SOLUTIONS(HOMEWORK 1)

1(a). Let
$$x \in A$$
Then,
$$x \in A \text{ or } x \in B$$
$$\Rightarrow x \in A \cup B$$
Hence,
$$A \subseteq A \cup B$$
1(b). Let $x \in A \cup (B \cap C)$. Then
$$x \in A \text{ or } x \in B \cap C$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$
$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$
$$\Rightarrow x \in (A \cup B) \cap (A \cup C).$$
Hence,
$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C).$$
Again, let $x \in (A \cup B) \cap (A \cup C)$. Then,
$$x \in A \cup B \text{ and } x \in A \cup C$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$
$$\Rightarrow x \in A \text{ or } x \in B \cap C$$
$$\Rightarrow x \in A \cup (B \cap C).$$
Hence,
$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$
So,
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
2(a). Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$. Then, $B \setminus C = \{2\}$ and hence $A \setminus (B \setminus C) = \{1\}$. On the other hand, $A \setminus B = \{1\}$. So, $(A \setminus B) \cup C = \{1, 3, 4\}$. Thus $A \setminus (B \setminus C) \neq (A \setminus B) \cup C$.

2(b). let $x \in A \setminus (B \cup C)$. This implies
$$x \in A \text{ and } x \notin B \cup C$$
$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$
$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \notin C$$
$$\Rightarrow x \in A \setminus B \text{ and } x \notin C$$

$$x \in (A \setminus B) \setminus C.$$

Hence, $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

Again, if $x \in (A \setminus B) \setminus C$, then

$$\begin{aligned} &\Rightarrow x \in A \setminus B \text{ and } x \notin C \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \notin C \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin B \cup C \\ &\Rightarrow x \in A \setminus (B \cup C). \end{aligned}$$

So,$(A \setminus B) \setminus C \subseteq A \setminus (B \cup C)$. Thus, we have

$$(A \setminus B) \setminus C = A \setminus (B \cup C).$$

3. Let y be such that $c + y = b$. Then by definition $y = b - c$. Now $a(b-c) + ac = ay + ac = a(y + c) = ab$. Hence, $a(b - c) = ab - ac$.

$a(ba^{-1}) = aa^{-1}b = 1.b = b$. So, $\frac{b}{a} = ba^{-1}$.

4.We first show that $x - y = x + (-y)$, for all $x, y \in \mathbb{R}$. Note that $y + (x + (-y)) = y + (-y) + x = 0 + x = x$. Hence by definition, $x - y = x + (-y)$.

Now we prove that $x(-y) = -xy$, for any $x, y \in \mathbb{R}$. In order to show that let us notice that

$$xy + x(-y) = x(y + (-y)) = x.0 = 0.$$

Hence, $x(-y) = -xy$.

Finally, we are in a position to prove the statement given in the question. If $a > b$, then

$$\begin{aligned} a - b &> 0 \\ \Rightarrow \ (a - b)c &> 0 \qquad (as,\ c > 0, \text{ by order axiom }) \\ \Rightarrow a + (-b)c &> 0 \\ \Rightarrow a + (-b)c &> 0 \\ \Rightarrow \ ac - bc &> 0 \\ \Rightarrow ac &> bc. \end{aligned}$$

5. If $x = 0$, then $x^2 = 0$. Hence,

$$x^2 = 0 + 1 = 1.$$

If $x \neq 0$, then x^2 is positive (If x is positive, then it is obvious from the order axioms.If $x < 0$ then $x = -y$, where y is positive. Now $x^2 = (-y)(-y) = y^2$, which is positive.). Hence $x^2 + 1$ is also positive. So, $x^2 + 1 \neq 0$.

6. Suppose $x \neq a$. Then, from the hypothesis we have $a < x$. i.e, $x - a$ is positive. So, by Archimedean property, there exists a positive integer n_0 such that

$$\begin{aligned} n_0(x - a) > y \\ \Rightarrow x > a + \frac{y}{n_0} \end{aligned}$$

which is a contradiction. So, $x = a$.

7. Since $x < y$, $y - x$ is positive. This implies, $\frac{y-x}{2}$ is also positive. Thus, we have

$$x < x + \frac{y-x}{2} < x + \frac{y-x}{2} + \frac{y-x}{2} = y. \text{ (Proved)}$$

8. Let $x = \frac{p}{q}$, where p and q are integer. Suppose $x + y \in \mathbb{Q}$. then there exists two integers p_1 and q_1 such that $x + y = \frac{p_1}{q_1}$. Now

$$y = (x + y) - x = \frac{p_1}{q_1} - \frac{p}{q} = \frac{qp_1 - pq_1}{qq_1} \in \mathbb{Q},$$

a contradiction. So, $x + y \in \mathbb{R} \setminus \mathbb{Q}$.

Again, suppose $\frac{x}{y} = \frac{p_2}{q_2} \in \mathbb{Q}$. Then, $\frac{y}{x} = \frac{q_2}{p_2}$. Hence

$$y = x.\frac{y}{x} = \frac{p}{q}.\frac{q_2}{p_2} = \frac{pq_2}{qp_2} \in \mathbb{Q},$$

a contradiction. So, $\frac{x}{y} \in \mathbb{R} \setminus \mathbb{Q}$.

Use similar arguments for $\frac{y}{x}$.

9.We first prove that the square of an odd number is odd. Let $a = 2m+1$, $m \in \mathbb{Z}$, be an odd number. Then $a^2 = (2m + 1)^2 = 4m^2 + 4m + 1$, an odd number.

Now, let x be a rational number whose square is 2. Let $x = \frac{p}{q}$, where p and q are integer and atleast one of them is odd. As $x^2 = 2$ we have $p^2 = 2q^2$. So p^2 is even which implies p is also even as square of an odd number is always odd. Let $p = 2n$. Then from the relation $p^2 = 2q^2$ we have $2n^2 = q^2$. Thus q^2 is even and hence q is also even, a contradiction. So, there is no rational number whose square is 2.