Solution Of Homework 12

1 .

2) Set of all rational functions form a vector space. The given set W(say) is non-empty as the constant function $0 \in W$. Let $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ be in W, i.e. $deg(f_i) \leq deg(g_i)$, $i = 1, 2$. Their sum is $\frac{f_1 g_2 + f_2 g_1}{g_1 g_2}$. Now see that $deg(f_1) \leq deg(g_1)$ which implies $deg(f_1).deg(g_2) \leq deg(g_1).deg(g_2)$. Similarly we have $deg(f_2).deg(g_1) \leq deg(g_1).deg(g_2)$. Hence W is closed under addition. It's clear that it is closed under scalar multiplication. Hence W is a subspace, so a vector space.

4) Set of all real valued functions having a common domain forms a vector space over $\mathbb{R}$. Let W be the set of all functions satisfying $2f(0) = f(1)$. Clearly $W \neq \phi$ as the constant $0 \in W$. If $f, g \in W$ then $2(f+g)(0) = 2f(0) + 2g(0) = f(1) + g(1) = (f+g)(1)$, i.e. $f + g \in W$. Let $c \in \mathbb{R}$ and $f \in W$ then $2(cf)(0) = c(2f)(0) = cf(1) = (cf)(1)$, i.e. $cf \in W$. So W is a subspace hence a vector space.

6) You have already seen while learning integration that sum of two step functions is a step function. It's clear that it is closed under scalar multiplication. Hence a subspace of the vector space of all real valued functions defined on $[0, 1]$.

9) Let f and g be odd functions. Then $(f+g)(-x) = f(-x)+g(-x) = -f(x) - g(x) = -(f+g)(x)$, i.e. $f + g$ is also odd. If $c \in \mathbb{R}$ and f is odd then $(cf)(-x) = cf(-x) = c(-f(x)) = -cf(x) = -(cf)(x)$, i.e. cf is also odd. So it's a vector space.

11) This set is not closed under scalar multiplication. If f is increasing(non constant) function then $(-1)f$ is decreasing and does not lie in the set of all increasing functions. Hence not a vector space.

2 .

13) If f and g are in the given concerned set(see that the set is non empty) then $f + g$ is also integrable on $[0, 1]$ and $\int_0^1 (f+g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0 + 0 = 0$, hence $f + g$ is in the concerned set. Similarly one can show that $\int_0^1 (cf)(x)dx = 0$ where $c \in \mathbb{R}$ and $\int_0^1 f(x)dx = 0$.

14) See that the set is not closed under scalar multiplication. Multiplication by any negative real would create problem. hence not a vector space.

3 .

18) The set is non empty. Think it as a subset of the vector space containing all real sequences. It is obvious that sum of two bounded sequence is also bounded. A bounded sequence multiplied by a real scalar is also bounded. Hence a vector space.

21) Consider the vector space V(say) consisting of all convergent real series. The set W(say) of all absolutely convergent series is a non empty subset of V. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be in W. Then by using triangle inequality we can show that the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is absolutely convergent. Also it is easy to see that W is closed under scalar multiplication. Hence W is a vector space.

4 .

23) $W = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \, or \, y = 0\}$. See that $(0, 1, 0)$ and $(1, 0, 0)$ are in W but $(0, 1, 0) + (1, 0, 0) = (1, 1, 0)$ is not W. Hence not a vector space.

24) $W = \{(x, y, z) \in \mathbb{R}^3 : y = 5x\}$. Clearly W is non empty and closed under addition and scalar multiplication. Hence a subspace of $\mathbb{R}^3$. Hence a vector space.

25) $W = \{(x, y, z) \in \mathbb{R}^3 : 3x + 4y = 1, z = 0\}$. Clearly the $(0, 0, 0)$ is not in W, hence not a vector space.

27) $W = \{(x, y, z) \in \mathbb{R}^3 : a_{11}x + a_{12}y + a_{13}z = 0, a_{21}x + a_{22}y + a_{23}z = 0, a_{31}x + a_{32}y + a_{33}z = 0\}$. Clearly W is non empty as $(0, 0, 0) \in W$. It is easy to see that W is closed under addition and scalar multiplication. Hence a subspace of $\mathbb{R}^3$. So it is a vector space.

5 .

$V = \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $F = \mathbb{R}$. Now see that

Closure Axioms

- Let $x, y \in V$. See that $x \oplus y = x.y$ is positive as $x, y > 0$.
- $c \in \mathbb{R}$ and $x \in V$. See that $c \otimes x = x^c$ is positive as $x > 0$.

