SOLUTIONS OF HOMEWORK 10

(1) Q.1 from the exercise 5.5
$f(x) = 5x^3$
Consider $P(x) = \frac{5}{4}x^4$ then $P'(x) = f(x)$
So, by second fundamental theorem, $\int_a^b f(x)dx = P(b) - P(a) = \frac{5}{4}(b^4 - a^4)$.

Q.3 from the exercise 5.5
$f(x) = (x+1)(x^3 - 2) = x^4 - x^3 - 2x + 2$
Consider $P(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4 - x^2 + 2x$ then $P'(x) = f(x)$.
So, by the second fundamental theorem ,
$\int_a^b f(x)dx = P(b) - P(a) = (\frac{1}{5}b^5 - \frac{1}{4}b^4 - b^2 + 2b) - (\frac{1}{5}a^5 - \frac{1}{4}a^4 - a^2 + 2a)$.

Q.6 from the exercise 5.5
$f(x) = \sqrt{2x} + \sqrt{\frac{x}{2}}$
Consider $P(x) = (\sqrt{2} + \frac{1}{\sqrt{2}})\frac{2}{3}x^{3/2}$
then $P'(x) = f(x)$
So, by the second fundamental theorem ,

$$\int_a^b f(x)dx = P(b) - P(a) = \{(\sqrt{2} + \tfrac{1}{\sqrt{2}})\tfrac{2}{3}b^{3/2}\} - \{(\sqrt{2} + \tfrac{1}{\sqrt{2}})\tfrac{2}{3}a^{3/2}\}$$

Q.9 from the exercise 5.5
$f(x) = 3\sin x + 2x^5$
Consider $P(x) = -3\cos x + \frac{1}{3}x^6$ then $P'(x) = f(x)$
So, by the second fundamental theorem ,
$\int_a^b f(x)dx = P(b) - P(a) = 3[\cos a - \cos b] + \frac{1}{3}[b^6 - a^6]$.

(2)

$$\begin{aligned}
\mathcal{A}_{[a,b]} &= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{c-a}{b-a}\frac{1}{c-a}\int_a^c f(x)dx + \frac{b-c}{b-a}\frac{1}{b-c}\int_c^b f(x)dx \\
&= \frac{c-a}{b-a}\mathcal{A}_{[a,c]} + \frac{b-c}{b-a}\mathcal{A}_{[c,b]}.
\end{aligned}$$

Take $t = \frac{c-a}{b-a}$.

(3) By the mean value theorem there exists a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{a}\int_0^a x^n dx = \frac{a^n}{n+1}$$

Hence, $c = \frac{a}{(n+1)^{\frac{1}{n}}}$.

(4) $\sqrt{1-x^2} \le 1$ on $[0, \frac{1}{2}]$. So, $\sqrt{1-x^2} \ge 1 - x^2$ for all $x \in (0, \frac{1}{2})$. Again, as $1 - x^2$ is a decreasing function on $[0, \frac{1}{2}]$, $\frac{1}{\sqrt{1-x^2}}$ is an increasing function on $[0, \frac{1}{2}]$. So, $\sqrt{1-x^2} = \frac{1-x^2}{\sqrt{1-x^2}} \le \frac{1-x^2}{\sqrt{1-(\frac{1}{2})^2}} = \sqrt{\frac{4}{3}}(1-x^2)$ for all $x \in (0, \frac{1}{2})$. Hence

$$\int_0^{\frac{1}{2}}(1-x^2)dx \le \int_0^{\frac{1}{2}} \sqrt{1-x^2}dx \le \sqrt{\frac{4}{3}}\int_0^{\frac{1}{2}}(1-x^2)dx$$

$$i.e \ \frac{11}{24} \le \int_0^{\frac{1}{2}} \sqrt{1-x^2}dx \le \frac{11}{24}\sqrt{\frac{4}{3}}$$

