SOLUTIONS FOR HOMEWORK 2

1. We only have to show that a convergent sequence of real numbers is bounded. Assume $(x_n)_{n\geqslant 0}$ is a convergent sequence, converging to $y \in \mathbb{R}$. Then by definition there is an $N \in \mathbb{N}$ such that for all $n \geqslant N$ one has $|x_n - y| \leqslant 1$. Now for all such n,

$$|x_n| - |y| \leqslant |x_n - y| \leqslant 1,$$

whence $|x_n| \leqslant |y| + 1$. Put $M := \max\big(\{|x_0|, |x_1|, \ldots, |x_{N-1}|, |y| + 1\}\big)$. Then, obviously, $|x_n| \leqslant M$ ($\forall\, n \in \mathbb{N}$), which means $(x_n)_{n\geqslant 0}$ is bounded.

2. It is enough to show that if $(x_n)_{n\geqslant 0}$ is a sequence of real numbers which converges to both $c \in \mathbb{R}$ and $d \in \mathbb{R}$, then $c = d$. Suppose, to get a contradiction that $c \neq d$; suppose, without loss of generality, that $c < d$. Choose an $\epsilon > 0$ such that $c + \epsilon < d - \epsilon$ (for example, $\epsilon := \frac{d-c}{3}$ will do). Then obviously $]c - \epsilon, c + \epsilon[\cap]d - \epsilon, d + \epsilon[= \varnothing$. By the assumption that $(x_n)_{n\geqslant 0}$ converges to c, we get an $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ one has $x_n \in]c - \epsilon, c + \epsilon[$; and by the assumption that $(x_n)_{n\geqslant 0}$ converges to d, we get an $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ one has $x_n \in]d - \epsilon, d + \epsilon[$. In particular, if $N := \max\big(\{N_1, N_2\}\big)$, then $x_N \in]c-\epsilon, c+\epsilon[$ and $x_N \in]d-\epsilon, d+\epsilon[$. But since $]c-\epsilon, c+\epsilon[\cap]d-\epsilon, d+\epsilon[= \varnothing$, this is a contradiction, which finishes the proof.

3.(a) Assume $\epsilon > 0$. Using the fact that $(a_n)_{n\geqslant 0}$ converges to A, choose $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ one has $|a_n - A| \leqslant \frac{\epsilon}{2}$; and using the fact that $(b_n)_{n\geqslant 0}$ converges to B, choose $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ one has $|b_n - B| \leqslant \frac{\epsilon}{2}$. If we put $N := \max\big(\{N_1, N_2\}\big)$, then clearly, for every $n \geqslant N$ one has

$$|(a_n + b_n) - (A + B)| \leqslant |a_n - A| + |b_n - B| \leqslant \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon,$$

which proves that $\lim_{n\to\infty}(a_n + b_n) = A + B$.

(b) This will follow from **(a)** and **(c)** (proved below.)

(c) If $c = 0$, the $(ca_n)_{n\geqslant 0}$ is the constant sequence 0, which converges to $0 = 0\cdot A$, so in this case there is nothing to prove. So suppose $c \neq 0$, and assume $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n \geqslant N$ one has $|a_n - A| \leqslant \frac{\epsilon}{|c|}$. Then for $n \geqslant N$ one also has

$$|ca_n - cA| = |c|\,|a_n - A| \leqslant |c|\frac{\epsilon}{|c|} = \epsilon,$$

which shows that $\lim_{n\to\infty}(ca_n) = cA$.

(d) Note that, for $n \in \mathbb{N}$ arbitrary, one has

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\ &\leqslant |a_n b_n - a_n B| + |a_n B - AB| \\ &= |a_n|\,|b_n - B| + |a_n - A|\,|B|. \end{aligned} \tag{1}$$

Now choose $M > 0$ such that $|a_n| \leqslant M$ for all $n \in \mathbb{N}$. Assume $\epsilon > 0$. Then choose $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ one has $|a_n - A| \leqslant \frac{\epsilon}{2(|B|+1)}$; also choose $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ one has $|b_n - B| \leqslant \frac{\epsilon}{2M}$. Put $N := \max\big(\{N_1, N_2\}\big)$. Then for all $n \geqslant N$ one has, by (1),

$$|a_n b_n - AB| \leqslant M\frac{\epsilon}{2M} + \frac{\epsilon}{2(|B|+1)}|B| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{n\to\infty}(a_n b_n) = AB$.

