Solution Set for Homework 5

1. a. Using the fact that "both multiplication and addition of two continuous functions are continuous", one can show that $25x^2+2$ and $75x^7-2$ are continuous. Also at $x=1$, $75x^7-2$ is non-zero and hence $\lim_{x\to 1}\frac{25x^2+2}{75x^7-2}$ exists and also equal to $\frac{25+2}{75-2}=\frac{27}{73}$.

   b. As x is tending to 0 from right side therefore $\frac{|x|}{x}$ is identically 1, which is a constant function. Using the fact that limit of a constant function, at any point, is equal to that constant itself, we conclude that $\lim_{x\to 0^+}\frac{|x|}{x}=1$.

   c. Using the same argument as of (a), we conclude that $\lim_{x\to a}\frac{x^2-a^2}{x^2+2ax+a^2}=0$.

   d. Since $(x+t)^2-t^2=x(x+2t)$ and $x\neq 0$ therefore $\frac{(x+t)^2-t^2}{x}=x+2t$. Now using the fact that sum of two continuous functions is continuous we conclude that $\lim_{x\to 0}\frac{(x+t)^2-t^2}{x}$ and equal to 0.

2.
$$f(x)=|x|=\begin{cases}x, & \text{if } x>0\\ 0, & \text{if } x=0\\ -x & \text{if } x<0\end{cases}$$

   Now,
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x=0$$
   and
$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} -x=0$$
   We get,
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0)=0$$
   So, f is continuous at 0.

3. $f:[0,\infty)\to\mathbb{R}$ satisfies $0\le f(x)\le x$ for all x in the domain of f. Taking $x=0$, we get $f(0)=0$. Also we have, $\lim\limits_{x\to 0^+} x=0$. So, from Sandwich Theorem, we get $\lim\limits_{x\to 0^+} f(x)=0$. Hence f is continuous at 0.

4. Let
$$f(x)=\begin{cases}\sin(\frac{1}{x}) & \text{if } x\neq 0\\ \alpha & \text{if } x=0\end{cases}$$
   where α is arbitrary but fixed constant. Let us define a sequence $\{x_n\}$ such that $x_n=\frac{1}{n\pi}$ for all $n\in\mathbb{N}$. Clearly $x_n\to 0$ as $n\to\infty$. Also, $f(x_n)=\sin(n\pi)=0$ for all $n\in\mathbb{N}$. Now let us consider another sequence $\{y_n\}$ such that $y_n=\frac{2}{(4n+1)\pi}$ for all $n\in\mathbb{N}$. Clearly $y_n\to 0$ as $n\to\infty$. But $f(y_n)=\sin(\frac{(4n+1)\pi}{2})=1$ for all $n\in\mathbb{N}$. So, $\lim\limits_{x\to 0^+} f(x)$ does not exist. Hence f can not be continuous at $x=0$.
   Now let $g(x)=x\sin(\frac{1}{x})$ for $x\neq 0$. We know that, $0\le |x\sin(\frac{1}{x})|\le |x|$ for all x. Also in Prob. 2, we have seen that, $\lim\limits_{x\to 0}|x|=0$. So from Sandwitch theorem, we get $\lim\limits_{x\to 0}|x\sin(\frac{1}{x})|=0$. Hence, $\lim\limits_{x\to 0} x\sin(\frac{1}{x})=0$. So if we define $g(0)=0$ then g is continuous 0.

5. To prove that $\sin x$ is continuous everywhere on $\mathbb{R}$, we shall use the inequality $|\sin x|\le|x|$ for all x. Let $c\in\mathbb{R}$ and $\varepsilon>0$ be given. Now
$$|\sin x-\sin c|=2\left|\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right|\le 2\left|\sin\left(\frac{x-c}{2}\right)\right|\le 2\left|\frac{x-c}{2}\right|=|x-c|$$
   So, $|\sin x-\sin c|<\varepsilon$ whenever $|x-c|<\varepsilon$. Hence $\sin x$ is continuous at c. Since c is arbitrary, $\sin x$ is continuous on $\mathbb{R}$.
   Let us define a function f on $\mathbb{R}$ such that $f(x)=x+\frac{\pi}{2}$ for all x. Clearly f is continuous on $\mathbb{R}$. Now $\sin(f(x))=\sin(x+\frac{\pi}{2})=\cos x$. Hence $\cos x$ is continuous on $\mathbb{R}$.