(5) $\frac{1}{1+x^2} = \frac{1-x^2+x^4}{1+x^6}$.
Now $1 + x^6$ is an increasing function on the interval $[0, a]$. So, $1 < 1 + x^6 < 1 + a^6$. This implies,

$$\frac{1}{1+a^6}\int_0^a (1-x^2+x^4)dx < \int_0^a \frac{1}{1+x^2}dx < \int_0^a (1-x^2+x^4)dx$$

$$i.e, \frac{1}{1+a^6}(a - \frac{a^3}{3} + \frac{a^5}{5}) < \int_0^a \frac{1}{1+x^2}dx < (a - \frac{a^3}{3} + \frac{a^5}{5})$$

(6) $\sin t$ changes its sign on the interval $[2\pi, 4\pi]$. So we can't apply weighted mean value theorem considering $\sin t$ as weight function.

(7) Suppose there exists a point $c \in [a, b]$ such that f is continuous at c and $f(c) > 0$. If $c \in (a, b)$ then there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \frac{f(c)}{2} \quad \forall\, x \in (c - \delta, c + \delta)$$

$$\Rightarrow f(x) > \frac{f(c)}{2} \quad \forall\, x \in (c - \delta, c + \delta)$$

Now,

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{c-\frac{\delta}{2}} f(x)dx + \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} f(x)dx + \int_{c+\frac{\delta}{2}}^{b} f(x)dx \\ &> 0 + \frac{f(c)}{2}\delta + 0 \\ &> 0\end{aligned}$$

a contradiction. So $f(c) = 0$

If $c = a$, then then there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{f(a)}{2} \qquad \forall\, x \in [a, a+\delta)$$

$$\Rightarrow f(x) > \frac{f(a)}{2}; \quad \forall\, x \in [a, a+\delta)$$

Now,

$$\int_a^b f(x)dx = \int_a^{a+\frac{\delta}{2}} f(x)dx + + \int_{a+\frac{\delta}{2}}^{b} f(x)dx \tag{0.1}$$

$$> \frac{f(a)}{2}\delta + 0 \tag{0.2}$$

$$> 0 \tag{0.3}$$

a contradiction. So, $f(c) = 0$.

Similarly, if $c = b$, then also $f(c) = 0$.

(8) Consider $x > 0$

then $\int\limits_0^x (t+|t|)^2 dt = \int\limits_0^x (2t)^2 dt = \int\limits_0^x 4t^2 dt = \frac{4}{3}x^3 = \frac{2x^2}{3}(x+|x|)$

Next if we consider $x < 0$

then $\int\limits_0^x (t+|t|)^2 dt = \int\limits_0^x (t-t)^2 dt = 0 = \frac{2x^2}{3}(x+|x|)$

$x = 0$ case is trivial. Hence we are done.

(9) A function f satisfying the following equation

$$\int\limits_c^x tf(t)dt = \sin x - x\cos x - \frac{1}{2}x^2 \ \ \forall x \in \mathbb{R}$$

Differentiating both sides wrt x

$$xf(x) - cf(c) = \cos x - \cos x + x\sin x - x = x(\sin x - 1)$$

Considering $c = 0$, we are getting

$$xf(x) = x(\sin x - 1) \ \ \forall x \in \mathbb{R}$$

Therefore $f(x) = \sin x - 1$ and a constant $c = 0$.

(10) It is given that f'' is continuous and $f(\pi) = 1$ and

$$\int_0^\pi (f(x) + f''(x)) \sin x dx = 0$$

Splitting the integral,

$$\int_0^\pi f(x) \sin x dx + \int_0^\pi f''(x) \sin x dx = 0$$

Again using integration by parts formula for the 2nd integral,

$$\int_0^\pi f(x) \sin x dx + (\sin \pi f'(\pi) - \sin 0 f'(0)) - \int_0^\pi f'(x) \cos x dx = 0$$

Using integration by parts formula for the last term,

$$\int_0^\pi f(x) \sin x dx - (\cos \pi f(\pi) - \cos 0 f(0)) - \int_0^\pi f(x) \sin x dx = 0$$

$$f(\pi) + f(0) = 0 \ \ therefore \quad f(0) = -1.$$