(e) Note that, by **(d)** above, it is sufficient to show that if $(a_n)_{n\geqslant 0}$ is a sequence of real numbers converging to $a \in \mathbb{R}$, and if $a \neq 0$, then $\left(\frac{1}{a_n}\right)_{n\geqslant k}$ (defined for sufficiently large n) is convergent, and converges to $\frac{1}{a}$. We may suppose, without loss of generality, that $a_n \neq 0$ for all $n \in \mathbb{N}$. Note that, for $n \in \mathbb{N}$ arbitrary, one has

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a_n a}\right| = \frac{1}{|a_n|}\frac{1}{|a|}\,|a - a_n|\,. \tag{2}$$

Now as $a_n \to a$, there is an $N \in \mathbb{N}$ such that for all $n \geqslant N$ one has $|a - a_n| \leqslant \frac{|a|}{2}$. So for all such n, $|a| - |a_n| \leqslant |a - a_n| \leqslant \frac{|a|}{2}$, and so $|a_n| \geqslant \frac{|a|}{2}$. Now assume $\epsilon > 0$ and choose $M \in \mathbb{N}$ such that $M \geqslant N$ and such that for all $n \geqslant M$ one has $|a - a_n| \leqslant \frac{|a|^2\epsilon}{2}$. Then for all $n \geqslant M$ one has, by (2),

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| \leqslant \frac{2}{|a|^2}\frac{|a|^2\epsilon}{2} = \epsilon.$$

This means $\lim_{n\to\infty}\left(\frac{1}{a_n}\right) = \frac{1}{a}$.

4.(a) The given sequence converges to 0; indeed, it is a sequence of positive real numbers which decreases to 0. The fact that each term of the sequence is positive follows from the fact that since

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\cdots\left(-\frac{1}{2} - (n-1)\right) = (-1)^n\left(\frac{1}{2}\right)\left(\frac{1}{2} + 1\right)\cdots\left(\frac{1}{2} + (n-1)\right),$$

therefore

$$(-1)^n\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\cdots\left(-\frac{1}{2} - (n-1)\right)}{n!} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2} + 1\right)\cdots\left(\frac{1}{2} + (n-1)\right)}{n!}.$$

Also,

$$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+(n-1)\right)}{n!} = \frac{\prod_{j=1}^{n}\left(\frac{1}{2}+(j-1)\right)}{\prod_{j=1}^{n} j} = \prod_{j=1}^{n}\frac{2j-1}{2j}.$$

So finally $x_n = \prod_{j=1}^{n}\frac{2j-1}{2j}$ ($\forall\, n \geqslant 1$), where $(x_n)_{n\geqslant 1}$ is the given sequence. It turns out that this form for x_n is extremely convenient. For example, one observes that $x_{n+1} = x_n\left(\frac{2n+1}{2n+2}\right)$ and since $\frac{2n+1}{2n+2} < 1$, therefore it follws immediately that $x_{n+1} < x_n$ ($\forall\, n \geqslant 1$), i.e., that $(x_n)_{n\geqslant 1}$ is strictly decreasing. So since $(x_n)_{n\geqslant 1}$ is a sequence of positive numbers, we can conclude immediately that it converges to some nonnegative real number L. The whole problem is to determine what L is. We propose to show that for all $n \geqslant 1$ one has $x_n \leqslant \frac{1}{\sqrt{n+1}}$. If this is shown, then it will follow trivially that $\lim_{n\to\infty} x_n = 0$, because then $0 \leqslant x_n \leqslant \frac{1}{\sqrt{n+1}}$, and $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$. We will prove the claim by induction on n. For $n = 1$, the statement is "$\frac{1}{2} \leqslant \frac{1}{\sqrt{2}}$", which is true. Assume that $n \in \mathbb{N}$ and that the statement holds for n; we have to prove that the statement holds also for $n+1$. Since $x_{n+1} = x_n\left(\frac{2n+1}{2n+2}\right)$, and since $x_n \leqslant \frac{1}{\sqrt{n+1}}$ by hypothesis, therefore we will be done if we can prove that $\frac{2n+1}{2n+2} \leqslant \frac{\sqrt{n+1}}{\sqrt{n+2}}$. But after squaring and cross-multiplying (we are dealing with positive quantities) we see that this last assertion is equivalent to the assertion that $(2n+1)^2(n+2) \leqslant 4(n+1)^3$, which in turn, after expanding out, turns out to be equivalent to the assertion that $-2 \leqslant 3n$, which is trivially true. So the induction step is complete, and with it the proof.

