SOLUTIONS OF HOMEWORK-3

(1) (a)

$$\frac{1}{4n^2-1} = \frac{1}{2}[\frac{1}{2n-1} - \frac{1}{2n+1}]$$

Therefore,

$$\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{2}[1 - \frac{1}{2n+1}]$$

i.e the partial sum $s_n = \frac{1}{2}[1 - \frac{1}{2n+1}]$.
Hence $s_n \to \frac{1}{2}$.
So, $\sum_n \frac{1}{4n^2-1} = \frac{1}{2}$.

(b) The partial sum $s_n = 2[1 + \frac{1}{3} + ... + \frac{1}{3^{n-1}}]$
So, $s_n = 3[1 - \frac{1}{3^n}]$
Therefore, $s_n \to 3$
Hence, $\sum_n \frac{2}{3^{n-1}} = 3$.

(c)

$$\frac{2^n + n + n^2}{2^{n+1}n(n+1)} = \frac{1}{2n(n+1)} + \frac{1}{2^{n+1}} = \frac{1}{2}[\frac{1}{n} - \frac{1}{n+1}] + \frac{1}{2^{n+1}}$$

Therefore, the partial sum

$$s_n = \frac{1}{2}[1 - \frac{1}{n+1}] + \frac{1}{2}$$

Hence, $s_n \to 1$. So, $\sum_n \frac{2^n+n+n^2}{2^{n+1}n(n+1)} = 1$.

(2) (a) Consider

$$s_n = 1 + 4x^2 + ... + (4x^2)^n = \frac{1-(4x^2)^{n+1}}{1-4x^2}$$

$s_n \to \frac{1}{1-4x^2}$ provided $4x^2 < 1$ i.e. $|x| < \frac{1}{2}$.

(b) Consider $|x| = 1$, then for x=1 the series converges since each term of the series is 0.
But for x=-1, the series converges to $-\infty$.
Since for $|x| < 1$ both the series $\sum_n x^n$ and $\sum_n x^{2n}$ converges. Hence the series

$\sum_n x^n - x^{2n}$ converges.
Consider

$$s_n = \sum_{k=1}^{n} x^k - x^{2k} = [x + x^2 + ... + x^n] - [x^2 + x^4 + ... + x^{2n}]$$

$$= x\frac{1-x^n}{1-x} - x^2\frac{1-x^{2n}}{1-x^2}$$

as $n \to \infty$, $s_n \to \frac{x}{1-x} - \frac{x^2}{1-x^2} = \frac{x}{1-x^2}$
For $|x| > 1$ notice that the series $\sum_n (x^n - x^{2n})$ diverges (why?).

(3) (a)

$$\frac{n}{(4n-1)(4n-3)} = \frac{n}{16n^2 - 16n + 3} \geq \frac{n}{16n^2 - 16n + 4} = \frac{n}{4(2n-1)^2}$$

Now

$$\frac{n}{4(2n-1)^2} = \frac{1}{8}\{\frac{1}{2n-1} + \frac{1}{(2n-1)^2}\}$$

As $\sum_n \frac{1}{2n-1}$ diverges and $\sum_n \frac{1}{(2n-1)^2}$ converges, so by comparison test we can conclude that the given series diverges.

(b) We know $2^{n+1} > (n+1)^3 \forall$ n ≥ 10
Hence $\frac{n+1}{2^n} \leq \frac{2}{(n+1)^2}$, so the series converges.

(c)

$$\frac{1+\sqrt{n}}{(1+n)^3 - 1} \leq \frac{1+\sqrt{n}}{3(n^2+n)} \leq \frac{1+\sqrt{n}}{n^2 + n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}$$

So by comparison test the given series converges.

(d) analogous to (b).

(4) (a) Observe that $a_n = \frac{(n!)^2}{(2n)!}$. Series' in which nth term contains factorial function we, in general, apply ratio test. Compute $\frac{a_{n+1}}{a_n}$ which in this case is $\frac{(n+1)^2}{(2n+2)(2n+1)}$. So $Lim_{n\to\infty}\frac{a_{n+1}}{a_n} = Lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{(2+\frac{2}{n})(2+\frac{1}{n})} = \frac{1}{4} < 1$. Hence the series is convergent.

(b) Here $a_n = \frac{2^n n!}{n^n}$. Now compute $Lim_{n\to\infty}\frac{a_{n+1}}{a_n} = Lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^{n+1} = Lim_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^n}\frac{1}{1+\frac{1}{n}} =$ $\frac{2}{e} < 1$. Hence by ratio test the series converges.

(c) Here $a_n = e^{-n^2}$. We use Root test here. So we compute first $(a_n)^{\frac{1}{n}} = e^{-n}$ which goes to zero as n tends to infinity. Hence by root test, the series converges.

(d) Write a_n as $a_n = \frac{n}{\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}}}$. Note that the denominator converges to 1 and numerator goes to infinity. Thus a_n does not converge to zero. Hence the series diverges.

(5) (a) $|a_n| = \frac{1}{2^n}$. So the series is absolutely convergent, and hence convergent also.

(b) $a_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \frac{(-1)^n\sqrt{n}}{n-1} - \frac{1}{n-1}$. Let $b_n = \frac{(-1)^n\sqrt{n}}{n-1}$ and $c_n = \frac{1}{n-1}$. The series Σc_n diverges. For the series Σb_n we use Dirichlet test to conclude that it is conditionally convergent. To see that observe $\Sigma(-1)^n$ has bounded partial sum. Now we'll prove that $\frac{\sqrt{n}}{n-1}$ decreases to zero. Write $\frac{\sqrt{n}}{n-1} = \frac{1}{\sqrt{n}-\frac{1}{\sqrt{n}}}$. Now check $\frac{1}{\sqrt{n+1}-\frac{1}{\sqrt{n+1}}} > \frac{1}{\sqrt{n}-\frac{1}{\sqrt{n}}}$ by squaring both side. Hence by Dirichlet test Σb_n converges conditionally. This shows that Σa_n does not converge conditionally (why?).

(c) Apply the Dirichlet test (as above) to conclude that the series is conditionally convergent. One point here to notice is that the sequence $\left(\frac{\sqrt{n}}{n+100}\right)$ decreases after finitely many terms. The series is not absolutely convergent. To see that notice $\frac{n}{n+100} > \frac{1}{2}, \forall n > 100$. Hence $\frac{\sqrt{n}}{n+100} > \frac{1}{2\sqrt{n}}, \forall n > 100$. Now use comparison test and the fact that $\Sigma\frac{1}{\sqrt{n}}$ diverges to conclude that the series diverges absolutely.

(d) Note that $|\sin\frac{1}{n}| \leq \frac{1}{n}, \forall n \in \mathbb{N}$. Hence $|\frac{\sin\frac{1}{n}}{n}| \leq \frac{1}{n^2}$. Now use comparison test and the fact that $\Sigma_{n=1}^{\infty}\frac{1}{n^2}$ is convergent to deduce that the series is absolutely convergent.