Homework 14 solution

Solution 1. Claim:

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad for\ all\ n \geq 1.$$

We will prove above claim by Principle of Induction. Assume

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad for\ all\ n = 1, 2, ..., k$$

Now note that

$$A^{k+1} = A^k.A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

Hence the claim is proved.

Solution 2, question 9 of 16.20.

The system of equation can be written as the following way

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

Now apply the Gauss-Jordan elimination process. In following paragraph $R_i - cR_j$ denote the operation, namely, substracting c times j^{th} row from i^{th} row of the matrix;

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 6 \end{bmatrix}$$

$$\xrightarrow{R_2\text{-}2R_1}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 5 & -1 & a & 6 \end{bmatrix}$$

$$\xrightarrow{R_3\text{-}5R_1}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & -4 \end{bmatrix}$$

$$\xrightarrow{\text{-}\frac{1}{3} R_2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & -6 & a-10 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3+6R_2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & a-8 & 0 \end{bmatrix} \qquad (1)$$

Now two cases arise. First let's study the case when $a \neq 8$.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & a-8 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{a-8} R_3}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In this case note that the solution is unique. We can get the solution recursively from the following system of equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

which gives us

$$x + y + 2z = 2$$
$$y + \frac{1}{3}z = \frac{2}{3}$$
$$z = 0$$

And this gives us the solution $x = \frac{4}{3}, y = \frac{2}{3}, z = 0$, which is unique.

Now let's study the case when $a = 8$. Following equation (1) we get our last matrix

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can get the solution from the following system of equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

which gives us

$$x + y + 2z = 2$$
$$y + \frac{1}{3}z = \frac{2}{3}$$

Note that in this case we have infinitely many solutions. Solution can be written in the following form

$$z = t$$

here t is an arbitrary real number.

$$y = \frac{2}{3} - \frac{1}{3}t$$
$$x = 2 - y - 2t = \frac{4}{3} - \frac{5}{3}t$$

Note solution can also be put in the following form

$$= (\frac{4}{3} - \frac{5}{3}t, \frac{2}{3} - \frac{1}{3}t, t)$$

$$= (\frac{4}{3}, \frac{2}{3}, 0) + t(-\frac{5}{3}, -\frac{1}{3}, 1)$$

where t is an arbitrary real number.

Solution 3. (1) **T is linear**:

$$(T(cp_1+p_2))(x) = xD(cp_1+p_2)(x) = x(cDp_1+Dp_2)(x) = cxDp_1(x)+xDp_2(x) = c(Tp_1)(x)+(Tp_2)(x)$$

implies

$$(T(cp_1 + p_2)) = c(Tp_1) + (Tp_2), \quad \forall p_1, p_2 \in V, \quad \forall c \in \mathbb{R}.$$

(2) Let $p \in V$ be such that $T(p) = p$. Let $p(x) = \sum_{n=0}^{N} a_n x^n$, which implies

$$Dp(x) = \sum_{n=1}^{N} n a_n x^{n-1}$$

So

$$xDp(x) = \sum_{n=1}^{N} n a_n x^n.$$

Now $p(x) = xDp(x)$ implies (by comparing the coefficients) $a_0 = 0$ and $a_n = na_n, \forall n = 1, 2, ..., N$. Which gives $a_n = 0$ $\forall n = 0, 2, ..., N$. So $p(x) = a_1 x$. Therefore if $T(p) = p$, then p has to be of the form $p(x) = a_1(x)$ for some $a_1 \in \mathbb{R}$. Conversely if $p(x) = a_1(x)$ for some $a_1 \in \mathbb{R}$, then clearly $T(p) = p$. So the polynomials p defined by $p(x) = a_1(x)$ for some $a_1 \in \mathbb{R}$, are the collection of all polynomials which satisfy $T(p) = p$.

(3) By the similar manner as above, we can show that the polynomials p defined by $p(x) = a_0 + a_2 x^2$ where $a_0, a_2 \in \mathbb{R}$ are the collection of all polynomials which satisfy the given equation.

Solution 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 * 2$ matrix such that $A^2 = 0$. But

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$$

implies

$$a^2 + bc = 0 \quad ...(1),$$

$$(a + d)b = 0 \quad ...(2),$$

$$(a + d)c = 0 \quad ...(3),$$

$$bc + d^2 = 0 \quad ...(4).$$

Now either $b = 0$ or $b \neq 0$.

Case 1. If $b = 0$, then (1) and (4) implies $a = 0$ and $d = 0$. Therefore

$$A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \quad for\ c \in \mathbb{R}.$$

Case 2. If $b \neq 0$, then $a = -d$ by (2). Now by (1) we have, $c = -\frac{a^2}{b}$. Therefore

$$A = \begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix} \quad for\ a, b \in \mathbb{R}\ and\ b \neq 0.$$

Solution 5. Given $A^2 = A$, therefore $A^n = A$ for all $n \geq 1$. Since the identity matrix I commutes with A, therefore

$$(A + I)^k = \sum_{r=0}^{k} \binom{k}{r} A^r I^{k-r} = \sum_{r=0}^{k} \binom{k}{r} A^r = I + \sum_{r=1}^{k} \binom{k}{r} A^r = I + \sum_{r=1}^{k} \binom{k}{r} A = I + (2^k - 1)A.$$

Solution 6. If A and B are two invertible $n * n$ matrices then $A + B$ need not be invertible. For example, take $A = I_n$ and $B = -I_n$. Where I_n is the identity matrix of order n.

Solution 7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then check that the matrix $B := \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the inverse of A.

If $ad - bc = 0$, then if b or d is equal to zero then it is easy to see (check this) that either a column or a row becomes zero. Hence A is not invertible.

If $b \neq 0$ and $d \neq 0$ then $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$. Therefore

$$-\frac{1}{b}(a, b) + \frac{1}{d}(c, d) = (0, 0)$$

implies that rows of A is linearly dependent set. Hence A is not invertible.