Axioms for addition

• Commutative Law : Since multiplication is commutative we have $x \oplus y = y \oplus x$ for all $x, y \in V$.

• Associative Law : Since multiplication is associative we have $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$.

• Existence of zero element : See that $1 \in V$ and $x \oplus 1 = x.1 = x$ for all $x \in V$.

• Existence of negatives : For given $x \in V$ see that $x \oplus ((-1) \otimes x) = x.x^{-1} = 1$.

Axioms for multiplication by numbers

• Associative Law : See that $a \otimes (b \otimes x) = a \otimes x^b = x^{ab} = (ab) \otimes x$

• Distributive Law For addition in V : See that $a \otimes (x \oplus y) = a \otimes xy = (xy)^a = x^a y^a = x^a \oplus y^a = a \otimes x \oplus a \otimes y$.

• Distributive Law For Addition of Numbers : See that $(a + b) \otimes x = x^{a+b} = x^a x^b = x^a \oplus x^b = (a \otimes x) \oplus (b \otimes x)$.

• Existence of Identity : See that $1 \otimes x = x^1 = x$ for all $x \in V$.

Hence V turns out be a vector space over F. And from the proof we see that the role of 0 in V is played by 1 and the role of 1 in F is played 1.

6 .

2) $S = \{(x, y, z) \in \mathbb{R}^3 : x+y = 0\}$. It is easy to see that this set(clearly non empty) is closed under addition and scalar multiplication. Hence it is a subspace of $\mathbb{R}^3$. So is a vector space. See that S is spanned by the independent set $\{(1, -1, 0), (0, 0, 1)\}$, hence $dim(S) = 2$.

5) $S = \{(x, y, z) \in \mathbb{R}^3 : x = y = z\}$. Clear that S is subspace of $\mathbb{R}^3$. See that S is spanned by $\{(1, 1, 1)\}$, so $dim(S) = 1$.

7) $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 = 0\}$ See that $(1, -1, 0), (1, 1, 0) \in S$ but their sum $(1, 0, 0)$ is not in S. S is not closed under addition, hence not a vector space.

8) $S = \{(x, y, z) \in \mathbb{R}^3 : x + y = 1\}$. Clearly $(0, 0, 0)$ does not belong to S. Hence S is not a vector space.

11) $S = \{f \in P_n : f(0) = 0\}$. See that the set $S(\neq \phi)$ is closed under addition and scalar multiplication. Hece a subspace of P_n, So is a vector

space. See that S is generated by the independent set $\{x, x^2, x^3, .., x^n\}$, hence is of dimension n.

16) $S = \{f \in P_n : f(0) = f(2)\}$. This set is non empty and closed under addition and multiplication, hence a subspace of P_n, so is a vector space. Let us define the linear map $L : P_n \to \mathbb{R}$ such that $L(f) = f(0) - f(2)$, hence we see that $S = ker(L)$, therefore by Rank-Nullity theorem we can conclude that $dim(S) = n$.

7 .

Let the vectors be linearly dependent for some t, i.e. there exists $c \in \mathbb{R}$ such that $(1+t, 1-t) = c(1-t, 1+t)$ which further implies $1 + t = c - ct$ and $1 - t = c + ct$. Solving we get $c = 1$ and $t = 0$. That means if they are dependent then t has to be 0, which means if $t \neq 0$ they can't be linearly depenedent, hence they must linearly independent. Therefore $(1+t, 1-t)$ and $(1-t, 1+t)$ are linearly independent iff $t \neq 0$.

8 .

Let S be a vector subspace of a vector space V over a field F. Clearly $S \subseteq L(S)$. Now let $v \in L(S)$, i.e. $v = \sum_{k=1}^{m} c_k x_k$ for some $c_k \in F$ and $x_k \in S$ where $k = 1, 2, ..., m$. Since S is a subspace S is closed under addition and scalar multiplication. Hence $v \in S$ as $x_k \in S$ for all $k = 1, 2, .., m$. Hence $L(S) \subseteq S$. So $L(S) = S$. Conversely if $L(S) = S$, then S is a subspace of V as $L(S)$ is a subspace of V by definition.