(b) Let us prove something more general but equally simple: suppose $(x_n)_{n\geqslant 0}$ is a bounded sequence and $(y_n)_{n\geqslant 0}$ is a sequence that tends to 0; then $(x_n y_n)_{n\geqslant 0}$ tends to 0. First, choose $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Then, if $\epsilon > 0$, use the convergence of $(y_n)_{n\geqslant 0}$ to 0 to choose $N \in \mathbb{N}$ such that for all $n \geqslant N$ one has $|y_n| \leqslant \frac{\epsilon}{M}$. Then, for $n \geqslant N$, one also has $|x_n y_n| = |x_n|\,|y_n| \leqslant M\frac{\epsilon}{M} = \epsilon$. This proves that $(x_n y_n)_{n\geqslant 0}$ converges to 0. We get the result we need by taking $x_n := (-1)^n$ ($\forall\, n \in \mathbb{N}$) and $y_n := \frac{1}{n}$ ($\forall\, n \in \mathbb{N}$): the given sequence converges to 0.

(c) Let us once again prove something more general. We will prove that if x is a real number *greater than* 1, then $\lim_{n\to\infty}\frac{x^n}{n} = +\infty$. Since $x > 1$, write $x = 1 + h$, with $h > 0$. Then, by the binomial formula,

$$x^n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \dots.$$

(Concentrate on the case $n \geqslant 2$; we are in any case interested in $n \to \infty$.) Note that each term that has been omitted is *positive*; when $n = 2$ then of course no terms have been omitted. So one clearly has

$$\frac{x^n}{n} \geqslant \frac{n(n-1)}{2n}h^2 = \frac{n-1}{2}h^2.$$

Now as h is fixed, $\left(\frac{n-1}{2}h^2\right)_{n\geqslant 0}$ obviously tends to $+\infty$. This clearly implies $\lim_{n\to\infty}\frac{x^n}{n}=+\infty$. In particular, $\lim_{n\to\infty}\frac{2^n}{n}=+\infty$. If the reader wants, he or she can write out the details of the special case $x=2$ to make matters clearer.

(d) The given sequence diverges to $+\infty$. One can see this by performing some trivial manipulations:

$$\frac{1}{\sqrt{n+1}-\sqrt{n}}=\frac{\sqrt{n+1}+\sqrt{n}}{n+1-n}=\sqrt{n+1}+\sqrt{n},$$

and obviously $\lim_{n\to\infty}\left(\sqrt{n+1}+\sqrt{n}\right)=+\infty$.

(e) The given sequence converges to $\frac{2}{5}$. We can see this by performing the following manipulation:

$$\frac{2n}{5n-7\sqrt{n}}=\frac{2}{5-\frac{7}{\sqrt{n}}}$$

and noting that since $\lim_{n\to\infty}\frac{7}{\sqrt{n}}=0$, one can apply the results of problem **3** step by step to conclude that the limit of the given sequence is $\frac{2}{5}$.

(f) The given sequence does not converge. Indeed, it is precisely the sequence $((-1)^n)_{n\geqslant 0}$, which clearly does not converge. The following more general remark holds: a sequence x is simply a mapping from $\mathbb{N}$ to $\mathbb{R}$; if it happens that there are $a,b\in\mathbb{R}$ such that $a\neq b$ and such that $x^{-1}(\{a\})$ and $x^{-1}(\{b\})$ are *both* infinite, then x does not converge. To prove this, suppose that x converges to $c\in\mathbb{R}$. If $c\neq a$, we can find an $\epsilon>0$ such that $a\notin\,]c-\epsilon,c+\epsilon[$. But by the definition of a limit, there is an $N\in\mathbb{N}$ such that for $n\geqslant N$ one has $x(n)\in\,]c-\epsilon,c+\epsilon[$. In particular, for all such n one has $x(n)\neq a$, so $x^{-1}(\{a\})\subseteq\{0,\ldots,N-1\}$, and is therefore finite, a contradiction. And if $c=a$, then we can find an $\epsilon>0$ such that $b\notin\,]a-\epsilon,a+\epsilon[$, and by an identical argument we will get that $x^{-1}(\{b\})$ is finite, which is another contradiction. So we invariably have a contradiction, so x cannot converge, which is what we wanted to prove.