6. a. $sin(x)$ and $cos(x)$ both are defined and continuous on $\mathbb{R}$. Hence, $f\circ g$ and $g\circ f$ are both defined and continuous on $\mathbb{R}$.

b. Being polynomial f and g both are defined and continuous on $\mathbb{R}$. Hence, $f \circ g$ and $g \circ f$ are both defined and continuous on $\mathbb{R}$.

c. Since range of f is $\mathbb{R}_+ \cup \{0\}$ and domain of g is $\mathbb{R}_+ \cup \{0\}$. Therefore the domain of $g \circ f$ is $\mathbb{R}$ and $g \circ f : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$g \circ f(x) = \sqrt{f(x)} = \sqrt{x^2} = |x| \text{ for all } x \in \mathbb{R}.$$

Since $|x|$ is continuous on $\mathbb{R}$, hence $g \circ f$ is continuous on $\mathbb{R}$.
The range of g is $\mathbb{R}_+ \cup \{0\}$ which is subset of domain of f. Therefore $f \circ g$ is defined on whole domain of g, i.e., $\mathbb{R}_+ \cup \{0\}$. The function $f \circ g : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is defined by

$$f \circ g(x) = (g(x))^2 = x \text{ for all } x \in \mathbb{R}_+ \cup \{0\}.$$

Since x is continuous, hence $f \circ g$ is continuous on $\mathbb{R}_+ \cup \{0\}$.

d. f maps $\mathbb{R}$ to $\mathbb{R}_+ \cup \{0\}$. g is well defined on $\mathbb{R}_- \cup \{0\}$. So $g \circ f$ is well defined on $f^{-1}\{0\} = \{0\}$ and $g \circ f(0) = 0$. Hence $g \circ f$ is continuous on $\{0\}$. On the other hand g maps $\mathbb{R}_- \cup \{0\}$ to $\mathbb{R}_+ \cup \{0\}$ which is a subset of the domain of f ($= \mathbb{R}$). Also, f and g are both continuous on their respective domain. So $f \circ g$ is defined and continuous on $\mathbb{R}_- \cup \{0\}$. In fact note that $f \circ g(x) = f \circ (\sqrt{-x}) = -x$ for $x \in \mathbb{R}_- \cup \{0\}$.

e. $f : \mathbb{R} \longrightarrow \{1, 0\}$ is defined by

$$f(x) = \begin{cases} 1 & \text{when } |x| \leq 1 \\ 0 & \text{when } |x| > 1 \end{cases}$$

and $g : \mathbb{R} \longrightarrow [-2, 2]$ is defined by

$$g(x) = \begin{cases} 2 - x^2 & \text{when } |x| \leq 2 \\ 2 & \text{when } |x| > 2 \end{cases}$$

hence $g \circ f : \mathbb{R} \longrightarrow \{1, 2\}$ is

$$g \circ f(x) = \begin{cases} g(1) = 1 & \text{when } |x| \leq 1 \\ g(0) = 2 & \text{when } |x| > 1 \end{cases}$$

Clearly $g \circ f$ is not continuous on $\{-1, 1\}$. But $g \circ f$ is constant in a small neighbourhood of every point in $\mathbb{R} \setminus \{-1, 1\}$, so it is clearly continuous on $\mathbb{R} \setminus \{-1, 1\}$.
Note that $f \circ g$ is well defined on the following domain.
$f \circ g : \mathbb{R} \longrightarrow \{0, 1\}$ is

$$f \circ g(x) = \begin{cases} f(2 - x^2) & \text{when } |x| \leq 2 \\ f(2) = 0 & \text{when } |x| > 2 \end{cases}$$

$$f \circ g(x) = \begin{cases} 0 & \text{when } |x| > \sqrt{3} \text{ and } x \in (-1, 1) \\ 1 & \text{when } x \in [-\sqrt{3}, -1] \cup [1, \sqrt{3}] \end{cases}$$

Clearly $f \circ g$ is not continuous on $\{-\sqrt{3}, -1, 1, \sqrt{3}\}$. But $f \circ g$ is constant in a small neighbourhood of every point in $\mathbb{R} \setminus \{-\sqrt{3}, -1, 1, \sqrt{3}\}$, so it is clearly continuous on $\mathbb{R} \setminus \{-\sqrt{3}, -1, 1, \sqrt{3}\}$.

f. Note that range of f is $\mathbb{R}_+ \cup \{0\}$ and domain of g is $\mathbb{R}$. So $g \circ f$ is well defined on $\mathbb{R}$. And $g \circ f$ is

$$g \circ f(x) = (f(x))^2 = (|x|)^2 = x^2, \text{ for all } x \in \mathbb{R}$$

. Therefore $g \circ f$ is continuous on $\mathbb{R}$.
The range of g is $\mathbb{R}$ which is domain of f. Therefore $f \circ g$ is defined on whole domain of g, i.e., $\mathbb{R}$. The function $f \circ g : \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$f \circ g(x) = |g(x)| = \begin{cases} |x| = -x & \text{when } x < 0 \\ x^2 & \text{when } x \geq 0 \end{cases}$$

Note that x, x^2 is everywhere continuous function and also observe that $\lim_{x \to 0+} f \circ g(x) = \lim_{x \to 0-} f \circ g(x) = f \circ g(0) = 0$. So $f \circ g$ is continuous on whole domain $\mathbb{R}$

7. Given that $p_n(x) = \sum_{k=0}^{n} C_k x^k$ is a nth degree polynomial with the property $C_0 C_n < 0$. Now there are two way in which $C_0 C_n$ could be negative.