9 .

a) V be the vector space of polynomials of degree at most 5. Given $S = \{1, x^2, x^5\}$. For any $c_i \in \mathbb{R}$ where $i = 1, 2, 3$ the polynomial $c_1 + c_2x^2 + c_3x^5$ has at most 5 roots when not all c_i's are 0. Hence for the polynomial to be the zero polynomial all c_i's must be zero. Hence S is linearly independent. So $dim(L(S)) = 3$.

b) If the set S is linearly dependent then there exists $c \in \mathbb{R}$ such that $(1+x) = c(1+x)^2$ for all $x \in \mathbb{R}$. Comparing the coefficients x^2 at both sides we get $c = 0$ and this further gives $x + 1 = 0$ for all $x \in \mathbb{R}$, i.e. we arrive at a contradiction. Hence S is linearly independent, so $dim(L(S)) = 2$.

c) If possible let the set S be linearly dependent. Then there exists $c_1, c_2, c_3 \in \mathbb{R}$ not all zero such that $c_1 + c_2e^{ax} + c_3xe^{ax} = 0$ for all $x \in \mathbb{R}$. Putting $x = 0$ gives $c_1 + c_2 = 0$, hence we have $c_1 - c_1e^{ax} + c_3xe^{ax} = 0$ for all $x \in \mathbb{R}$. Consider the case when $c_3 = 0$. Then $c_1(1 - e^{ax}) = 0$ for all $x \in \mathbb{R}$. Hence $c_1 = 0$, so $c_2 = 0$. Hence $c_1 = c_2 = c_3 = 0$. So we arrive at a contradiction. Now if $c_3 \neq 0$ then we have $c_1 - e^{ax}(c_1 - c_3 x) = 0$ for all $x \in \mathbb{R}$. Putting $x = \frac{c_1}{c_3}$ we get $c_1 = 0$ and hence we have $c_2 = 0$. This further implies $c_3 x e^{ax} = 0$ for all $x \in \mathbb{R}$, but this implies c_3 must be 0. Hence we again arrive at a contradiction. Therefore the set is linearly independent. Hence $dim(L(S)) = 3$.

d) Let the set S be linearly dependent. Then there exist $c_1, c_2, c_3 \in \mathbb{R}$ not all zero such that $c_1 + c_2 cos2x + c_3 cos3x = 0$ for all $x \in \mathbb{R}$. Putting $x = 0$ we get $c_1 + c_2 = 0$ and putting $x = \frac{\pi}{4}$ we get $c_1 + \frac{c_2}{\sqrt{2}} = 0$ and putting $x = \frac{\pi}{2}$ we get $c_1 - c_2 + c_3 = 0$. Solving these three equations we get $c_1 = c_2 = c_3 = 0$. We arrive at a contradiction, as according to our assumption c_1, c_2, c_3 were not all zero.

10 .

Solution 1: (countablity argument) If V is a finite dimensional(say n) vector space over a field F then there is a bijective correspondence between V and F^n, i.e. there cardinality is same. Now if $\mathbb{R}$ has dimension n over the field $\mathbb{Q}$, then $\mathbb{R}$ and $\mathbb{Q}^n$ have the same cardinality. But we know that $\mathbb{R}$ is uncountable and $\mathbb{Q}^n$ being a finite product of a countable set is countable. Hence we arrive at a contradiction. So the dimension of $\mathbb{R}$ over $\mathbb{Q}$ is infinite.

Solution 2: (existence of an infinite linearly independent set) Let $\mathbb{P} = \{P_i : P_i \text{ is a prime number}\}$ be the set of all prime numbers. Consider the set $A = \{log(P_i) : P_i \in \mathbb{P}\}$. We claim that W is a linearly independent subset. Basically we will show that every finite subset of A is linearly independent. Consider the set $\{log(p_1), log(P_2), ..., log(P_k)\}$. Suppose $c_1 log(p_1) + c_2 log(P_2) + ... + c_k log(P_k) = 0$ for some $c_i \in \mathbb{Q}$ where $i = 1, 2, ..., k$. After multiplying by a large integer we can have $d_1 log(p_1) + d_2 log(P_2) + ... + d_k log(P_k) = 0$ where $d_i \in \mathbb{Z}$ for all $i = 1, 2, .., k$. This implies $P_1^{d_1}.P_2^{d_2}.....P_k^{d_k} = 1$. Since P_i's are all distinct primes we must have $d_i = 0$ for all $i = 1, 2, ..., k$. Hence $c_i = 0$ for all $i = 1, 2, ..., k$. Hence $\{log(p_1), log(P_2), ..., log(P_k)\}$ is linearly independent $\forall k \in \mathbb{N}$, so the set A is linearly independent.(see that A is infinite as there are infinitely many primes)

11 .

$V = \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and $B = \{(1, 0), (0, 1)\}$