## Solution Of Homework 6

**1.** It is easy to see that the function is continuous and it is piece-wise increasing, hence it is an increasing function.

Sincd the function is bijective its inverse exists and the inverese is

$$f^{-1}(x) = \begin{cases} x & x < 1 \\ x^{\frac{1}{2}} & 1 \leq x \leq 16 \\ \frac{x^2}{64} & x > 16 \end{cases}$$

See that the inverse function is continuous and piece-wise increasing, hence it is also increasing.

**2.**

**a) True :** Since the function is continuous (as it can be written as a composition of continuous functions) and defined on a closed and bounded interval.

**b) False :** We can only gurantee the existence of extremum using the extreme value theorem when the given function is continuous on a closed and bounded interval, but here the domain of the function is not closed.

**c) False :** The extremum value theorem gurantees the existence of maximum and minimum values only.

**3.**

3. If $f(x) = x^2 + 3x + 2$ then $f^{'}(x) = 2x + 3$
4. If $f(x) = x^4 + sinx$ then $f^{'}(x) = 4x^3 + cosx$
5. If $f(x) = x^4 sinx$ then $f^{'}(x) = 4x^3 sinx + x^4 cosx$
6. If $f(x) = \frac{1}{x+1}$ and $x \neq -1$ then $f^{'}(x) = \frac{-1}{(x+1)^2}$
7. If $f(x) = \frac{1}{x^2+1} + x^5 cosx$ then $f^{'}(x) = \frac{-2x}{(x^2+1)^2} + 5x^4 cosx - x^5 sinx$
8. If $f(x) = \frac{x}{x-1}$ and $x \neq 1$ then $f^{'}(x) = \frac{-1}{(x-1)^2}$
9. If $f(x) = \frac{1}{2+cosx}$ then $f^{'}(x) = \frac{sinx}{(2+cosx)^2}$
10. If $f(x) = \frac{x^2+3x+2}{x^4+x^2+1}$ then $f^{'}(x) = \frac{-2x^5-9x^4-8x^3-3x^2-2x+3}{(x^4+x^2+1)^2}$

11. If $f(x) = \frac{2-sinx}{2-cosx}$ then $f^{'}(x) = \frac{1-2(sinx+cosx)}{(2-cosx)^2}$

12. If $f(x) = \frac{xsinx}{1+x^2}$ then $f^{'}(x) = \frac{x^3cosx-x^2sinx+xcosx+sinx}{(1+x^2)^2}$

**4.** Let us define a collection of functions as $f_k(x) = x^k$ where $k \in \mathbb{N}$. See that

$$\sum_{k=1}^{n} kx^k = x\sum_{k=1}^{n} kx^{k-1} = x\sum_{k=1}^{n} f_k^{'}(x) = x\frac{d}{dx}(\sum_{k=1}^{n} f_k(x))$$

Now see that $\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} x^k = \frac{x^{n+1}-x}{x-1}$, therefore we have

$$\sum_{k=1}^{n} kx^k = x\frac{d}{dx}(\frac{x^{n+1}-x}{x-1}) = x\frac{n.x^{n+1}-(n+1)x^n+1}{(x-1)^2}$$

Now see that

$$\sum_{k=1}^{n} kx^k = (\frac{x}{x-1})^2 n.x^n - (\frac{1}{x-1})^2(n+1)x^{n+1} + \frac{x}{(x-1)^2}$$

Since the series $\sum_{n=1}^{\infty} nx^n$ is convergent when $|x| < 1$ (use *Ratio Test*), we must have $\lim_{n\to\infty} nx^n = 0$ whenever $|x| < 1$. Therefore we conclude that when $|x| < 1$

$$\lim_{n\to\infty}\sum_{k=1}^{n} kx^k = \frac{x}{(x-1)^2}$$

Clearly when $|x| = 1$ the series does not converge. And also when $|x| > 1$ the series is divergent as the terms of the series do not converge to 0.

**5.** Let us define $g_k(x) = f_1(x)f_2(x)...f_k(x)$ for $k \in \{1, 2, ..n\}$. Therefore

$$g_n(x) = f_1(x)f_2(x)...f_n(x) = g_{n-1}(x)f_n(x)$$

Now

$$\begin{aligned} g_n^{'}(x) &= f_n^{'}(x)g_{n-1}(x) + f_n(x)g_{n-1}^{'}(x) \\ &= f_n^{'}(x)g_{n-1}(x) + f_n(x)(f_{n-1}^{'}(x)g_{n-2}(x) + f_{n-1}(x)g_{n-2}^{'}(x)) \end{aligned}$$

