Homework 5

Analysis and Linear Algebra I (Autumn 2018)
Indian Institute of Science

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1. Compute the limits and prove that the limits are correct using the limit theorems. Clearly state which theorem you are using.

   (a) $\lim_{x\to 1} \dfrac{25x^2+2}{75x^7-2}$ (b) $\lim_{x\to 0+} \dfrac{|x|}{x}$

   (c) $\lim_{x\to a} \dfrac{x^2-a^2}{x^2+2ax+a^2}$ (d) $\lim_{x\to 0} \dfrac{(x+t)^2-t^2}{x}$

2. Prove that the function $f(x) = |x|$ is continuous at 0.

3. Suppose $f : [0,\infty) \to \mathbb{R}$ satisfies $0 \le f(x) \le x$ for all x in the domain of f. Show that f is continuous at 0.

4. Suppose $f(x) = \sin(1/x)$ for $x \neq 0$. Prove that no matter how you define $f(0)$, f cannot be continuous at 0. On the other hand, if $g(x) = x\,\sin(1/x)$ for $x \neq 0$, then prove that you can define $g(0)$ in a way that makes g continuous at 0. What is $g(0)$ in this case?

5. Prove that $\sin x$ and $\cos x$ are continuous everywhere. (Hints are given in Problem 26 of Section 3.6).

6. For the functions f and g below, determine the domain and region of continuity of both $u = f\circ g$ and $v = g\circ f$. Unless otherwise specified, the domains of f and g are to be considered as $\mathbb{R}$.

   (a) $f(x) = \sin x$, $g(x) = \cos x$ (b) $f(x) = x^2 - 2x$, $g(x) = x+1$

   (c) $f(x) = x^2$, $g(x) = \sqrt{x}, \quad x \ge 0$ (d) $f(x) = x^2$, $g(x) = \sqrt{-x}, \quad x \le 0$

   (e) $f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1, \end{cases}$ $g(x) = \begin{cases} 2-x^2, & |x| \le 2, \\ 2, & |x| > 2. \end{cases}$

   (f) $f(x) = |x|$, $g(x) = \begin{cases} x, & x < 0, \\ x^2, & x \ge 0. \end{cases}$

7. Let $p(x) = \sum_{k=0}^{n} c_k x^k$ be a polynomial of degree n such that $c_0 c_n < 0$. Prove that $p(x)$ has at least one positive **root**, i.e. $\exists\, t > 0$ so that $p(t) = 0$.

8. If $f : [a, b] \to [a, b]$ is a continuous function, prove that f has a fixed point in $[a, b]$.

9. Let $f(x) = \tan x$ be defined on $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. Recall that $f\left(\frac{\pi}{4}\right) = 1$ and $f\left(\frac{3\pi}{4}\right) = -1$. By Bolzano's theorem, there should be a point $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ such that $f(x) = 0$. Find x.

10. Prove that there is some real number x so that $\sin x = x - 1$.