# SOLUTIONS FOR HOMEWORK 13

1. **Exercise 1 of Section 15.12**
   **(a)** The given formula does *not* define an inner product. For example, it is not symmetric; $\langle x, y\rangle = 1 \neq -1 = \langle y, x\rangle$, where $x := (1, 0, \dots, 0),\ y := (-1, 0, \dots, 0)$.
   **(c)** The given formula does *not* define an inner product. It is not positive-definite: $\langle x, x\rangle = 0$, where $x := (1, -1, 0, \dots, 0)$ (we are of course assuming $n \geqslant 2$). We note that the product given here is linear in the first input, symmetric, and even satisfies $\langle x, x\rangle \geqslant 0$ for all $x \in V = \mathbb{R}^n$.
   **Exercise 4 of Section 15.12**
   Assume $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Expanding out the LHS, we see that we have $\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2$, so $\langle x, y\rangle = 0$. Conversely, if $\langle x, y\rangle = 0$, then, again expanding using the definition, $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2$.

2. We simply expand the LHS out using the definition. One has

$$
\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \\
&= 2\|x\|^2 + 2\|y\|^2,
\end{aligned}
$$

   as required.

3. **Exercise 18 of Section 16.4**

   $T(x, y, z) = (x, 2y, 3z)$.

   Check it is closed under addition and scalar multiplication to conclude it is a linear transformation.

   Now, $(x, 2y, 3z) = (0, 0, 0) \implies x = y = z = 0$. Thus null space of $T$ is $\{(0, 0, 0)\}$. So, nullity of $T$ is $o$. Hence, by rank nullity theorem, rank of $T$ is 3. Hence, range of $T$ is $\mathbb{R}^3$.

   **Exercise 20 of Section 16.4**

   $T(x, y, z) = (x + 1, y + 1, z - 1)$. We know that, if $S$ is a linear transformation on $\mathbb{R}^3$ then $S(0, 0, 0) = (0, 0, 0)$. But $T(0, 0, 0) = (1, 1, -1)$. Hence, $T$ is not a linear transformation.

4. **Exercise 15 of Section 16.4**

   $T(r, \theta) = (r, 2\theta)$. Suppose $T$ is a linear transformation. It is easy to see that null space of $T$ is $\{(0, 0, 0)\}$. Thus, $T$ is invertible. So, $\{T(1, 0), T(1, \frac{\pi}{2})\}$ is a basis for $\mathbb{R}^2$. But, $T(1, 0) + T(1, \frac{\pi}{2}) = (1, 0) + (1, \pi) = 0$. Hence, $T$ is not a linear transformation.

5. **Exercise 24 of Section 16.4**

   We need to show that $T(p(x)+q(x)) = T(p(x)) + T(q(x))$ and $T(c.p(x)) = c.T(p(x))$ $\forall p, q \in V$ and $\forall c \in \mathbb{R}$. Let $h(x) = p(x)+q(x)$. Then, $h(x+1) = p(x+1)+q(x+1)$. Thus,$T(p(x)+q(x)) = T(p(x))+T(q(x))$ $\forall p, q \in V$. Now, let $g(x) = c.p(x)$. So, $g(x+1) = c.p(x+1)$. Thus, $T(c.p(x)) = c.T(p(x))$ $\forall c \in \mathbb{R}$. Hence, $T \in \mathcal{L}(V)$.

6. **Exercise 11 of Section 16.8**

   $T(x, y) = (x-y, x+y) = (0, 0) \implies x = y = 0$. Hence, $T$ is injective. So, $T(V_2) = V_2$.

   Let $(x, y) = T^{-1}(u, v)$. Then $u = x-y$ and $v = x+y$. This gives $T^{-1}(u, v) = (\frac{u+v}{2}, \frac{v-u}{2})$.

7. Multiplication of transformations is defined as $T^{m+1} = T^m \cdot T$. We show that $T^{m+n} = T^m \cdot T^n$ $\forall m, n$. We proceed by induction on $n$. For $n = 1$, it follows from definition. Suppose, $T^{m+k} = T^m \cdot T^k$. Then $T^{m+k+1} = T^{m+k} \cdot T = (T^m \cdot T^k) \cdot T = T^m \cdot T^{k+1}$ (Since composition operation is associative). Hence, the result follows.

   Check $T^n \cdot (T^{-1})^n = (T^{-1})^n \cdot T^n = id$.

9. Let $I$ be the transformation that takes a vector $v$ to itself. It is easy to see that composition with $I$ is commutative. Also, $S^{-1} \cdot S = I$ and $T^{-1} \cdot T = I$. Thus, $T^{-1}S^{-1} \cdot ST = T^{-1} \cdot (S^{-1} \cdot S) \cdot T = T^{-1} \cdot I \cdot T = (T^{-1} \cdot T) \cdot I = I$. Similarly, show that $ST \cdot T^{-1}S^{-1} = I$ to conclude the result.

10. $S(x, y, z) = (z, y, x)$ and $T(x, y, z) = (x, x+y, x+y+z)$.

    $S(x, y, z) = 0 \implies x = y = z = 0$. Also, from $T(x, y, z) = 0$, we get $x = 0$, $x + Y = 0$, $x+y+z = 0$, i.e. $x = y = z = 0$. Since, nullity of $S$ and $T$ are 0, we can conclude that $S$ and $T$ are injective.

    Check that $S^{-1}(x, y, z) = (z, y, x)$, $T^{-1}(x, y, z) = (x, y-x, z-y)$, $ST(x, y, z) = (x+y+z, x+y, x)$, $TS(x, y, z) = (z, y+z, z+y+x)$, $(ST)^{-1}(x, y, z) = (z, y-z, x-y)$, and $TS^{-1}(x, y, z) = (z-y, y-x, x)$.

11. **Exercise 4 of Section 16.12**

    Reflection about y-axis gives $(x, y) \longrightarrow (-x, y)$ and doubling length means $(-x, y) \longrightarrow (-2x, 2y)$. Thus, $T(x, y) = (-2x, 2y)$. So, $T(1, 0) = (-2, 0)$ and $T(0, 1) = (0, 2)$. So. $m(T) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$. Check that $m(T^2) = (m(T))^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.