Proceeding in this way we will get

$(f_1.f_2...f_n)^{'}(x) = f_1^{'}(x)f_2(x)f_3(x)...f_n(x)+f_1(x)f_2^{'}(x)f_3(x)....f_n(x)+f_1(x)f_2(x)f_3^{'}(x)...f_n(x)+ ... + f_1(x)f_2(x)f_3(x)...f_n^{'}(x)$

**6.** Let $f(x) = x^2 + ax + b$ and $g(x) = x^3 - c$. Therefore $f^{'}(x) = 2x + a$ and $g^{'}(x) = 3x^2$. Now the condition $f(1) = g(1)$ tells us that $a + b + c = 0$. And the condition $f^{'}(1) = g^{'}(1)$ gives $a = 1$. Hence we have $a = 1$ and $b+c = -1$. If $1 + a + b = 1 - c = 2$ the $c = -1$ and therefore we have $b = 0$.

**7.** Deriving formula for sum

$$\begin{aligned} D^*(f+g)(x) &= \lim_{h\to 0} \frac{(f+g)^2(x+h) - (f+g)^2(x)}{h} \\ &= \lim_{h\to 0} \frac{[(f+g)(x+h)]^2 - [(f+g)(x)]^2}{h} \\ &= D^*f(x) + D^*g(x) + 2[f(x)Dg(x) + g(x)Df(x)] \end{aligned}$$

Similarly for product of functions one can derive that

$$D^*(fg)(x) = f^2(x)D^*(g)(x) + g^2(x)D^*(f)(x)$$

And for the quotient of functions we have

$$D^*(\frac{f}{g})(x) = \frac{g^2(x)D^*(f)(x) - f^2(x)D^*(g)(x)}{g^4(x)}$$

Now we observe that

$$\begin{aligned} D^*f(x) &= \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\{f(x+h) + f(x)\} \\ &= 2f(x)Df(x) \end{aligned}$$

Therefore using the above relation we can conclude that the equation $D^*(f)(x) = Df(x)$ is satisfied only when $f$ is a constant function.

**8.**

1. $f(x) = cos2x - 2sinx$ and $f^{'}(x) = -2sin2x - 2cosx$
2. $f(x) = \sqrt{1+x^2}$ and $f^{'}(x) = \frac{x}{\sqrt{1+x^2}}$
3. $f(x) = (2-x^2)cosx^2 + 2xsinx^3$ and $f^{'}(x) = (-2x)cosx^2 + (2-x^2)2x(-sinx^2) + 2sinx^3 + 6x^3cosx^3$
4. $f(x) = sin(cos^2x).cos(sin^2(x))$ and $f^{'}(x) = cos(cos^2x).[-2sinxcosx].cos(sin^2(x)) + sin(cos^2x).(-sin(sin^2(x)))2sinxcosx$
5. $f(x) = sin^n(x).cos(nx)$ and $f^{'}(x) = nsin^{n-1}(x)cosx.cos(nx) + nsin^n(x)(-sin(nx))$
6. $f(x) = sin[sin(sinx)]$ and $f^{'}(x) = cos[sin(sinx)].cos(sinx).cosx$
7. $f(x) = \frac{sin^2(x)}{sinx^2}$ and $f^{'}(x) = \frac{2.sinx^2.sinx.cosx - 2x.sin^2(x).cosx^2}{(sinx^2)^2}$
8. $f(x) = tan(\frac{x}{2}) - cot(\frac{x}{2})$ and $f^{'}(x) = \frac{1}{2}sec^2(\frac{x}{2}) + \frac{1}{2}cosec^2(\frac{x}{2})$
9. $f(x) = sec^2(x) + cosec^2(x)$ and $f^{'}(x) = 2sec^2x.tanx - 2cosec^2x.cotx$
10. $f(x) = x\sqrt{1+x^2}$ and $f^{'}(x) = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}$
11. $f(x) = \frac{x}{\sqrt{4-x^2}}$ and $f^{'}(x) = \frac{\sqrt{4-x^2} - x^2(4-x^2)^{\frac{-1}{2}}}{4-x^2}$
12. $f(x) = (\frac{1+x^3}{1-x^3})^{\frac{1}{3}}$ and $f^{'}(x) = \frac{1}{3}(\frac{1+x^3}{1-x^3})^{\frac{-2}{3}}\frac{6x^2}{(1-x^3)^2}$
13. $f(x) = \frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})}$ and $f^{'}(x) = \frac{2x+\sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}}{(\sqrt{1+x^2}(x+\sqrt{1+x^2}))^2}$
14. $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$ and $f^{'}(x) = \frac{1}{8}\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\frac{1}{\sqrt{x+\sqrt{x}}}\frac{1}{\sqrt{x}}$