HOMEWORK-4 SOLUTION

1. The convergent serieses in Problem 5 of Homework 3 are 1)$\sum \frac{(-1)^{n(n-1)/2}}{2^n}$, 2) $\sum \frac{(-1)^n\sqrt{n}}{n+100}$, 3) $\sum \frac{\sin(1/n)}{n}$.

The first series converges absolutely beacause $|\frac{(-1)^{n(n-1)/2}}{2^n}| = \frac{1}{2^n}$ and the series $\sum \frac{1}{2^n}$ is convergent. The second series is not absolutely convergent. To see that notice $\frac{n}{n+100} > \frac{1}{2}, \forall\, n > 100$. Hence $\frac{\sqrt{n}}{n+100} > \frac{1}{2\sqrt{n}}, \forall\, n > 100$. Now use comparison test and the fact that $\Sigma \frac{1}{\sqrt{n}}$ diverges to conclude that the series does not converge absolutely.

The third series is absolutely convergent. Note that $|\sin\frac{1}{n}| \leq \frac{1}{n}, \forall\, n \in \mathbb{N}$. Hence $|\frac{\sin\frac{1}{n}}{n}| \leq \frac{1}{n^2}$. Now use comparison test and the fact that $\Sigma_{n=1}^{\infty}\frac{1}{n^2}$ is convergent to deduce that the series is absolutely convergent.

2. 2(a). $b_n - \frac{b_{n+1}a_{n+1}}{a_n} \geq r\,\forall\, n \geq N$ which implies $a_nb_n - b_{n+1}a_{n+1} \geq ra_n$. For every $k \geq N$ we have $r\Sigma_{j=N}^{k} a_j < \Sigma_{j=N}^{k}(a_jb_j - a_{j+1}b_{j+1}) = a_Nb_N - a_{k+1}b_{k+1} < a_Nb_N$. Hence the partial sums are bounded and they form an increasing sequence.

2(b). As $c_n \leq 0$ we have $a_nb_n - a_{n+1}b_{n+1} \leq 0, \forall\, n \geq N$. Again taking the partial sum we get $a_Nb_N - a_{k+1}b_{k+1} \leq 0, \forall\, k \geq N$. So $a_{k+1} \geq \frac{a_Nb_N}{b_{k+1}}$. Now take the partial sum both side and using comparison test and the fact that $\Sigma\frac{1}{b_k}$ diverges we are done.

3. The series $\sum a_n$ converges absolutely. Hence $\sum |a_n|$ converges. So $\lim |a_n| = 0$. Choose $\epsilon = 1 > 0$, then $\exists N \in \mathbb{N}$ such that $|a_n| < 1\ \forall n \geq N$. So for all $n \geq N$ we have $a_n^2 \leq |a_n|$. Now using comparison test we conclude that $\sum a_n^2$ converges.

The converse is not true. Consider the sequence $a_n = 1/n$ and the corresponding series.

4. Suppose $\lim_{x\to p^+} f(x) = a$ and $\lim_{x\to p^-} f(x) = b$, where $a \neq b$. Also suppose $\lim_{x\to p} f(x) = l$ for some finite number l and $l \neq a$ (WLOG). Then for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - a| < \epsilon \text{ whenever } p < x < p + \delta$$
$$|f(x) - l| < \epsilon \text{ whenever } p - \delta < x < p + \delta$$

and hence

$$|f(x) - a| + |f(x) - l| < 2\epsilon \text{ whenever } p < x < p + \delta.$$

Observe that for any $p < x < p + \delta$

$$\begin{aligned}|l - a| &\leq |f(x) - a| + |f(x) - l|\\ &< 2\epsilon\end{aligned}$$

and hence $l = a$, which is a contradiction to the assumption that $l \neq a$.

5. Let $\epsilon > 0$ be given

a. Choose $\delta = \frac{\epsilon}{1+\epsilon}$, then

$$
\begin{aligned}
&|x-1| < \delta \\
\Rightarrow &1-\delta < x < 1+\delta \\
\Rightarrow &1-\frac{\epsilon}{1+\epsilon} < x < 1+\frac{\epsilon}{1+\epsilon} \\
\Rightarrow &\frac{1}{1+\epsilon} < x < \frac{1+2\epsilon}{1+\epsilon} \\
\Rightarrow &\frac{1+\epsilon}{1+2\epsilon} < \frac{1}{x} < 1+\epsilon \\
\Rightarrow &-\epsilon < -\frac{\epsilon}{1+2\epsilon} < \frac{1}{x} - 1 < \epsilon \\
\Rightarrow &|\frac{1}{x} - 1| < \epsilon
\end{aligned}
$$

b. Choose $\delta = \min\{2 - \frac{1}{\sqrt{\frac{1}{4}+\epsilon}}, \frac{1}{\sqrt{\frac{1}{4}-\epsilon}} - 2\}$(here $\epsilon < 1/4$), then

$$
\begin{aligned}
&|x-2| < \delta \\
\Rightarrow &2-\delta < x < 2+\delta \\
\Rightarrow &\frac{1}{\sqrt{\frac{1}{4}+\epsilon}} < x < \frac{1}{\sqrt{\frac{1}{4}-\epsilon}} \\
\Rightarrow &\frac{1}{\frac{1}{4}+\epsilon} < x^2 < \frac{1}{\frac{1}{4}-\epsilon} \\
\Rightarrow &\frac{1}{4} - \epsilon < \frac{1}{x^2} < \frac{1}{4} + \epsilon \\
\Rightarrow &|\frac{1}{x^2} - \frac{1}{4}| < \epsilon.
\end{aligned}
$$

c. Choose $\delta = \min\{1-(1-\epsilon)^2, (1+\epsilon)^2 - 1\}$, then

$$
\begin{aligned}
&|x-1| < \delta \\
\Rightarrow &1-\delta < x < 1+\delta \\
\Rightarrow &(1-\epsilon)^2 < x < (1+\epsilon)^2 \\
\Rightarrow &(1-\epsilon) < \sqrt{x} < (1+\epsilon) \\
\Rightarrow &|\sqrt{x} - 1| < \epsilon.
\end{aligned}
$$

d. Choose any $\delta > 0$, then

$$
\begin{aligned}
&|x| < \delta;\ x \neq 0 \\
\Rightarrow &-\delta < x < \delta;\ x \neq 0 \\
\Rightarrow &|1-1| < \epsilon(\ \textit{because}\ \frac{x}{x} = 1\ \forall x \neq 0).
\end{aligned}
$$