Case 1: $C_0 < 0, \ \ C_n > 0$.

Note that $p_n(0) = C_0 < 0$. Now suffices to show $p_n(t) > 0$ for some $t \in \mathbb{R}_+$ In that case, p_n being a continuous function on $\mathbb{R}$, we can apply Bolzano's theorem to get a point $c \in (0, t)$ so that $p_n(c) = 0$. Note that

$$\begin{aligned} p_n(x) &= x^n \left(C_n + \frac{C_{n-1}}{x} + \cdots + \frac{C_1}{x^{n-1}} + \frac{C_0}{x^n} \right) \text{ for all } x > 0. \\ \iff p_n(x) &= x^n r_n(x) \text{ for all } x > 0. \end{aligned}$$

where $r_n(x)$ is given by the following equation

$$r_n(x) = \left(C_n + \frac{C_{n-1}}{x} + \cdots + \frac{C_1}{x^{n-1}} + \frac{C_0}{x^n} \right) \text{ for all } x > 0.$$

Note that for $j = 1, 2, ..., n$;

$$\frac{|C_{n-j}|}{x^j} < \frac{C_n}{2n} \text{ whenever } x > \left(\frac{2n|C_{n-j}|}{C_n} \right)^{\frac{1}{j}}.$$

Now take $R = \max\{ \left(\frac{2n|C_{n-j}|}{C_n} \right)^{\frac{1}{j}} : j = 1, 2, ..., n\}$.

For $x > R$, we have $|r_n(x) - C_n| \leq \sum_{j=1}^{n} \frac{|C_{n-j}|}{x^j} \leq \frac{C_n}{2}$.

So $r_n(x) > \frac{C_n}{2}$ for all $x > R$. Also we know $x^n > 1$ for all $x > 1$.

Now Take M =max$\{R, 1\}$, Then for $x > M$ we have

$$p_n(x) = x^n r_n(x) > \frac{C_n}{2} > 0.$$

So we have $p_n(t) > 0$ for all $t > M$.

Case 2: $C_0 > 0, \ \ C_n < 0$.

Consider $q_n(x) = -p_n(x)$. Then apply case 1 to $q_n(x)$ to get a point $c \in \mathbb{R}_+$ so that $q_n(c) = 0$ which gives $p_n(c) = 0$.

8. Given that $f : [a, b] \longrightarrow [a, b]$ is a continuous function. Now Consider the function $g : [a, b] \longrightarrow \mathbb{R}$ defined by

$$g(x) = f(x) - x; \quad \text{for all } x \in [a, b].$$

Note that g is a continuous function on $[a, b]$. Also note that as

$$a \leq f(x) \leq b \text{ for all } x \in [a, b], \text{ we have}$$

$$g(a) = f(a) - a \geq 0; \ \ g(b) = f(b) - b \leq 0$$

Now if $g(a) = 0$ then a is a fixed point of f. Similarly if $g(b) = 0$ then b is a fixed point for f. And if neither $g(a)$ nor $g(b)$ is zero then we have $g(a) > 0$ and $g(b) < 0$. Also g is continuous on $[a, b]$. Hence using Bolzano's theorem we will have a point $c \in (a, b)$ such that $g(c) = 0$. In this case c is a fixed point for f. Hence f has a fixed point.

9. Note that $f(x) = tan(x)$ for $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ is not a continuous function on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$. Note $sin(x)$ is continuous bounded function with $sin(\frac{\pi}{2}) = 1$ and $cos(x)$ is continuous everywhere with $cos(\frac{\pi}{2}) = 0$. Also observe $cos(\frac{\pi}{2} + h) < 0$ and $cos(\frac{\pi}{2} - h) > 0$ for sufficiently small $h > 0$, we then have $lim_{x \to \frac{\pi}{2}+} tan(x) = -\infty$ and $lim_{x \to \frac{\pi}{2}-} tan(x) = +\infty$. So f is not continuous at $\frac{\pi}{2}$. Hence Bolzano theorem is not applicable to the given function on the given interval.

10. Let us define a function f on $\mathbb{R}$ such that $f(x) = \sin(x) - x + 1$ for all x. Clearly f is continuous everywhere on $\mathbb{R}$. Now $f(0) = 1 > 0$ and $f(3) = \sin(3) - 3 + 1 < 0$. By Bolzano's theorem, there exists a real number c such that $f(c) = 0$ i.e. $\sin(c) = c - 1$.