Homework 10

Analysis and Linear Algebra I (Autumn 2018)
Indian Institute of Science

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1. Exercises 1, 3, 6, 9 of Section 5.5 in the textbook.
2. Let $A_{[a,b]}(f)$ be the average value of f in $[a,b]$. If $a < c < b$, prove that there exists a $t \in (0,1)$ such that $A_{[a,b]}(f) = tA_{[a,c]}(f) + (1-t)A_{[c,b]}(f)$.
3. Find $c \in [0,a]$ satisfying the mean value theorem for integrals for the function x^n where $n \in \mathbb{N}$.
4. Show that
$$\frac{11}{24} \leq \int_0^{1/2} \sqrt{1-x^2}\,\mathrm{d}x \leq \frac{11}{24}\sqrt{\frac{4}{3}}.$$
5. Show that, for an $a > 0$,
$$\frac{1}{1+a^6}\left(a - \frac{a^3}{3} + \frac{a^5}{5}\right) \leq \int_0^a \frac{\mathrm{d}x}{1+x^2} \leq \left(a - \frac{a^3}{3} + \frac{a^5}{5}\right).$$
6. What is the error in the following argument? The integral
$$\int_{2\pi}^{4\pi} \frac{\sin t}{t}\,\mathrm{d}t = 0$$
by the weighted mean value theorem for integrals because there exists a $c \in [2\pi, 4\pi]$ such that it equals $\frac{1}{c}\int_{2\pi}^{4\pi} \sin t\,\mathrm{d}t$, which is zero no matter what c is.
7. If f is integrable and nonnegative on $[a,b]$ and $\int_a^b f(x)\,\mathrm{d}x = 0$, then $f(x) = 0$ at all continuity points of f.
8. Exercises 1, 3, 6, 9 of Section 5.5 in the textbook.
9. Show that, for all $x \in \mathbb{R}$,
$$\int_0^x (t+|t|)^2\,\mathrm{d}t = \frac{2x^2}{3}(x+|x|).$$
10. Find a function f and a constant c such that, for all $x \in \mathbb{R}$,
$$\int_c^x t\,f(t)\,\mathrm{d}t = \sin x - x\cos x - \frac{1}{2}x^2.$$
11. Suppose f'' is continuous. Find $f(0)$ if $f(\pi) = 1$ and
$$\int_0^\pi \big(f(x) + f''(x)\big)\sin x\,\mathrm{d}x = 0.$$