Homework 2

Analysis and Linear Algebra I (Autumn 2018)
Indian Institute of Science

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1. Prove that an unbounded sequence cannot have a limit.
2. Prove that the limit of a sequence, if it exists, is unique.
3. Suppose $\{a_n\}$ and $\{b_n\}$ are two convergent sequences, such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then prove the following:
   (a) $\lim_{n\to\infty} (a_n + b_n) = A + B$
   (b) $\lim_{n\to\infty} (a_n - b_n) = A - B$
   (c) For $c \in \mathbb{R}$, show that $\lim_{n\to\infty} (ca_n) = cA$
   (d) $\lim_{n\to\infty} a_n\, b_n = A\, B$
   (e) Assuming further that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $B \neq 0$, $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$
4. For the following sequences, prove either that they have a limit (in which case determine the limit) or that they do not.
   (a) $\left\{(-1)^n \binom{-1/2}{n}\right\}$, where $\binom{a}{n} = \dfrac{a(a-1)\cdots(a-n+1)}{n!}$ for $a \in \mathbb{R}, n \in \mathbb{N}$.
   (b) $\left\{\dfrac{(-1)^n}{n}\right\}$
   (c) $\left\{\dfrac{2^n}{n}\right\}$
   (d) $\left\{\dfrac{1}{\sqrt{n+1} - \sqrt{n}}\right\}$
   (e) $\left\{\dfrac{2n}{5n - 7\sqrt{n}}\right\}$
   (f) $\{\cos n\pi\}$
5. Prove that any convergent sequence is bounded.