Homework 6

Analysis and Linear Algebra I (Autumn 2018)
Indian Institute of Science

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1. Show that $f(x)$ given below is strictly monotonic. Calculate $f^{-1}(x)$ and conclude that it is also strictly monotonic.

$$f(x) = \begin{cases} x, & x < 1, \\ x^2, & 1 \leq x \leq 4, \\ 8\sqrt{x}, & x > 4. \end{cases}$$

2. State whether the following statements are **True** or **False** giving justification in each case.

   (a) The extreme value theorem guarantees that the function $f(x) = \sqrt{|\cos(x^3)|}$ has a minimum on the interval $[-2, 7]$.

   (b) The extreme value theorem guarantees that the function $f(x) = 1 - x^2$ has a maximum on the interval $(-1, 1)$.

   (c) The extreme value theorem guarantees that the function $f(x) = x^3 - 1$ has a zero on the interval $[0, 2]$.

3. Exercises 3 to 12 of Section 4.6 in the textbook.

4. Evaluate $\displaystyle\sum_{k=0}^{n} kx^k$ using differentiation. Take the limit as $n \to \infty$ carefully to compute the series $\displaystyle\sum_{n} nx^n$. For what values of x does the series converge?

5. Generalise the product rule by giving a formula for the derivative of $f_1 \cdot f_2 \cdot \ldots \cdot f_n$.

6. Find relations between a, b and c such that both the values and the derivatives of the functions $x^2 + ax + b$ and $x^3 - c$ are equal at $x = 1$. If the value equals 2, what more relations do you get?

7. Define a new derivative operator D^* as follows

$$D^* f(x) = \lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{h},$$

   where $f^2(x) = [f(x)]^2$. Derive formulas for computing D^* of sums, differences, products and quotients of functions. For what functions does $D^* f = Df$?

8. Exercises 1 to 14 of Section 4.12 in the textbook.