}(x)^2$
(since $\frac{d}{df(x^2)} f(f(x^2)) = f^{'}(x)$ and $\frac{d}{dx^2} f(x^2) = f^{'}(x)$)
And all other sums i.e. (a),(b),(c) can be done in this way.

(2) (30)(a) we know

$$x^3 + y^3 = 1$$

Differentiate wrt x both sides and we will get

$$x^2 + y^2 y^{'} = 0$$

(b) Again differentiating wrt x both sides

$$2x + 2y(y^{'})^2 + y^2 y'' = 0$$

Putting $y' = -\frac{x^2}{y^2}$,

$$2x + 2y\frac{x^4}{y^4} + y^2 y^{''} = 0$$

Again solving this equation and substituting $x^3 + y^3 = 1$ we will get

$$y^{''} = -2xy^{-5}$$

(3) $f'(x) = 4x - 7$. So, check that $f'(3.5) = \frac{f(5)-f(2)}{3}$.

(4) Let f(x)=$px^2 + qx + r$, then $f^{'}(x) = 2px + q$, which is the slope of the tangent at $(x, f(x))$.
So the slope of the tangent at the point $(\frac{a+b}{2}, f(\frac{a+b}{2})\ )$ = p(a+b)+q.
Now the slope of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is $\frac{f(b)-f(a)}{b-a} = p(a+b) + q$. Hence we are done.

(5) Consider $f(x) = c_0 x + \frac{c_1}{2} x^2 + ... + \frac{c_n}{n+1} x^{n+1}$.
Now f is continuous on [0,1] and $f(0) = f(1) = 0$.
So by Rolle's Theorem $\exists$ a $c \in (0, 1) \ni f^{'}(c) = 0$ and $f^{'}(x) = c_0 + c_1 x + ... + c_n x^n$.
Hence f has a real root.

(6) Consider f(x)= sin(x) and use the mean-value theorem on the interval [x,y] assuming x<y. Since $\frac{|sin(y)-sin(x)|}{|y-x|} = |cos(c)| \leq 1$ for some $c \in (x, y)$, we are getting the inequality.

(7) Let $f_{(a,\,c)}$ denotes the function f with domain restricted to $(a,\, c)$. Let m be the slope of the line passing through the points $(a,\, f(a))$ and $(b,\, f(b))$. Then, using **LMVT**, there exists a point $x_1 \in (a,\, c)$ such that $(f_{(a,\,c)})'(x_1) = m$. Proceeding similarly, we find a point $x_2 \in (c,\, b)$ such that $(f_{(c,\,b)})'(x_2) = m$. Note that f' is a differentiable function. Now, using **Rolle's Theorem**, there exists $x_0 \in (x_1,\, x_2)$ such that $f''(x_0) = 0$.

(8) Let $M := \sup\{f(x) : x \in [a,\, b]\}$. Then, by the extreme value theorem, there exists a point $x_0 \in [a,\, b]$ such that $f(x_0) = M$. We'll prove that $M = 0$. As $f(a) = f(b) = 0$, $M \geq 0$. Also if $x_0 \in \{a,\, b\}$ then $M = 0$. Now if $M > 0$, $x_0 \in (a,\, b)$. As x_0 is a point of global maxima, and f is twice differentiable, we have $f'(x_0) = 0$, and $f''(x_0) \leq 0$. Using the functional equation:

$$f''(x) + f'(x)g(x) - f(x) = 0$$

at the point x_0 we get $f''(x_0) = f(x_0) = M$. This implies that $M \leq 0$. Hence we get a contradiction. Thus, $M = 0$.
Proceeding similarly, we can show that that the $\inf\{f(x) : x \in [a,\, b]\} = 0$. Hence, the result.

(9) Let's see problem 12. Given $f(x) = x - \sin x$, then $f'(x) = 1 - \cos x$. Hence the set $\{2n\pi : n \in \mathbb{N}\}$ is the zero set of f'. As $|\cos x| \in [-1,\, 1]$, so we have $f'(x) \geq 0$ for all x. Hence f is monotonic on $\mathbb{R}$. Now $f''(x) = \sin x$. So $f''(x) \geq 0$ when $x \in [2m\pi,\, (2m+1)\pi]$ for some $m \in \mathbb{Z}$. So f' is monotonic in $\cup_{m \in \mathbb{Z}}[2m\pi,\, (2m+1)\pi]$.